ON THE CIRCULAR SUMMATION OF THE ELEVENTH POWERS OF RAMANUJAN'S THETA FUNCTION

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Abstract. Let $f(a,b)$ denote Ramanujan's theta series. In his "Lost Notebook", Ramanujan claimed that the "circular" summation of n^{th} powers of f satisfies a factorization of the form $f(a^n,b^n)F_n(a^nb^n)$ where $F_n(q)=1+2nq^{\frac{n-1}{2}}+\cdots$. Moreover, he listed explicit closed formulas for F_2, F_3, F_4, F_5, and F_7. Berndt and Son have asked for a similar expression for any other F_n. Here we obtain such an expression for $F_{11}(q)$.

For $|ab|<1$ let $f(a,b)$ denote Ramanujan's theta function

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{\frac{k(k+1)}{2}} b^{\frac{k(k-1)}{2}}. \tag{1}$$

In his "Lost Notebook" [p. 54, R], Ramanujan claims (without proof) that if $n\geq 2$, then

$$\sum_{r=0}^{n-1} U_r^n f^n\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right) = f(a^n,b^n)F_n(a^nb^n) \tag{2}$$

where $F_n(q)=1+2nq^{\frac{n-1}{2}}+\cdots$, and

$$U_j := a^{j(j+1)/2}b^{j(j-1)/2} \quad \text{and} \quad V_j := a^{j(j-1)/2}b^{j(j+1)/2}.$$

Throughout, we shall assume that $|q|<1$. Rangachari [Ra] proved Ramanujan's claim using the theory of modular forms, and recently Son [S] obtained an elementary proof of the claim.

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In Ramanujan's notation [R], let $f(-q), \psi(q)$, and $\phi(q)$ denote the theta functions

$$f(-q) := \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k+1)}{2}}, \quad \psi(q) := \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}}, \quad \text{and} \quad \phi(q) := \sum_{k=-\infty}^{\infty} q^{k^2}.$$

In [R], Ramanujan claims the following identities:

$$F_2(q) = \phi(\sqrt{q}),$$

$$F_3(q) = \left(\frac{f^9(-q)}{f^3(-q^3)} + 27q \cdot \frac{f^9(-q^3)}{f^3(-q)} \right)^{1/3},$$

$$F_4(q) = \phi^3(q^2) + (2\sqrt{q})^3 \cdot \psi^3(q^4),$$

$$F_5(q) = \frac{f^5(-q)}{f(-q^5)} + 5q \cdot \frac{f^5(-q^5)}{f(-q)},$$

$$F_7(q) = \frac{f^7(-q)}{f(-q^7)} + 7q^2 \cdot \frac{f^7(-q^7)}{f(-q)} + 7q \cdot f^3(-q) f^3(-q^7).$$

These identities have now been proved by Rangachari and Son (see [Ra], [S]). Son's proofs are completely elementary and do not depend on the theory of modular forms.

Berndt and Son have asked for similar "elegant" expressions for any other $F_n(q)$. Here we present such an expression for $F_{11}(q)$; however, first we shall analyze some of the summands which appear in Ramanujan's identities. For instance, the expression for $F_5(q)$ contains the Eisenstein series (see [p. 129, H])

$$\text{(3a)} \qquad q \cdot \frac{f^5(-q^5)}{f(-q)} = \sum_{n=1}^{\infty} \sum_{d|n} \left(\frac{d}{5} \right) \frac{n}{d} \cdot q^n = q + q^2 + 2q^3 + 3q^4 + 5q^5 + \cdots,$$

and the expression for $F_7(q)$ contains the following bivariate theta function (i.e. "CM" form)

$$\text{(3b)} \qquad q \cdot f^3(-q) f^3(-q^7) = \sum_{\substack{x,y \geq 1 \text{ odd} \\ x^2+7y^2 \equiv 0 \pmod 8}} (-1)^{\frac{x+y}{2}-1} xy q^{\frac{x^2+7y^2}{8}}.$$

Given a modular form $f(z) \in M_k(\Gamma_0(N), \chi)$, one often typically tries to express f in terms of well known "canonical forms" which lie in the space. Such forms include the Eisenstein series, like the form in (3a), and forms with complex multiplication like the form in (3b). In our result, we shall require $A(q)$, an Eisenstein series, and $B(q)$, a CM form. CM forms have been known for a long time for many can be obtained by products of classical q-series identities (e.g. as in (3b)). For more on CM forms the reader may consult [Sh].

