A NOTE ON THE IRREDUCIBILITY OF HECKE POLYNOMIALS

Kevin James and Ken Ono

Let S_{2k} denote the space of modular cusp forms of weight $2k$ for $SL_2(\mathbb{Z})$. If $f(z) := \sum_{n=1}^{\infty} a_f(n)q^n \in S_{2k}$ (here $q := e^{2\pi iz}$ throughout), and p is prime, then the Hecke operator T_p^{2k} is the linear transformation on S_{2k} given by

$$T_p^{2k} f := \sum_{n=1}^{\infty} \left(a_f(np) + p^{2k-1} a_f(n/p) \right) q^n.$$

Let $T_p^{2k}(x)$ denote the characteristic polynomial of the action of T_p^{2k} on S_{2k}. It is well known that $T_p^{2k}(x) \in \mathbb{Z}[x]$ and has degree d_k where $d_k := \dim(S_{2k})$. However, much more is conjectured to be true. Maeda has conjectured that the Hecke algebra of S_{2k} over $\mathbb{Q}$ is simple, and that its Galois closure over $\mathbb{Q}$ has Galois group S_{d_k}. Recently there have been numerous investigations regarding the irreducibility of characteristic polynomials of Hecke operators on S_{2k}. The existence of such polynomials have proven to be useful in proving nonvanishing theorems for central values of level 1 modular L-functions, and in constructing base changes to totally real number fields for level 1 eigenforms (see [C-F,H-M,Ko-Z]).

In this note we show that a "positive proportion" of the Hecke polynomials $T_p^{2k}(x)$ are irreducible if there are two distinct primes ℓ and q for which $T_q^{2k}(x)$ is irreducible over $\mathbb{F}_\ell$, the finite field with ℓ elements. Throughout p will denote a prime number.

Theorem 1. *If there are distinct primes ℓ and q for which the Hecke polynomial $T_q^{2k}(x)$ is irreducible in $\mathbb{F}_\ell[x]$, then*

$$\#\left\{ p < X \mid T_p^{2k}(x) \ \ \textit{is irreducible in } \mathbb{Q}[x] \right\} \gg_k \frac{X}{\log X}.$$

This result follows immediately from the following more general result.

The second author supported by NSF grant DMS-9508976 and NSA grant MSPR-97Y012.

As usual, we will let $S_k(N,\chi)$ denote the space of modular cusp forms of weight k, level N and character χ. For $f(z)=\sum_{n=1}^{\infty} a_f(n)q^n \in S_k(N,\chi)$ and $p \nmid N$ the Hecke operator $T_{N,p}^{k,\chi}$ is defined by

$$T_{N,p}^{k,\chi} f = \sum_{n\geq 1} \left(a_f(np) + \chi(p)p^{k-1}a_f(n/p)\right) q^n.$$

Let $T_{N,p}^{k,\chi}(x)$ denote the characteristic polynomial of $T_{N,p}^{k,\chi}$ on $S_k^{\mathrm{new}}(N,\chi)$. Moreover, let $K_{k,\chi,N}$ denote the finite extension of $\mathbb{Q}$ obtained by adjoining the roots of all of the $T_{N,p}^{k,\chi}(x)$ with $p \nmid N$.

Theorem 2. *Let q and ℓ be distinct primes not dividing N, and let $\mathcal{L}$ denote a prime ideal of $\mathbb{K}_{k,\chi,N}$ lying above ℓ. Then*

$$\#\left\{p < X \quad | \quad T_{N,p}^{k,\chi}(x) \equiv T_{N,q}^{k,\chi}(x) \pmod{\mathcal{L}}\right\} \gg_{N,\chi,k} \frac{X}{\log X}.$$

Proof. Let $\{f_1,\ldots,f_d\}$ be a basis of $S_k^{\mathrm{new}}(N,\chi)$ such that each f_i is an eigenform for all of the $T_{N,p}^{k,\chi}$ where $p \nmid N$. Let $\lambda_{f_i}(p)$ be the eigenvalue of $T_{N,p}^{k,\chi}$ corresponding to the eigenform f_i (i.e. $T_{N,p}^{k,\chi} f_i = \lambda_{f_i}(p) f_i$). By the work of Deligne, Serre and Shimura [D,D-S,Sh], there exist continuous representations

$$\rho_{f_i,\mathcal{L}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{\mathcal{L}})$$

satisfying the following conditions:

(i) $\rho_{f_i,\mathcal{L}}$ is unramified for $p \nmid N\ell$,
(ii) $\mathrm{trace}\,(\rho_{f_i,\mathcal{L}}(\mathrm{Frob}_p)) = \lambda_{f_i}(p)$ for $p \nmid N\ell$,
(iii) $\det\,(\rho_{f_i,\mathcal{L}}(\mathrm{Frob}_p)) = \chi(p)p^{k-1}$ for $p \nmid N\ell$.

Here $\mathcal{O}$ denotes the ring of integers of $K_{k,\chi,N}$, $\mathcal{O}_{\mathcal{L}}$ denotes its completion at $\mathcal{L}$ and Frob_p denotes any Frobenius element for p. Let $\mathfrak{l}$ denote a uniformizer for $\mathcal{O}_{\mathcal{L}}$. By reducing the representations $\rho_{f_i,\mathcal{L}}$ modulo $\mathfrak{l}$, we obtain representations from $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to $\mathrm{GL}_2(\mathcal{O}_{\mathcal{L}}/\langle\mathfrak{l}\rangle)$. These representations have finite image and therefore factor to yield representations:

$$\bar{\rho}_{f_i,\mathcal{L}} : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{\mathcal{L}}/\langle\mathfrak{l}\rangle),$$

where K is a finite extension of $\mathbb{Q}$.

By the Chebotarev density theorem, there is a set S of rational primes having positive Dirichlet density and having the property that for each $p \in S$, $\mathrm{Frob}_{\mathfrak{P}}$ is conjugate to $\mathrm{Frob}_{\mathfrak{Q}}$, where $\mathfrak{P}$ and $\mathfrak{Q}$ are prime ideals of K lying above p and q respectively, and therefore have the property that for each $1 \leq i \leq d$

$$\mathrm{trace}(\bar{\rho}_{f_i,\mathcal{L}}(\mathrm{Frob}_{\mathfrak{P}})) \equiv \lambda_{f_i}(p) \pmod{\mathfrak{l}} \quad \text{if } p \nmid N\ell.$$

Since the trace is conjugation invariant, it follows that for $p \in S$,

$$\lambda_{f_i}(p) \equiv \lambda_{f_i}(q) \pmod{\mathfrak{l}}.$$

Since the $\lambda_{f_i}(p)$ $(1 \leq i \leq d)$ are precisely the roots of $T_{N,p}^{k,\chi}(x)$, the theorem follows. $\square$

Example. Here we shall give a simple example that illustrates Theorem 2. We consider the Hecke polynomials on the three dimensional space $S_5^{\text{new}}(11, \chi_{-11})$. For convenience, let $T_p(x)$ denote the characteristic polynomial for the Hecke operator $T_{11,p}^{5,\chi_{-11}}$. The first few terms of the Fourier expansions of the three newforms are

$$N_1(z) = \sum_{n=1}^{\infty} a_1(n)q^n = q + 7q^3 + 16q^4 - 49q^5 - 32q^9 + \cdots,$$
$$N_2(z) = \sum_{n=1}^{\infty} a_2(n)q^n = q + \sqrt{-30}q^2 - 3q^3 - 14q^4 + 31q^5 - 3\sqrt{-30}q^6 - 10\sqrt{-30}q^7 + \cdots,$$
$$N_3(z) = \sum_{n=1}^{\infty} a_3(n)q^n = q - \sqrt{-30}q^2 - 3q^3 - 14q^4 + 31q^5 + 3\sqrt{-30}q^6 + 10\sqrt{-30}q^7 - \cdots.$$

It is easy to verify that if $p \neq 11$ is prime, then

$$a_2(p) = a_3(p) \in \mathbb{Z} \quad \text{if } \left(\frac{p}{11}\right) = 1 \tag{1}$$

and

$$a_2(p) = -a_3(p) \quad \text{if } \left(\frac{p}{11}\right) = -1. \tag{2}$$

Moreover, if $\left(\frac{p}{11}\right) = -1$, then

$$\frac{a_2(p)}{\sqrt{-30}} \in \mathbb{Z}. \tag{3}$$

These all follow from standard facts about eigenvalues of Hecke operators (e.g. [Kob]).

