NON-VANISHING OF QUADRATIC TWISTS OF MODULAR L-FUNCTIONS

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1. Introduction

Let $F(z) := \sum_{n=1}^{\infty} a(n)q^n \in S_{2k}(M, \chi_0)$ ($q := e^{2\pi iz}$ as usual) be a newform of weight $2k$ with trivial Nebentypus character χ_0, and let $L(F, s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ be its L-function. If $D \neq 0$ is a fundamental discriminant, then let χ_D denote the Kronecker character for the field $\mathbb{Q}(\sqrt{D})$. The D-quadratic twist of F, denoted F_D, is the newform corresponding to the twist of F by the character χ_D. If $(D, M) = 1$, then $F_D(z) := \sum_{n=1}^{\infty} \chi_D(n)a(n)q^n$. The central critical values $L(F_D, k)$ have been the subject of much study, both because of their intrinsic interest and because of the prominent role they have played in Kolyvagin's work on the Birch and Swinnerton-Dyer Conjecture (see [B-F-H], [I], [J], [Ko], [Ma-M], [M-M1], [O-S], [P-P]).

Waldspurger proved a fundamental theorem [Théorème 1, W1] relating these central critical values to the Fourier coefficients of half-integral weight cusp forms. For notational convenience, if D is a fundamental discriminant of a quadratic number field, then define D_0 by

$$D_0 := \begin{cases} |D| & \text{if } D \text{ is odd,} \\ |D|/4 & \text{if } D \text{ is even.} \end{cases} \tag{1}$$

Waldspurger's theorem then guarantees the existence of a $\delta(F) \in \{\pm 1\}$, an integer N, and a non-zero eigenform $0 \neq g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(N)$ of half-integral weight $k + \frac{1}{2}$ such that $4M|N$ and for each fundamental discriminant D for which $\delta(F)D > 0$,

$$b(D_0)^2 = \begin{cases} \varepsilon_D \frac{L(F_D,k)D_0^{k-\frac{1}{2}}}{\Omega} & \text{if } (D_0, N) = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{2a}$$

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where Ω is some non-zero complex period of F and ε_D is an algebraic number. Moreover,

(2b) the $b(D_0)$'s are algebraic integers in some finite extension of $\mathbb{Q}$.

(For a proof that the existence of such a g is a consequence of Waldspurger's work see the beginning of §2 below.) This result is at the heart of this paper's study of the values $L(F_D, k)$.

There have been numerous papers focusing on the non-vanishing of $L(F_D, k)$. The works of Bump, Friedberg, Hoffstein [B-F-H], [F-H], Luo [H-L], M.R. Murty and V. K. Murty [M-M2], Mai [Ma-M], Ono [O1] and Waldspurger [W1] [W2], among others, guarantee the existence of infinitely many fundamental discriminants D for which $L(F_D, k) \neq 0$. In this note we too focus on questions pertaining to the non-vanishing of the values $L(F_D, k)$.

Definition. *Let P be the set of fundamental discriminants. If $\pi = \{p_1, p_2, \ldots, p_t\}$ is an arbitrary finite set of distinct primes, and if $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots \varepsilon_t) \in \{\pm 1\}^t$, then define sets of fundamental discriminants $P(r), P(\varepsilon, \pi)$ and $P(\varepsilon, \pi, r)$ by*

$$\begin{aligned} P(r) :=& \{D \in P \mid D \ \ \textit{square-free with exactly } r \ \ \textit{prime factors}\}, \\ P(\varepsilon, \pi) :=& \{D \in P \mid D \ \ \textit{square-free, } \chi_D(p_i) = \varepsilon_i \ \ \textit{for each } i\}, \\ P(\varepsilon, \pi, r) :=& \{D \in P(\varepsilon, \pi) \mid D \textit{ has exactly } \ r \ \ \textit{prime factors}\}. \end{aligned}$$

If $F \in S_{2k}(M, \chi_0)$ is a newform, then for any $P(\varepsilon, \pi, r)$ we consider the following question.

Question. *How many $0 < |D| \leq X$ in $P(\varepsilon, \pi, r)$ have the property that*

$$L(F_D, k) \neq 0?$$

This question for $r = 1$ (i.e., prime twists of F) has been asked by H. Iwaniec.

