ON “GOOD” HALF-INTEGRAL WEIGHT MODULAR FORMS

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1. Introduction and Statement of Results

If \( k \) is a positive integer, let \( S_k(N) \) denote the space of cusp forms of weight \( k \) on \( \Gamma_1(N) \), and let \( S_k^{cm}(N) \) denote the subspace of \( S_k(N) \) spanned by those forms having complex multiplication (see [Ri]). For a non-negative integer \( k \) and any positive integer \( N \equiv 0 \mod 4 \), let \( M_{k+\frac{1}{2}}(N) \) (resp. \( S_{k+\frac{1}{2}}(N) \)) denote the space of modular forms (resp. cusp forms) of half-integral weight \( k + \frac{1}{2} \) on \( \Gamma_1(N) \). Similarly, if \( k \in \frac{1}{2}\mathbb{N} \), then let \( M_k(N, \chi) \) (resp. \( S_k(N, \chi) \)) denote the space of modular (resp. cusp) forms with respect to \( \Gamma_0(N) \) and Nebentypus character \( \chi \). Throughout this note we shall refer to classical facts which may be found in [Ko, Mi, S-S, Sh].

If \( i = 0 \) or \( 1 \), \( 0 \leq r < t \), and \( a \geq 1 \), then let \( \theta_{a,i,r,t}(z) \) denote the Shimura theta function

\[
\theta_{a,i,r,t}(z) := \sum_{n \equiv r \mod t} n^i q^{an^2}
\]

(Note: \( q := e^{2\pi i z} \) throughout). Each \( \theta_{a,i,r,t}(z) \) is a holomorphic modular form of weight \( i + \frac{1}{2} \). If \( \Theta(N) \) is the set of modular forms generated by such functions of level dividing \( N \), then the Serre-Stark Theorem [S-S] implies

\[
\Theta(N) = M_{\frac{1}{2}}(N) \cup \left\{ \text{subspace of } M_{\frac{3}{2}}(N) \text{ spanned by those } \theta_{a,1,r,t}(z) \text{ on } \Gamma_1(N) \right\}.
\]
If \( g(z) \in M_{k + \frac{1}{2}}(N_1) \) and \( h(z) \in \Theta(N_2) \), then let \( g_h(n) \) denote the Fourier coefficient of \( q^n \) of the modular form
\[
g(z) \cdot h(z) = \sum_{n=0}^{\infty} g_h(n) q^n.
\]
Moreover, let \( G_h(z) \) denote the modular form
\[
G_h(z) := \sum_{\gcd(n, N_1 N_2) = 1} g_h(n) q^n.
\]

It follows from [Lemma 4, S-S] that \( G_h(z) \) is a modular form on \( \Gamma_1(N_1^2 N_2^2) \) of integral weight \( k + 1 \) or \( k + 2 \).

**Definition.** A modular form \( g(z) \in M_{k + \frac{1}{2}}(N_1) \) is **good** if there is an integer \( N_2 \) and a function \( h(z) \in \Theta(N_2) \) for which

(i) \( G_h(z) \) is a nonzero cusp form.

(ii) \( G_h(z) \notin S_{k+1}^{\text{cm}}(N_1^2 N_2^2) \cup S_{k+2}^{\text{cm}}(N_1^2 N_2^2) \).

There have been a number of recent papers on the non-vanishing of Fourier coefficients of half-integral weight modular forms modulo primes \( \ell \) (see [B2, J, O-S1]), and in this direction the first author and C. Skinner were able to prove the following theorem for “good” forms.

**Theorem.** [p. 454, O-S1] Let \( g(z) = \sum_{n=0}^{\infty} c(n) q^n \in M_{k + \frac{1}{2}}(N) \) be an eigenform whose coefficients are algebraic integers. If \( g(z) \) is good, then for all but finitely many primes \( \ell \) there are infinitely many square-free integers \( m \) for which \( |c(m)|_\ell = 1 \).

Here \( | \cdot |_\ell \) denotes an extension of the usual \( \ell \)-adic valuation to an algebraic closure of \( \mathbb{Q} \).

In [O-S1], the first author and Skinner made the following natural conjecture:

**The “Good” Conjecture.** [p. 468, O-S1] Every form in \( M_{k + \frac{1}{2}}(N) \) \( \setminus \Theta(N) \) is good.

In this note we prove:

**Theorem 1.** The “Good” Conjecture is true.

In a recent preprint, W. McGraw [M] obtains another proof of Theorem 1.

