A NOTE ON THE IRREDUCIBILITY OF HECKE POLYNOMIALS

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Let $S_{2k}$ denote the space of modular cusp forms of weight $2k$ for $SL_2(\mathbb{Z})$. If $f(z) := \sum_{n=1}^{\infty} a_f(n)q^n \in S_{2k}$ (here $q := e^{2\pi iz}$ throughout), and $p$ is prime, then the Hecke operator $T_p^{2k}$ is the linear transformation on $S_{2k}$ given by

$$T_p^{2k} f := \sum_{n=1}^{\infty} \left( a_f(np) + p^{2k-1} a_f(n/p) \right) q^n.$$ 

Let $T_p^{2k}(x)$ denote the characteristic polynomial of the action of $T_p^{2k}$ on $S_{2k}$. It is well known that $T_p^{2k}(x) \in \mathbb{Z}[x]$ and has degree $d_k$ where $d_k := \dim(S_{2k})$. However, much more is conjectured to be true. Maeda has conjectured that the Hecke algebra of $S_{2k}$ over $\mathbb{Q}$ is simple, and that its Galois closure over $\mathbb{Q}$ has Galois group $S_{d_k}$. Recently there have been numerous investigations regarding the irreducibility of characteristic polynomials of Hecke operators on $S_{2k}$. The existence of such polynomials have proven to be useful in proving nonvanishing theorems for central values of level 1 modular $L$-functions, and in constructing base changes to totally real number fields for level 1 eigenforms (see [C-F,H-M,Ko-Z]).

In this note we show that a “positive proportion” of the Hecke polynomials $T_p^{2k}(x)$ are irreducible if there are two distinct primes $\ell$ and $q$ for which $T_q^{2k}(x)$ is irreducible over $\mathbb{F}_\ell$, the finite field with $\ell$ elements. Throughout $p$ will denote a prime number.

**Theorem 1.** If there are distinct primes $\ell$ and $q$ for which the Hecke polynomial $T_q^{2k}(x)$ is irreducible in $\mathbb{F}_\ell[x]$, then

$$\# \left\{ p < X \mid T_p^{2k}(x) \text{ is irreducible in } \mathbb{Q}[x] \right\} \gg_k \frac{X}{\log X}.$$ 

This result follows immediately from the following more general result.

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As usual, we will let $S_k(N, \chi)$ denote the space of modular cusp forms of weight $k$, level $N$ and character $\chi$. For $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(N, \chi)$ and $p \nmid N$ the Hecke operator $T_{N,p}^{k,\chi}$ is defined by

$$T_{N,p}^{k,\chi} f = \sum_{n \geq 1} \left( a_f(np) + \chi(p)p^{k-1}a_f(n/p) \right) q^n.$$ 

Let $T_{N,p}^{k,\chi}(x)$ denote the characteristic polynomial of $T_{N,p}^{k,\chi}$ on $S_{new}^{k,N,\chi}$. Moreover, let $K_{k,\chi,N}$ denote the finite extension of $\mathbb{Q}$ obtained by adjoining the roots of all of the $T_{N,p}^{k,\chi}(x)$ with $p \nmid N$.

**Theorem 2.** Let $q$ and $\ell$ be distinct primes not dividing $N$, and let $\mathcal{L}$ denote a prime ideal of $\mathbb{K}_{k,\chi,N}$ lying above $\ell$. Then

$$\# \left\{ p < X \mid T_{N,p}^{k,\chi}(x) \equiv T_{N,q}^{k,\chi}(x) \pmod{\mathcal{L}} \right\} \gg N, \chi, k \frac{X}{\log X}.$$ 

**Proof.** Let $\{f_1, \ldots, f_d\}$ be a basis of $S_{new}^{k,N,\chi}$ such that each $f_i$ is an eigenform for all of the $T_{N,p}^{k,\chi}$ where $p \nmid N$. Let $\lambda_{f_i}(p)$ be the eigenvalue of $T_{N,p}^{k,\chi}$ corresponding to the eigenform $f_i$ (i.e. $T_{N,p}^{k,\chi}f_i = \lambda_{f_i}(p)f_i$). By the work of Deligne, Serre and Shimura [D,D-S,Sh], there exist continuous representations

$$\rho_{f_i,\mathcal{L}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(O_{L})$$

satisfying the following conditions:

(i) $\rho_{f_i,\mathcal{L}}$ is unramified for $p \nmid N\ell$,

(ii) $\text{trace}(\rho_{f_i,\mathcal{L}}(\text{Frob}_p)) = \lambda_{f_i}(p)$ for $p \nmid N\ell$,

(iii) $\text{det}(\rho_{f_i,\mathcal{L}}(\text{Frob}_p)) = \chi(p)p^{k-1}$ for $p \nmid N\ell$.

Here $O$ denotes the ring of integers of $K_{k,\chi,N}$, $O_{\mathcal{L}}$ denotes its completion at $\mathcal{L}$ and $\text{Frob}_p$ denotes any Frobenius element for $p$. Let $l$ denote a uniformizer for $O_{\mathcal{L}}$. By reducing the representations $\rho_{f_i,\mathcal{L}}$ modulo $l$, we obtain representations from $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $GL_2(O_{\mathcal{L}}/(l))$. These representations have finite image and therefore factor to yield representations:

$$\tilde{\rho}_{f_i,\mathcal{L}} : \text{Gal}(K/\mathbb{Q}) \to GL_2(O_{\mathcal{L}}/(l)),$$

where $K$ is a finite extension of $\mathbb{Q}$.

By the Chebotarev density theorem, there is a set $S$ of rational primes having positive Dirichlet density and having the property that for each $p \in S$, $\text{Frob}_\mathfrak{P}$ is conjugate to $\text{Frob}_\mathfrak{Q}$, where $\mathfrak{P}$ and $\mathfrak{Q}$ are prime ideals of $K$ lying above $p$ and $q$ respectively, and therefore have the property that for each $1 \leq i \leq d$

$$\text{trace}(\tilde{\rho}_{f_i,\mathcal{L}}(\text{Frob}_\mathfrak{P})) \equiv \lambda_{f_i}(p) \pmod{l} \text{ if } p \nmid N\ell.$$
Since the trace is conjugation invariant, it follows that for \( p \in S \),
\[
\lambda_{f_i}(p) \equiv \lambda_{f_i}(q) \pmod{1}.
\]
Since the \( \lambda_{f_i}(p) \) (\( 1 \leq i \leq d \)) are precisely the roots of \( T_{N,p}^{k,\chi}(x) \), the theorem follows. \( \square \)

**Example.** Here we shall give a simple example that illustrates Theorem 2. We consider the Hecke polynomials on the three dimensional space \( S_{5}^{\text{new}}(11, \chi_{-11}) \). For convenience, let \( T_p(x) \) denote the characteristic polynomial for the Hecke operator \( T_{11,5}^{5,\chi_{-11}} \). The first few terms of the Fourier expansions of the three newforms are

\[
N_1(z) = \sum_{n=1}^{\infty} a_1(n)q^n = q + 7q^3 + 16q^4 - 49q^5 - 32q^9 + \cdots ,
\]
\[
N_2(z) = \sum_{n=1}^{\infty} a_2(n)q^n = q + \sqrt{-30}q^2 - 3q^3 - 14q^4 + 31q^5 - 3\sqrt{-30}q^6 - 10\sqrt{-30}q^7 + \cdots ,
\]
\[
N_3(z) = \sum_{n=1}^{\infty} a_3(n)q^n = q - \sqrt{-30}q^2 - 3q^3 - 14q^4 + 31q^5 + 3\sqrt{-30}q^6 + 10\sqrt{-30}q^7 - \cdots .
\]

It is easy to verify that if \( p \neq 11 \) is prime, then

\[
(1) \quad a_2(p) = a_3(p) \in \mathbb{Z} \quad \text{if} \quad \left( \frac{p}{11} \right) = 1
\]

and

\[
(2) \quad a_2(p) = -a_3(p) \quad \text{if} \quad \left( \frac{p}{11} \right) = -1.
\]

Moreover, if \( \left( \frac{p}{11} \right) = -1 \), then

\[
(3) \quad \frac{a_2(p)}{\sqrt{-30}} \in \mathbb{Z}.
\]

These all follow from standard facts about eigenvalues of Hecke operators (e.g. [Kob]).