In a recent preprint, Hoffstein and Luo [H-L] have proved that there are infinitely many D in $\bigcup_{r=1}^{4} P(\varepsilon, \pi, r)$ for which $L(F_D, k) \neq 0$. Inspired by some ideas in [O2], we prove a fundamental lemma that implies that under a certain (mild) condition a *positive proportion* of $D \in P(\varepsilon, \pi, r)$ have the property that $L(F_D, k) \neq 0$. We expect that this condition holds for $P(\varepsilon, \pi, 1)$ for every newform F of even weight and trivial Nebentypus character. In other words, we expect that our methods always prove that there are infinitely many nonvanishing prime twists. For evidence supporting this, see Corollary 2 below and the example involving Ramanujan's Delta function.

Goldfeld conjectured that a positive proportion of $0 < |D| \leq X$ have the property that $L(F_D, k) \neq 0$ (cf. [G] and [K-S]). This has only been proved for very exceptional F by James [Ja], Kohnen [K2], and Vatsal [V]. Apart from these forms, the best result to date is due to Perelli and Pomykala [P-P] who show that the number of $0 < |D| \leq X$ for which $L(F_D, k) \neq 0$ is $\gg X^{1-\varepsilon}$. Our fundamental lemma combined with a result of Friedberg and Hoffstein implies the stronger result that the number of $0 < |D| \leq X$ for which $L(F_D, k) \neq 0$ is $\gg X/\log X$.

For modular elliptic curves, these results shed light on the distribution of quadratic twists having rank zero. Recall that if $E/\mathbb{Q}$ is an elliptic curve given by

$$E: \quad y^2 = x^3 + Ax + B,$$

then $E(D)$, its D-quadratic twist, is the curve given by

$$E(D): \quad y^2 = x^3 + AD^2x + BD^3.$$

Although it is widely believed that a positive proportion of twists $E(D)$ have rank zero, this is only known for special curves. Heath-Brown [HB] and Wong [Wo] obtain such results under special circumstances. When $r = 1$, the above Question is connected to the following conjecture which was brought to our attention by J. Silverman.

Conjecture. *If $E/\mathbb{Q}$ is an elliptic curve, then there are infinitely many primes p for which either $E(p)$ or $E(-p)$ has rank zero.*

Our results follow from the following lemma.

Fundamental Lemma. *Let $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(N)$ be an eigenform for which*

(*i*) *$b(m) \neq 0$ for at least one square-free $m > 1$* coprime to $4N$,

(*ii*) *the coefficients $b(n)$ are algebraic integers contained in a number field K,*

and let $\delta \in \{\pm 1\}$. Let v be a place of K over 2, and for each s let

$$B_s := \{\delta m \mid m > 1 \ \ \textit{square-free, } (m, 4N) = 1, \ \textit{and } \mathrm{ord}_v(b(m)) = s\}.$$

Let s_0 be the smallest integer for which $B_{s_0} \neq \emptyset$. If $B_{s_0} \cap P(r) \neq \emptyset$, then

$$\#\{m \in B_{s_0} \cap P(r) \mid |m| \leq X\} \gg \frac{X}{\log X}(\log\log X)^{r-1}.$$

Using this lemma we obtain the following results, valid for arbitrary $P(\varepsilon, \pi)$ and $P(\varepsilon, \pi, r)$.

Corollary 1. *Suppose $F \in S_{2k}(M, \chi_0)$ is a newform. Let $g(z) := \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(N)$ be an eigenform satisfying* (2a,b). *Let K be the extension of $\mathbb{Q}$ generated by the $b(n)$'s, and v a place of K over 2. Define u_0 by*

$$u_0 := \min\{u \mid \mathrm{ord}_v(b(|D|) = u \textit{ for some } \ D \in P(\varepsilon) \textit{ coprime to } 4N, \ \delta(F)D > 0\}.$$

If there exists a $D_1 \in P(\varepsilon, \pi, r)$ coprime to $4N$, $\delta(F)D_1 > 0$, for which $\mathrm{ord}_v(b(|D_1|)) = u_0$, *then a positive proportion of the $D \in P(\varepsilon, \pi, r)$ satisfy $L(F_D, k) \neq 0$.*

Corollary 2. *If $E/\mathbb{Q}$ is an elliptic curve with conductor* ≤ 100, *then either $E(-p)$ or $E(p)$ has rank zero for a positive proportion of primes p.*

Corollary 3. *If $F \in S_{2k}(M, \chi_0)$ is a newform, then the number of $0 < |D| \leq X$ in $P(\varepsilon, \pi)$ for which $L(F_D, k) \neq 0$ is $\gg X/\log X$. In particular, if $E/\mathbb{Q}$ is a modular elliptic curve, then the number of $0 < |D| \leq X$ for which $E(D)$ has rank zero is $\gg X/\log X$.*

2. Proofs

For each positive integer k, let $S_{k+\frac{1}{2}}(N)$ be the space of cusp forms of half-integral weight $k+\frac{1}{2}$ on $\Gamma_1(4N)$, and let $S_k(M)$ (resp. $S_k(M,\chi_0)$) be the space of cusp forms of weight k on $\Gamma_1(M)$ (resp. $\Gamma_0(M)$ with trivial Nebentypus character χ_0).