To prove the conjecture, we employ a well known result of M.-F. Vignéras, the Fundamental Lemma from [pp. 653-654, O-S2], and Brun’s sieve.
2. Proof of Theorem 1

Here we begin by recalling a well-known result due to M.-F. Vignéras [V] (see [B1] for a new elementary proof).

**Theorem 2.** [Th. 3, V] Suppose that \( f(z) = \sum_{n=0}^{\infty} a(n)q^n \) is in \( M_{k+\frac{1}{2}}(N) \). If there are finitely many square-free integers \( d_1, d_2, \ldots, d_j \) such that \( a(n) = 0 \) for every \( n \) not of the form \( d_im^2 \) with \( 1 \leq i \leq j \) and \( m \in \mathbb{Z}^+ \), then \( f(z) \in \Theta(N) \). 

We begin by combining Theorem 2 and [Fund. Lemma, pp. 653-654, O-S2] to obtain a lower bound for the number of non-zero coefficients of any modular form \( f(z) \in M_{k+\frac{1}{2}}(N, \chi) \setminus \Theta(N) \).

**Theorem 3.** Suppose that \( f(z) = \sum_{n=0}^{\infty} a(n)q^n \) is a modular form in \( M_{k+\frac{1}{2}}(N, \chi) \setminus \Theta(N) \). If \( f(z) \) is an eigenform of the Hecke operators \( T(p^2) \) for every prime \( p \nmid N \), then

\[
\# \{n \leq X : a(n) \neq 0 \} \gg \frac{X}{\log X}.
\]

**Proof.** By [Lemma 8, S-S], we may assume that all of the Fourier coefficients \( a(n) \) and the eigenvalues of the Hecke operators \( T(p^2) \), for primes \( p \nmid N \), are algebraic integers in a fixed number field \( K \). Let \( v \) be a place in \( K \) over 2.

By Theorem 2 there are infinitely many square-free positive integers \( d_1 < d_2 < \ldots \) for which there are positive integers \( n \) with \( a(d_in^2) \neq 0 \). Let \( s_0 \) be the smallest integer for which there is a square-free integer \( d > 1 \), with \( d \nmid N \), and a positive integer \( n \) for which \( \text{ord}_v(a(dn^2)) = s_0 \). Moreover, let \( d_0 \) be such a \( d \) and let \( n_0 \) be a positive integer for which \( \text{ord}_v(a(d_0n_0^2)) = s_0 \). Since \( d_0 \nmid N \), there are square-free integers \( D_0 > 1 \) and \( D_1 \) for which \( d_0 = D_0D_1 \) and \( D_1 \mid N \) and \( \gcd(D_0, N) = 1 \). Similarly, let \( m_0 \) and \( m_1 \) denote the unique positive integers for which \( n_0 = m_0m_1, \gcd(m_0, N) = 1 \), and every prime \( p \mid m_1 \) also divides \( N \).

Now recall the action of the Hecke operators. If \( p \) is prime, then

\[
(4) \quad f(z) \mid T(p^2) := \sum_{n=0}^{\infty} \left( a(p^2n) + \chi(p)\left(\frac{(-1)^k}{p}\right) p^{k-1}a(n) + \chi(p^2)p^{2k-1}a(n/p^2) \right) q^n.
\]

Suppose that \( d \) is a positive integer and \( p \nmid N \) is a prime for which \( p^2 \nmid d \). Since \( f(z) \) is an eigenform, it is easy to see that \( a(d) \mid a(dp^{2i}) \). As a consequence, it turns out that \( a(D_0D_1m_1^2) \neq 0 \) and \( \text{ord}_v(a(D_0D_1m_1^2)) = s_0 \).

If \( p \mid N \) is prime, then by [Lemma 1, S-S] it is known that

\[
(5) \quad f(z) \mid U(p) = \sum_{n=0}^{\infty} a(pn)q^n.
\]
is a cusp form in $M_{k+\frac{1}{2}}(N, \chi \cdot (\frac{4p}{\cdot}))$. Therefore, if $j$ is any positive integer for which every prime $p \mid j$ also divides $N$, then $f(z) \mid U(j) = \sum_{n=0}^{\infty} a(jn)q^n \in M_{k+\frac{1}{2}}(N, \chi \cdot (\frac{4j}{\cdot}))$. Now define $f_0(z) \in M_{k+\frac{1}{2}}(N, \chi \cdot (\frac{4D_0}{\cdot}))$ by

$$f_0(z) = \sum_{n=0}^{\infty} b(n)q^n := f(z) \mid U(D_1m_1^2) = \sum_{n=0}^{\infty} a(D_1m_1^2n)q^n.$$ 

By construction, we have that $b(D_0) = a(D_0D_1m_1^2) \neq 0$ and $\text{ord}_v(b(D_0)) = s_0$.

