q-SERIES IDENTITIES AND VALUES OF CERTAIN L-FUNCTIONS

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1. Introduction and Statement of Results.

As usual, define Dedekind's eta function $\eta(z)$ by the infinite product

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \qquad (q := e^{2\pi i z} \ \text{ throughout}) \tag{1.1}$$

In a recent paper, Zagier [Z; Theorem 2] proved that (note. empty products equal 1 throughout)

$$\sum_{n=0}^{\infty}\left(\eta(24z) - q(1-q^{24})(1-q^{48})\cdots(1-q^{24n})\right) = \eta(24z)D(q) + E(q) \tag{1.2}$$

where the series $D(q)$ and $E(q)$ are defined by

$$D(q) = -\frac{1}{2} + \sum_{n=1}^{\infty}\frac{q^{24n}}{1-q^{24n}} = -\frac{1}{2} + \sum_{n=1}^{\infty} d(n)q^{24n} = -\frac{1}{2} + q^{24} + 2q^{48} + 2q^{72} + 3q^{96} + \dots,$$

$$E(q) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{12}{n}\right)nq^{n^2} = \frac{1}{2}q - \frac{5}{2}q^{25} - \frac{7}{2}q^{49} + \frac{11}{2}q^{121} + \dots.$$

Here $d(n)$ denotes the number of positive divisors of n. This identity plays an important role in Zagier's work on Vassiliev invariants in knot theory [Z].

Two other similar identities were known, and they were noticed by the first author in connection with one of Ramanujan's mock theta functions. In [A2], the first author proved that

$$\sum_{n=0}^{\infty}\left(\frac{\eta(48z)}{\eta(24z)} - q(1+q^{24})(1+q^{48})\cdots(1+q^{24n})\right) = \frac{\eta(48z)}{\eta(24z)}D(q) + \frac{M_1(q)}{2}, \tag{1.3}$$

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$$\sum_{n=0}^{\infty}\left(\frac{\eta(48z)}{\eta(24z)}-\frac{q}{(1-q^{24})(1-q^{72})\cdots(1-q^{24(2n+1)})}\right)=\frac{\eta(48z)}{\eta(24z)}D(q^2)+\frac{M_1(q)}{2} \tag{1.4}$$

where $M_1(q)$ is the mock theta function given by

$$M_1(q)=q+\sum_{n=1}^{\infty}\frac{q^{12n^2+12n+1}}{(1+q^{24})(1+q^{48})\cdots(1+q^{24n})}=q+q^{25}-q^{49}+2q^{73}-\cdots. \tag{1.5}$$

The q-series of the function $M_1(q)$ was the focus of two extensive studies [A-D-H, C]. Although $M_1(q)$ is not the Fourier expansion of a modular form, these works show that the coefficients of $M_1(q)$ are given by a Hecke character for the quadratic field $\mathbb{Q}(\sqrt{6})$. In particular, $M_1(q)$ enjoys nice properties that one expects for certain weight 1 cusp forms. For these reasons, we shall refer to $M_1(q)$ and $M_2(q)$ (defined in (1.8)) as mock theta functions although they do not exactly fit Ramanujan's original definition [A3; p. 291].

In view of identities (1.2-4), it is natural to investigate the general behavior of q-series which are obtained by summing the iterated differences between an infinite product and its truncated products. Here we establish two general theorems which yield infinitely many such identities, and we illustrate how such identities are useful in determining the values at negative integers for certain L-functions.

We shall employ the standard notation

$$(A;q)_n=\prod_{j=0}^{\infty}\frac{(1-Aq^j)}{(1-Aq^{n+j})}, \tag{1.6}$$

and throughout we assume that $|q|<1$ and that the other parameters are restricted to domains that do not contain any singularities of the series or products under consideration.

Theorem 1.

$$\begin{aligned}\sum_{n=0}^{\infty}&\left(\frac{(t;q)_\infty}{(a;q)_\infty}-\frac{(t;q)_n}{(a;q)_n}\right)\\ &=\sum_{n=1}^{\infty}\frac{(q/a;q)_n(a/t)^n}{(q/t;q)_n}\\ &\quad+\frac{(t;q)_\infty}{(a;q)_\infty}\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}+\sum_{n=1}^{\infty}\frac{q^nt^{-1}}{1-q^nt^{-1}}-\sum_{n=0}^{\infty}\frac{tq^n}{1-tq^n}-\sum_{n=0}^{\infty}\frac{aq^nt^{-1}}{1-aq^nt^{-1}}\right).\end{aligned}$$

Theorem 2.