Proof of (2a,b). While Waldspurger's theorem is quite general, two technical hypotheses, H1 and H2 in the notation of [W1], intervene in an attempt to apply it to an arbitrary form F. However, there is a twist F_ψ of F, satisfying these hypotheses. One can construct such a character ψ as follows. Choose ψ to be a product of an even character of conductor a large power of 2 and odd characters of conductor either ℓ or ℓ^2 for each prime ℓ for which $\varepsilon_\ell(F)=-1$ (ℓ can equal 2). Here, $\varepsilon_\ell(F)$ is the local root number at ℓ. That F_ψ satisfies Hypothesis H1 is a consequence of the characterization of local root numbers. The large power of 2 dividing the conductor of ψ ensures that F_ψ satisfies Hypothesis H2. A similar construction is carried out in more detail in [Section 6, J]. Now put

$$\delta(F)=(-1)^k\psi(-1) \quad \text{and} \quad \chi=\begin{cases} \psi\left(\frac{-1}{\cdot}\right) & \text{if } \psi(-1)=-1,\\ \psi & \text{otherwise,}\end{cases}$$

and apply [Théorème 1, W1] to the form F_ψ and character χ (which is even by construction and satisfies $\chi^2=\psi^2$). The existence of an N and g satisfying (2a) can be seen by inspecting the explicit formulae given in [W1] for the functions $c_p(n)$ (notation as in [I, 4, W1]). Moreover, N can be chosen so that it is divisible by the conductor $\mathrm{cond}(\chi)$ of χ. This is just a straight-forward case-by-case analysis. The ε_D's can be taken to be the root numbers $W(\chi^{-1}\chi_{-4}^k\chi_D)$ if $\delta(F)=1$ or $W(\chi^{-1}\chi_{-4}^{k+1}\chi_D)$ if $\delta(F)=-1$. By the theory of modular symbols (cf. [M-T-T] and [Theorem 3.5.4, G-S]) there is a complex period Ω_0 such that $L(F_D,k)D_0^{k-\frac{1}{2}}/\Omega_0$ is in the ring of integers of some finite extension of $\mathbb{Q}$ for any discriminant D for which $\delta(F)D>0$ and $(4M,D_0)=1$. It is a simple consequence of the definition of the root numbers ε_D and the assumption that $(D_0,N)=1$ that $\mathrm{cond}(\chi)^{\frac{1}{2}}\varepsilon_D$ is an algebraic integer lying in a fixed, finite extension of $\mathbb{Q}$. Property (2b) follows from combining these two observations with (2a) (take $\Omega=\Omega_0\mathrm{cond}(\chi)^{-\frac{1}{2}}$). Note that we are not making a precise claim about the nature of N. To do so would require much more care than is needed for either statements (2a,b) or the applications in this paper.

Q.E.D.

By the theory of newforms, every $F\in S_k(M)$ can be uniquely expressed as a linear combination

$$F(z)=\sum_{i=1}^{r}\alpha_iA_i(z)+\sum_{j=1}^{s}\beta_jB_j(\delta_jz),$$

where $A_i(z)$ and $B_j(z)$ are newforms of weight k and level a divisor of M, and where each δ_j is a non-trivial divisor of M. Let

$$F^{\mathrm{new}}(z):=\sum_{i=1}^{r}\alpha_iA_i(z) \quad \text{and} \quad F^{\mathrm{old}}(z):=\sum_{j=1}^{s}\beta_jB_j(\delta_jz)$$

be, respectively, the *new part of* F and the *old part of* F.

If $F(z) := \sum_{n=1}^{\infty} a(n)q^n \in S_k(M)$ is a newform, then the $a(n)$'s are algebraic integers and generate a finite extension of $\mathbb{Q}$, say K_F. If K is any finite extension of $\mathbb{Q}$ containing K_F, and if $\mathcal{O}_v$ is the completion of the ring of integers of K at any finite place v with residue characteristic, say ℓ, then by the work of Shimura, Deligne, and Serre ([Sh], [D], [D–S]) there is a (not necessarily unique) continuous representation

$$\rho_{F,v} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_v)$$

for which

$(R1)$ $\rho_{F,v}$ is unramified at all primes $p \nmid M\ell$.