Also by construction, if there is an integer $s < s_0$ and an integer $n$ for which $\text{ord}_v(b(n)) = s$, then $\gcd(n, N) \neq 1$. This follows from the minimality of $s_0$. If this is the case, then define $f_1(z) \in M_{k+\frac{1}{2}}(N^2, \chi \cdot (\frac{4D_1}{\cdot}))$ (see [Lemma 4, S-S]) by

$$f_1(z) = \sum_{n=1}^{\infty} c(n)q^n := \sum_{\gcd(n, N)=1} b(n)q^n.$$ 

If there is no such $s$, then let $f_1(z) = \sum_{n=0}^{\infty} c(n)q^n := f_0(z)$.

In either case, $f_1(z) = \sum_{n=0}^{\infty} c(n)q^n$ is in $M_{k+\frac{1}{2}}(N^2, \chi \cdot (\frac{4D_1}{\cdot}))$ and has the property that $s_0$ is indeed the smallest integer for which there is an $n$ with $\gcd(v(c(n)) = s_0$. Moreover, the square-free integer $D_0$ which is coprime to $N^2$ is such an $n$. By the Fundamental Lemma [pp. 653-654, O-S2], if $f_1(z)$ is a cusp form, then

$$\#\{n \leq X : \gcd(n, N^2) = 1 \text{ and } a(D_1m_1^2n = c(n) \neq 0\} \gg f_1 \frac{X}{\log X}.$$ 

Although the Fundamental Lemma is stated for eigenforms which are cusp forms, it is easy to modify the argument to apply to forms $f_1(z)$ which are not cuspidal. Following the proof of the Fundamental Lemma, consider the integer weight form

$$F(z) := f_1(z) \cdot \left(1 + 2\sum_{n=1}^{\infty} q^{n^2}\right),$$

and decompose it into a cusp form $C(z)$ and a linear combination of Eisenstein series $E(z)$. By construction, the coefficient of $q^{D_0}$ in $F(z)$ has minimal 2-adic valuation $s_0$, and is determined by a linear combination of generalized divisor functions related to the Eisenstein series in $E(z)$ (see [Mi]) and the collection of 2-adic Galois representations associated to the newforms constituting $C(z)$. By Dirichlet’s Theorem on primes in arithmetic progressions, the Chebotarev Density theorem, and the multiplicativity of the coefficients of newforms, it follows that a ‘positive proportion’ of the square-free integers $D$ with the same number of prime factors as $D_0$ have the property that the coefficient of $q^D$ in $F(z)$ have minimal 2-adic valuation $s_0$. As in the proof of the Fundamental Lemma, this implies that

$$\#\{1 \leq n \leq X : c(n) \neq 0\} \gg \frac{X}{\log X} (\log \log X)^{r-1}.$$
where $D_0$ has exactly $r$ prime factors.

As a corollary, we obtain the following result (see [O] for a similar result).

**Corollary 4.** If $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ is a modular form in $M_{k+\frac{1}{2}}(N, \chi)\setminus \Theta(N)$, then

$$\#\{n \leq X : a(n) \neq 0\} \gg f \frac{X}{\log X}.$$  

**Proof.** If $w = \sum_{n=0}^{\infty} a_w(n)q^n$ is a formal power series in $q$, then define

$$M_w(X) := \#\{0 \leq n \leq X : a_w(n) \neq 0\}.$$  

Now suppose that $M_f(X) = o(X/\log X)$. In view of (4), it is easy to see that if $p \nmid N$ is prime, then

$$(7) \quad M_f|T(p^2)(X) \leq M_f(p^2X) + 2M_f(X).$$

By (7), $p \nmid N$ is prime, then $M_f|T(p^2)(X) = o(X/\log X)$.

If $w_1$ and $w_2$ are formal power series, then it is obvious that

$$M_{w_1+w_2}(X) \leq M_{w_1}(X) + M_{w_2}(X).$$

Therefore, if $\mathbb{T}$ is the Hecke algebra generated by the Hecke operators $T(p^2)$ and $\mathbb{X} = \mathbb{T}f$, then for every $u(z) \in \mathbb{X}$ we have that $M_u(X) = o(X/\log X)$.

Since $\mathbb{T}$ is commutative, every simple submodule of $\mathbb{X}$ is generated by an eigenform. If $u(z)$ is such an eigenform, then Theorem 3 contradicts the conclusion that $M_u(X) = o(X/\log X)$. Therefore, it must be that $M_f(X) \gg f X/\log X$.

Q.E.D.