$$\begin{aligned}\sum_{n=0}^{\infty}&\left(\frac{(a;q)_\infty(b;q)_\infty}{(q;q)_\infty(c;q)_\infty}-\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}\right)\\ &=\frac{(b;q)_\infty(a;q)_\infty}{(c;q)_\infty(q;q)_\infty}\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}-\sum_{n=0}^{\infty}\frac{aq^n}{1-aq^n}-\sum_{n=1}^{\infty}\frac{(c/b;q)_nb^n}{(a;q)_n(1-q^n)}\right).\end{aligned}$$

Many interesting specializations of these two theorems yield identities for modular forms that are eta-products (including identities (1.2-4)). Here we highlight ten of these identities. First we fix notation. We let $\sqrt{\Theta}$ be the operator defined by

$$\sqrt{\Theta}\left(\sum_{n=0}^{\infty} a(n)q^n\right) = \sum_{n=0}^{\infty} \sqrt{n}a(n)q^n. \tag{1.7}$$

It is easy to see that the series $E(q)$ in (1.2) is given by

$$E(q) = \sqrt{\Theta}\left(\eta(24z)\right)/2.$$

In addition to the mock theta function $M_1(q)$, we shall require the mock theta function $M_2(q)$ defined by

$$M_2(q) = \sum_{n=1}^{\infty} \frac{(-1)^n q^{24n^2-1}}{(1-q^{24})(1-q^{72})\cdots(1-q^{24(2n-1)})} = -q^{23} - q^{47} - \cdots . \tag{1.8}$$

See [A-D-H] for a detailed study of this function.

The ten eta-products $F_1(z), F_2(z), \ldots, F_{10}(z)$ we consider are of the form

$$F_i(z) = \eta^{a_i}(\delta_i z)\eta^{b_i}(2\delta_i z)$$

with $a_i \neq 0$. Obviously, each $F_i(z)$ is a modular form of weight $(a_i + b_i)/2$. For each $F_i(z)$ we define quantities c_i and $f_i(j)$, which are not necessarily unique, for which

$$F_i(z) = c_i \prod_{j=1}^{\infty} f_i(j). \tag{1.9}$$

These are listed in the table below.

Table 1.

i	$F_i(z)$	δ_i	c_i	$f_i(j)$
1	$1/\eta(24z)$	24	q^{-1}	$1/(1-q^{24j})$
2	$\eta(2z)/\eta^2(z)$	1	1	$(1+q^j)/(1-q^j)$
3	$\eta(8z)/\eta^2(16z)$	8	q^{-1}	$(1-q^{16j-8})/(1-q^{16j})$
4	$\eta(48z)/\eta(24z)$	24	q	$1+q^{24j}$
5	$\eta(48z)/\eta(24z)$	24	$q/(1-q^{24})$	$1/(1-q^{24(2j+1)})$
6	$\eta(24z)/\eta(48z)$	24	q^{-1}	$1/(1+q^{24j})$
7	$\eta(24z)/\eta(48z)$	24	q^{-1}	$1-q^{24(2j-1)}$
8	$\eta(24z)$	24	q	$1-q^{24j}$
9	$\eta^2(z)/\eta(2z)$	1	1	$(1-q^j)/(1+q^j)$
10	$\eta^2(16z)/\eta(8z)$	8	$q/(1-q^8)$	$(1-q^{16j})/(1-q^{16j+8})$

If $\delta \in \{1, 8, 24\}$, then let $d_\delta(n)$ be the divisor function defined by

$$d_\delta(n) = \begin{cases} d(n) = \sum_{d|n} 1 & \text{if } \delta = 24, \\ \sum_{d|n} (-1)^d & \text{if } \delta = 8, \\ \sum_{d|n \text{ odd}} 1 & \text{if } \delta = 1. \end{cases} \tag{1.10}$$

Also, for each i define α_i by

$$\alpha_i = \begin{cases} -\frac{1}{2} & \text{if } (a_i + 2b_i)\delta_i = 24, \\ 0 & \text{otherwise.} \end{cases} \tag{1.11}$$

Notice that $\alpha_i = -1/2$ if and only if the order of vanishing of $F_i(z)$ at ∞ is 1. The last quantity we require is γ_i which is defined by

$$\gamma_i = \begin{cases} 2 & \text{if } i = 5, 7, \\ 1 & \text{otherwise.} \end{cases} \tag{1.12}$$