$(R2)$ trace $\rho_{F,v}(\mathrm{frob}_p) = a(p)$ for all primes $p \nmid M\ell$.

Here frob_p denotes any Frobenius element for the prime p.

Proof of Lemma. Let s_0 be the smallest integer such that $B_{s_0} \neq \emptyset$, and let $m_0 > 1$ be some square-free integer coprime to $4N$ for which $\mathrm{ord}_v(b(m_0)) = s_0$. It is clear that by taking combinations of quadratic twists (an possibly twists of twists) one can find a cusp form $g'(z) := \sum_{n=1}^{\infty} b'(n)q^n$ of weight $k+\frac{1}{2}$ and level N' coprime to m_0 for which $b'(m_0) = b(m_0)$ and, for any square-free integer m, $b'(m) = 0$ if $(m, 4N') \neq 1$ or $m = 1$, and otherwise $b'(m)$ is either $b(m)$ or 0. Since g is an eigenform, it follows that

$$\mathrm{ord}_v(b'(n)) \geq s_0 \tag{3}$$

for all n. Let $G(z) := \sum_{n=1}^{\infty} c(n)q^n := g'(z) \cdot \left(1 + 2\sum_{n=1}^{\infty} q^{n^2}\right)$, so

$$c(n) = b'(n) + 2 \sum_{\substack{mx^2+y^2=n,\, y>0 \\ m \text{ square-free}}} b'(mx^2). \tag{4}$$

Then G is a cusp form of integer weight $k+1$ on $\Gamma_1(4N')$. Write $G = G^{\mathrm{new}} + G^{\mathrm{old}}$. Since $\mathrm{ord}_v(b'(m_0)) = s_0$, it follows from (3) and (4) that $c(m_0) \neq 0$. Since m_0 is coprime to the level of G, it must be that G^{new} is not identically zero. Write

$$G^{\mathrm{new}} = \sum_{i=1}^{h} \alpha_i f_i(z), \quad \alpha_i \neq 0,$$

where each $f_i(z) := \sum_{n=1}^{\infty} a_i(n)q^n$ is a newform of level dividing $4N'$. If $(n, 2N') = 1$, then

$$c(n) = \sum_{i=1}^{h} \alpha_i a_i(n). \tag{5}$$

Let L be a finite extension of $\mathbb{Q}$ containing K, the Fourier coefficients of each f_i, and the α_i's. Let w be a place of L over v, let e be the ramification index of w over v, let $\mathcal{O}_w$ be the completion of the ring of integers of L at the place w, and let λ be a uniformizer for $\mathcal{O}_w$. Let

$$E = \max_{1\leq i\leq h} |\mathrm{ord}_w(4\alpha_i)|, \tag{6}$$

and let $\rho_{f_i,w} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_w)$ be a representation as in the preceding discussion. Finally, let $\varepsilon : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{O}^\times$ be the cyclotomic character giving the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on all 2^nth power roots of unity. Consider the representation

$$\rho = \varepsilon \bigoplus_{i=1}^{h} \rho_{f_i,w} \mod \lambda^{E+es_0+1}.$$

Write

$$m_0 = p_1 \cdots p_r, \qquad p_j \text{ a prime }.$$

By the Chebotarev Density Theorem, for each j there are $\gg X/\log X$ primes q less than X for which $\rho(\mathrm{frob}_q) = \rho(\mathrm{frob}_{p_j})$. By (R2), for such a prime, $a_i(q) \equiv a_i(p_j) \mod \lambda^{E+es_0+1}$ for all i. Also, $q = \varepsilon(\mathrm{frob}_q) \equiv \varepsilon(\mathrm{frob}_{p_j}) = p_j \bmod 4$. It follows from these observations and the multiplicativity of the Fourier coefficients of newforms that there are $\gg \frac{X}{\log X}(\log\log X)^{r-1}$ square-free integers $\delta m = \delta q_1 \cdots q_r \in P(r)$, $m < X$, such that

$$(m, 4N') = 1 \text{ and } a_i(m) \equiv a_i(m_0) \bmod \lambda^{E+es_0+1}.$$

For any such m, it follows from (5) and (6) that $c(m) \equiv c(m_0) \bmod \lambda^{es_0+1}$. By our choice of s_0, $\mathrm{ord}_w(c(m_0)) = es_0$, so $\mathrm{ord}_w(c(m)) = es_0$. It follows from (3) and (4) that $\mathrm{ord}_w(b'(m)) = es_0$ (equivalently, $\mathrm{ord}_v(b'(m)) = s_0$), whence $\mathrm{ord}_v(b(m)) = s_0$. In other words, $\delta m \in B_{s_0}$. This proves the lemma.