Now we employ Brun’s sieve to obtain an important technical result regarding the prime divisors of a shifted set of integers. As usual, $p^a||n$ means that $a$ is the exact power of $p$ dividing $n$.

**Lemma 5.** Let $\ell$ be a fixed prime, and let $1 \leq r < t$ be integers for which $\gcd(r, t) = 1$. If $A$ is a set of non-negative integers for which

$$\#\{n \leq X : n \in A\} \gg \frac{X}{\log X},$$

then there is a positive integer $E$ and at least one integer $n \in A$ with $n < \ell^E$ such that $p||n + \ell^E$ for some prime $p \equiv r$ (mod $t$).
Proof. If $\phi(\bullet)$ denotes the usual Euler phi-function, then define the polynomial $F(n)$ by
\begin{equation}
F(n) = (n + \ell)(n + \ell^2) \cdots (n + \ell^{\phi(t)+1}).
\end{equation}

Let $\mathcal{A}_X$ denote the set of integers
\begin{equation}
\mathcal{A}_X := \{F(n) : n \leq X\}
\end{equation}
and let $P_X$ denote the set
\begin{equation}
P_X := \{p \equiv r \pmod{t} \text{ prime : } \log^2 X < p < X\}.
\end{equation}

It is easy to see that if $X$ is sufficiently large, then every prime $p \in P_X$ has the property that the multiplicative order of $\ell$ in $\mathbb{Z}/p\mathbb{Z}^\times$ is larger than $\phi(t) + 1$. Therefore, if $n$ is an integer and $p \in P_X$ is any prime for which $F(n) \equiv 0 \pmod{p}$, then there is exactly one integer $1 \leq i \leq \phi(t) + 1$ for which
\begin{equation}
n + \ell^i \equiv 0 \pmod{p}.
\end{equation}
Moreover, it is obvious that if $p \in P_X$, then there are $\phi(t) + 1$ distinct residue classes $n \pmod{p}$ for which $F(n) \equiv 0 \pmod{p}$.

Now we consider the function $S(\mathcal{A}_X, P_X; X)$ which is defined by
\begin{equation}
S(\mathcal{A}_X, P_X; X) := \#\{1 \leq n \leq X : \gcd(F(n), p) = 1 \text{ for every } p \in P_X\}.
\end{equation}

By a straightforward application of Brun’s sieve method [Theorem 2.2, H-R] we find that
\begin{equation}
S(\mathcal{A}_X, P_X; X) \ll X \prod_{p \in P_X} \left(1 - \frac{\phi(t) + 1}{p}\right).
\end{equation}

Using the well known fact [p. 605, R] that
\[\prod_{p \equiv r \pmod{t} \atop p \leq X} \left(1 - \frac{1}{p}\right) \ll \frac{1}{(\log X)^{1/\phi(t)}},\]
it is easy to deduce
\begin{equation}
S(\mathcal{A}_X, P_X; X) \ll \frac{X}{(\log X)^{1 + 1/2\phi(t)}}.
\end{equation}

Therefore, if $X$ is sufficiently large, then there are integers $n \in A$ with $n \leq X$ for which there is at least one prime $p \in P_X$ with $F(n) \equiv 0 \pmod{p}$. In particular, in view of (14) we find that
\begin{equation}
\#\{n \leq X : n \in A \text{ and } F(n) \equiv 0 \pmod{p} \text{ for some prime } p \in P_X\} \gg \frac{X}{\log X}.
\end{equation}
However, the number of positive integers \( n \leq X \) which are divisible by \( p^2 \) for some prime \( p \in P_X \) is

\[
\ll X \sum_{\log^2 X < p < X} \frac{1}{p^2} < \frac{X}{\log^2 X} \sum_{p < X} \frac{1}{p} \ll \frac{X}{(\log X)^{1+1/2}},
\]

since \( \sum_{p \leq X} 1/p \ll \log \log X \). Therefore, by (11) and (15) we find that the number of integers \( n \leq X \) and \( n \in A \) for which there is at least one prime \( p \in P_X \) and an integer \( 1 \leq e \leq \phi(t) + 1 \) such that \( p|n + \ell^e \) is \( \gg X/\log X \).

To conclude the proof, we note that if \( p||(n + \ell^e) \), then \( p||(n + \ell^{E(j)}) \) where \( E(j) := e + p(p-1)(p(p-1)+1)j \) and \( j \geq 0 \). To see this, note that \( n + \ell^{E(j)} = n + \ell^e + (\ell^{E(j)} - \ell^e) \), \( \ell^{p-1} \equiv 1 \mod p \) and \( \ell^{p(p-1)} \equiv 1 \mod p^2 \). Therefore if \( j \) is sufficiently large, then \( n < \ell^E \).