The form \( N_1(z) \) has complex multiplication by \( \mathbb{Q}(\sqrt{-11}) \) in the sense of Ribet (see [R]). By construction, there is exactly one such form in this space. In particular if \( p \neq 11 \) is prime, then

\[
a_1(p) = \begin{cases} 
0 & \text{if } \left( \frac{p}{11} \right) = -1, \\
\frac{2x^4 - 132x^2y^2 + 242y^4}{16} & \text{if } \left( \frac{p}{11} \right) = 1 \text{ and } 4p = x^2 + 11y^2.
\end{cases}
\]
This implies that if $p \neq 11$ is prime, then

\[
(4) \quad a_1(p) \equiv \begin{cases} 
0 \pmod{11} & \text{if } (\frac{p}{11}) = -1, \\
2p^2 \pmod{11} & \text{if } (\frac{p}{11}) = 1.
\end{cases}
\]

Now if $B(z) = \sum_{n=1}^{\infty} b(n)q^n$ is defined by

\[
B(z) := \frac{15 + \sqrt{-30}}{30} \cdot N_2(z) + \frac{15 - \sqrt{-30}}{30} \cdot N_3(z) = q - 2q^2 - 3q^3 - \cdots,
\]

then the methods of Swinnerton-Dyer [S-D] suggest that $B(z)$ may satisfy a congruence with a linear combination of twisted Eisenstein series. Using a theorem of Sturm [St], we verify indeed that there is such a congruence modulo 11, and it turns out that

\[
(5) \quad b(n) \equiv \left(8n + 4n\left(\frac{n}{11}\right)\right) \sum \frac{d^2}{d|n} \pmod{11}.
\]

By combining (1-5), if $p \neq 11$ is prime, then

\[
T_p(x) \equiv \begin{cases} 
x^3 + 5x^2 + x + 3 \pmod{11} & \text{if } p \equiv 1 \pmod{11}, \\
x^3 + 8x \pmod{11} & \text{if } p \equiv 2 \pmod{11}, \\
x^3 + 10x^2 + 3 \pmod{11} & \text{if } p \equiv 3 \pmod{11}, \\
x^3 + 8x^2 + 4 \pmod{11} & \text{if } p \equiv 4 \pmod{11}, \\
x^3 + 9x^2 + 2x + 9 \pmod{11} & \text{if } p \equiv 5 \pmod{11}, \\
x^3 + 10x \pmod{11} & \text{if } p \equiv 6 \pmod{11}, \\
x^3 + 8x \pmod{11} & \text{if } p \equiv 7 \pmod{11}, \\
x^3 + 7x \pmod{11} & \text{if } p \equiv 8 \pmod{11}, \\
x^3 + x^2 + 6x + 3 \pmod{11} & \text{if } p \equiv 9 \pmod{11}, \\
x^3 \pmod{11} & \text{if } p \equiv 10 \pmod{11}.
\end{cases}
\]

If $p$ is a prime for which $\left(\frac{p}{11}\right) = 1$, then by (1) and (4) we see that $T_p(x)$ factors into three linear factors in $\mathbb{Z}[x]$. If $p$ is a prime for which $\left(\frac{p}{11}\right) = -1$ and $a_2(p) \neq 0$, then by (2) and (3) it follows that $T_p(x)$ factors into irreducibles in $\mathbb{Z}[x]$ as

\[
T_p(x) = x(x^2 + a_2(p))^2.
\]

By (5) one easily finds that $a_2(p) \neq 0$ for every such $p \equiv 2, 6, 7, 8 \pmod{11}$.
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References


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