The form $N_1(z)$ has complex multiplication by $\mathbb{Q}(\sqrt{-11})$ in the sense of Ribet (see [R]). By construction, there is exactly one such form in this space. In particular if $p \neq 11$ is prime, then

$$a_1(p) = \begin{cases} 0 & \text{if } \left(\frac{p}{11}\right) = -1, \\ \frac{2x^4 - 132x^2y^2 + 242y^4}{16} & \text{if } \left(\frac{p}{11}\right) = 1 \text{ and } 4p = x^2 + 11y^2. \end{cases}$$

This implies that if $p \neq 11$ is prime, then

$$a_1(p) \equiv \begin{cases} 0 \pmod{11} & \text{if } \left(\frac{p}{11}\right) = -1, \\ 2p^2 \pmod{11} & \text{if } \left(\frac{p}{11}\right) = 1. \end{cases} \tag{4}$$

Now if $B(z) = \sum_{n=1}^{\infty} b(n)q^n$ is defined by

$$B(z) := \frac{15+\sqrt{-30}}{30} \cdot N_2(z) + \frac{15-\sqrt{-30}}{30} \cdot N_3(z) = q - 2q^2 - 3q^3 - \cdots,$$

then the methods of Swinnerton-Dyer [S-D] suggest that $B(z)$ may satisfy a congruence with a linear combination of twisted Eisenstein series. Using a theorem of Sturm [St], we verify indeed that there is such a congruence modulo 11, and it turns out that

$$b(n) \equiv \left(8n + 4n\left(\frac{n}{11}\right)\right) \sum_{d|n} d^7 \pmod{11}. \tag{5}$$

By combining (1-5), if $p \neq 11$ is prime, then

$$T_p(x) \equiv \begin{cases} x^3 + 5x^2 + x + 3 \pmod{11} & \text{if } p \equiv 1 \pmod{11}, \\ x^3 + 8x \pmod{11} & \text{if } p \equiv 2 \pmod{11}, \\ x^3 + 10x^2 + 3 \pmod{11} & \text{if } p \equiv 3 \pmod{11}, \\ x^3 + 8x^2 + 4 \pmod{11} & \text{if } p \equiv 4 \pmod{11}, \\ x^3 + 9x^2 + 2x + 9 \pmod{11} & \text{if } p \equiv 5 \pmod{11}, \\ x^3 + 10x \pmod{11} & \text{if } p \equiv 6 \pmod{11}, \\ x^3 + 8x \pmod{11} & \text{if } p \equiv 7 \pmod{11}, \\ x^3 + 7x \pmod{11} & \text{if } p \equiv 8 \pmod{11}, \\ x^3 + x^2 + 6x + 3 \pmod{11} & \text{if } p \equiv 9 \pmod{11}, \\ x^3 \pmod{11} & \text{if } p \equiv 10 \pmod{11}. \end{cases}$$

If p is a prime for which $\left(\frac{p}{11}\right) = 1$, then by (1) and (4) we see that $T_p(x)$ factors into three linear factors in $\mathbb{Z}[x]$. If p is a prime for which $\left(\frac{p}{11}\right) = -1$ and $a_2(p) \neq 0$, then by (2) and (3) it follows that $T_p(x)$ factors into irreducibles in $\mathbb{Z}[x]$ as

$$T_p(x) = x(x^2 + a_2(p)^2).$$

By (5) one easily finds that $a_2(p) \neq 0$ for every such $p \equiv 2, 6, 7, 8 \pmod{11}$.

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Department of Mathematics, Penn State University, University Park, Pa. 16802
E-mail address: `klj@math.psu.edu`

Department of Mathematics, Penn State University, University Park, Pa. 16802
E-mail address: `ono@math.psu.edu`