Q.E.D.

*Proof of Theorem 1.* Here we recall the essential facts regarding modular forms with complex multiplication (see [Ri]). If \( \phi(z) = \sum_{n=1}^{\infty} a_{F}(n) q^n \in S_k(N, \chi) \) is a newform with complex multiplication by the imaginary quadratic field \( K = \mathbb{Q}(\sqrt{d}) \), where \( d \) is the discriminant of \( K \), then \( d \mid N \), and if \( p \) is a prime for which \( (\frac{d}{p}) = -1 \), then \( a_{F}(p) = 0 \).

Now suppose that \( F(z) = \sum_{n=1}^{\infty} a_{F}(n) q^n \) is an integer weight cusp form in \( S_w(N, \psi) \). There are finitely many fundamental discriminants of imaginary quadratic fields, say \( d_1, d_2, \ldots, d_j \) for which \( d_i \mid N \). Therefore, it is easy to construct an arithmetic progression \( r \mod t \) with \( \gcd(r, t) = 1 \) such that every prime \( p \equiv r \mod t \) has the property that \( (\frac{d_i}{p}) = -1 \) for each \( 1 \leq i \leq j \). Therefore, by the multiplicativity of the Fourier coefficients of newforms, \( F(z) \) cannot be a linear combination of forms with complex multiplication if there is a positive integer \( n \) and a prime \( p \equiv r \mod t \) for which \( p\mid n \) and \( a_{F}(n) \neq 0 \).

Now we prove Theorem 1 by considering two different cases.

**Case I.** Suppose that \( g(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_{k+\frac{1}{2}}(N, \chi_1) \setminus \Theta(N) \). By Corollary 4, we know that

\[
\#\{n \leq X : a(n) \neq 0\} \gg g \frac{X}{\log X}.
\]

Now let \( \ell \mid 576N \) be prime, and let \( r \mod t \) with \( \gcd(r, t) = 1 \) be an arithmetic progression such that \( (\frac{d_i}{p}) = -1 \) for every prime \( p \equiv r \mod t \) and every fundamental discriminant of an imaginary quadratic field \( d_i \mid 576N \). By Lemma 5, there exists an integer \( n < \ell^E \) for which \( a(n) \neq 0 \), a prime \( p \equiv r \mod t \), and a positive integer \( E \) such that \( p\mid n + \ell^E \).

Now consider the cusp form \( g(z) \cdot \eta(24\ell^E z) \), where \( \eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \) denotes Dedekind’s eta-function. It is well known that \( \eta(24z) = q + \cdots \in S_{1/2}(576, \chi_{12}) \), where \( \chi_{12} \) is the non-trivial quadratic character with conductor 12. Obviously, \( \eta(24\ell^E z) \in \Theta(576\ell^E) \), and so \( g(z)\eta(24\ell^E z) \in S_{k+1}(576N\ell^E) \). The coefficient of \( q^{n+\ell^E} \) of this form is \( a(n) \neq 0 \). Since every fundamental discriminant of an imaginary quadratic field \( d \mid
576N\ell E already divides 576N, we find that \(g(z) \eta(24\ell Ez)\) cannot be a linear combination of forms with complex multiplication (i.e. \(g(z)\) is good).

**Case II.** Suppose that \(g(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_{k+\frac{1}{2}}(N)\setminus \Theta(N)\). It is well known that if \(w \in \frac{1}{2}\mathbb{Z}\), then
\[
M_w(N) = \bigoplus_{\chi} M_w(N, \chi),
\]
where the direct sum is over Dirichlet characters \(\chi \mod N\). Therefore, we may decompose \(g(z)\) as
\[
g(z) = \sum_{\chi} \alpha_{\chi} g_{\chi}(z).
\]
If \(\chi\) is a character for which \(\alpha_{\chi} g_{\chi}(z) \neq 0\), then by Case I there is a weight 1/2 cusp form \(\theta(z) \in S_{1/2}(N, \Psi)\) for which \(g_{\chi}(z)\theta(z)\) is a weight \(k+1\) cusp form which is not a linear combination of forms with complex multiplication.

If \(\chi_1\) and \(\chi_2\) are distinct characters \(\mod N\), then \(g_{\chi_1}(z)\theta(z)\) and \(g_{\chi_2}(z)\theta(z)\) will lie in different spaces of weight \(k+1\) cusp forms with Nebentypus. Therefore, it follows immediately that \(g(z)\theta(z)\) is good.

Q.E.D.

**References**


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