Q.E.D.

Proof of Corollary 1. By taking combinations of quadratic twists of g (and possibly twists of twists), one obtains an eigenform $g^*(z) := \sum_{n=1}^{\infty} b^*(n)$ of level coprime to D_1 whose coefficients are supported on integers $m > 1$ such that $\chi_{\delta(F)m}(p_i) = \varepsilon_i$ for each i and for which $b^*(|D_1|) = b(|D_1|) \neq 0$. Furthermore, $b^*(m)$ is either $b(m)$ or 0. The corollary now follows from the Fundamental Lemma applied to the eigenform $g^*(z)$ with $\delta = \delta(F)$ (note: $s_0 = u_0$).

Q.E.D.

Proof of Corollary 2. This result is an application of the previous corollary with $r = 1$ and $\pi = \emptyset$ and of the work of Kolyvagin [Ko]. For each isogeny class of elliptic curves over $\mathbb{Q}$, Basmaji [B] computed a basis from which a relevant eigenform $g(z) = \sum_{n=1}^{\infty} b(n)q^n$ can be constructed. The rest of the proof involves checking the condition of Corollary 1.

Q.E.D.

Proof of Corollary 3. Let $g(z) := \sum_{n=1}^{\infty} b(n)q^n$ be an eigenform satisfying (2a,b). By [Theorem B(i), F-H], there is a $D' \in P(\varepsilon, \pi)$ coprime to $4N$ for which $b(|D'|) \neq 0$. One now proceeds as in the proof of Corollary 1, with D' playing the role of D_1. The application to modular elliptic curves follows from the work of Kolyvagin [Ko].

Q.E.D.

Remark 1. The theorem of Friedberg and Hoffstein was a critical ingredient in the proof of Corollary 3. However, the proof does not require the existence of infinitely many non-vanishing critical values, only a single suitable non-zero value.

Remark 2. The fundamental lemma is a result about the coefficients of an eigenform $g(z)$ of half-integral weight. However, the result can be applied in a slightly different setting. Let M be an odd square-free integer, and let $F(z) \in S_{2k}(M, \chi_0)$ be a newform. Kohnen and Zagier [K1], [K–Z] have constructed an explicit cusp form $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(M)$ for which $b(n) = 0$ unless $(-1)^k n \equiv 0, 1 \pmod 4$ and for which

$$L(F_D, k) = 2^{-\nu(M)}|D|^{\frac{1}{2}-k} \frac{\pi^k}{(k-1)!} \frac{\langle F, F \rangle}{\langle g, g \rangle} |b(|D|)|^2 \tag{7}$$

for any fundamental discriminant D for which $(-1)^k D > 0$ and $\chi_D(\ell) = w_\ell$, the eigenvalue of the Atkin-Lehner involution at ℓ, for each prime ℓ dividing N. This $g(z)$ is an eigenform for operators similar to the classical Hecke operators. The conclusion and proof of the fundamental lemma applies to these forms as well.

Example. Let $\Delta(z) := \sum_{n=1}^{\infty} \tau(n)q^n \in S_{12}(1)$ be Ramanujan's delta function, and let $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{\frac{13}{2}}(4, \chi_0)$ be the eigenform given in [K–Z] satisfying (7). It turns out that $g(z) \equiv \sum_{n=0}^{\infty} q^{(2n+1)^2} \pmod 8$, but modulo 16 it is

$$g(z) \equiv q + 8q^4 + 8q^5 + 9q^9 + 8q^{13} + \dots \pmod{16}.$$

By the analog of Corollary 1 (see Remark 2), we find that $u_0 = 4$ since $b(5) \equiv 0 \pmod 8$ but $b(5) \not\equiv 0 \pmod{16}$. Since $5 \in P(1)$, there is a positive proportion of primes p for which $L(\Delta_p, 6) \neq 0$. In fact, Kohnen and Zagier [Corollary 2, K-Z] first noticed that if $p \equiv 5 \pmod 8$ is prime, then $L(\Delta_p, 6) \neq 0$.

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