In view of Ramanujan's identities, (3a), and (3b), one hopes to obtain an expression for $F_{11}(q)$ in terms of $f(-q), \phi(q), \psi(q)$, and q-series resembling (3a) and (3b). The following theorem asserts such an identity.

Theorem. *If $p(N)$ denotes the usual partition function, then let $P_{11}(q)$ denote the q-series*

$$P_{11}(q) := \sum_{n=0}^{\infty} p(11n+6)q^{n+1} = 11q + 297q^2 + 3718q^3 + 31185q^4 + 204226q^5 + \cdots .$$

Define the q-series $A(q)$ and $B(q)$ by:

$$A(q) := \sum_{n=1}^{\infty} \sum_{d|n} \left(\frac{d}{11}\right) \cdot \frac{n^4}{d^4} \cdot q^n = q + 15q^2 + 82q^3 + 241q^4 + 626q^5 + \cdots ,$$

$$B(q) := \sum_{\substack{x,y\in\mathbb{Z} \\ x^2+11y^2\equiv 0 \pmod 4}} \frac{(2x^4 - 132x^2y^2 + 242y^4)}{64} \cdot q^{\frac{x^2+11y^2}{4}}$$

$$= q + 7q^3 + 16q^4 - 49q^5 - \cdots ,$$

In this notation, $F_{11}(q)$ satisfies the identity

$$F_{11}(q) = \frac{f^{11}(-q)}{f(-q^{11})} + 880q^5 \cdot \frac{f^{11}(-q^{11})}{f(-q)} - 9P_{11}(q)f^{11}(-q) + \frac{308}{3}A(q) + \frac{22}{3}B(q).$$

Proof. By [Ra], it is evident that if $p > 3$ is an odd prime, then $F_p(q) = \sum_{n=1}^{\infty} a_p(n)q^n$ ($q := e^{2\pi iz}$) is a holomorphic modular form of weight $\frac{p-1}{2}$ with respect to $\Gamma_0(p)$. In our case, the form

$$F_{11}(q) = 1 + 22q^5 + 110q^9 + 110q^{11} + 330q^{12} + 660q^{14} + 924q^{15} + \cdots \in M_5\left(\Gamma_0(11), \chi\right),$$

the space of weight 5 modular forms with respect to $\Gamma_0(11)$ with Nebentypus character $\chi := \left(\frac{\bullet}{11}\right)$. This space is 5 dimensional, and so verifying the identity reduces to checking it for the first 5 terms, together with the verification that each summand is a form in $M_5\left(\Gamma_0(11), \chi\right)$.

That $\frac{f^{11}(-q)}{f(-q^{11})}, q^5 \cdot \frac{f^{11}(-q^{11})}{f(-q)} \in M_5\left(\Gamma_0(11), \chi\right)$ is well known, and follows immediately from the interpretation of f in terms of the Dedekind eta-function. If U_{11} is the Atkin operator (see [p. 161, Ko]) giving the action of

$$U_{11} | \sum_{n=0}^{\infty} a(n)q^n = \sum_{n=0}^{\infty} a(11n)q^n,$$

then $U_{11}: M_5\left(\Gamma_0(11), \chi\right) \rightarrow M_5\left(\Gamma_0(11), \chi\right)$. That $P_{11}(q) f^{11}(-q) = U_{11} | q^5 \cdot \dfrac{f^{11}(-q^{11})}{f(-q)}$ is an easy exercise. The fact that $A(q) \in M_5\left(\Gamma_0(11), \chi\right)$ follows from the work of Hecke (see [p. 129, H]). The fact that $B(q) \in M_5\left(\Gamma_0(11), \chi\right)$ follows from the fact that $B(q)$ is the weight 5 form with complex multiplication by $\mathbb{Q}(\sqrt{-11})$ and trivial character (see [p. 205, Sh]).

□

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