Theorem 3. *If $1 \leq i \leq 10$, then*

$$\sum_{n=0}^{\infty} \left(F_i(z) - c_i \prod_{j=1}^{n} f_i(j) \right) = (1 + [1/\delta_i])\, F_i(z) D_i(q) + G_i(q)$$

where $[\bullet]$ denotes the greatest integer function,

$$D_i(q) = \alpha_i + \sum_{n=1}^{\infty} d_{\delta_i}(n) q^{\delta_i \gamma_i n}$$

and

$$G_i(q) = \begin{cases} 0 & \text{if } i = 1, 2, 3, \\ M_1(q)/2 & \text{if } i = 4, 5, \\ 2M_2(q)/\gamma_i & \text{if } i = 6, 7, \\ (-\alpha_i + [2/\delta_i]) \sqrt{\Theta}\,(F_i(z)) & \text{if } i = 8, 9, 10. \end{cases}$$

The three forms $F_1(z), F_2(z)$ and $F_3(z)$ have weight -1/2 and the four forms $F_4(z)$, $F_5(z)$, $F_6(z)$ and $F_7(z)$ have weight 0. The remaining three forms have weight 1/2. The series $G_4(z)$, $G_5(z)$, $G_6(z)$ and $G_7(z)$ are mock theta functions, whereas $G_8(q), G_9(q)$ and $G_{10}(q)$ are the half-derivatives of $F_8(z), F_9(z)$ and $F_{10}(z)$. In other words, the "error series" $G_i(q)$ in Theorem 3 satisfy

$$F_i(z) \quad \longrightarrow \quad G_i(q) \sim \begin{cases} 0 & \text{if } F_i(z) \text{ has weight -1/2}, \\ \text{Mock Theta function} & \text{if } F_i(z) \text{ has weight } 0, \\ \sqrt{\Theta}(F_i(z)) & \text{if } F_i(z) \text{ has weight } 1/2. \end{cases}$$

Although these identities are elegant in their own right, they are also often useful in calculating the values of L-functions at negative integers. In particular, they lead to analogs of the classical result

$$\frac{t}{e^t-1} = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \zeta(1-n) \cdot \frac{t^n}{(n-1)!},$$

where $\zeta(s)$ is the Riemann zeta-function. In this direction, Zagier used (1.2) to show that

$$-e^{-t/24} \sum_{n=0}^{\infty} (1-e^{-t})(1-e^{-2t}) \cdots (1-e^{-nt}) = \frac{1}{2} \sum_{n=0}^{\infty} (-1/24)^n \cdot L(\chi_{12}, -2n-1) \cdot \frac{t^n}{n!}, \tag{1.13}$$

where χ_{12} is the Dirichlet character with modulus 12 defined by

$$\chi_{12}(n) := \begin{cases} 1 & \text{if } n \equiv 1, 11 \pmod{12}, \\ -1 & \text{if } n \equiv 5, 7 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

Here we illustrate the generality of this phenomenon by proving the following theorems.

Theorem 4. *As a power series in t, we have*

$$-\frac{1}{4} \cdot \sum_{n=0}^{\infty} \frac{(1-e^{-t})(1-e^{-2t}) \cdots (1-e^{-nt})}{(1+e^{-t})(1+e^{-2t}) \cdots (1+e^{-nt})} = \sum_{n=0}^{\infty} (-1)^n (4^{n+1}-1) \cdot \zeta(-2n-1) \cdot \frac{t^n}{n!}.$$

In addition to $\zeta(s)$, we shall consider the Dirichlet L-function

$$L(\chi_2, s) := \sum_{n=1}^{\infty} \frac{\chi_2(n)}{n^s}, \tag{1.14}$$

where

$$\chi_2(n) := \begin{cases} 1 & \text{if } n \equiv 1, 7 \pmod{8}, \\ -1 & \text{if } n \equiv 3, 5 \pmod{8}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.15}$$

Theorem 5. *As a power series in t, we have*

$$-2e^{-t/8} \sum_{n=0}^{\infty} \frac{(1-e^{-2t})(1-e^{-4t}) \cdots (1-e^{-2nt})}{(1+e^{-t})(1+e^{-3t}) \cdots (1+e^{-(2n+1)t})} = \sum_{n=0}^{\infty} (-1/8)^n \cdot L(\chi_2, -2n-1) \cdot \frac{t^n}{n!}.$$

We shall also consider the Hecke L-function

$$L(\rho, s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} := \sum_{\mathfrak{a} \subseteq \mathbb{Z}[\sqrt{6}]} \chi(\mathfrak{a}) N\mathfrak{a}^{-s} \tag{1.16}$$

where χ is the order 2 character of conductor $4(3+\sqrt{6})$ on ideals in $\mathbb{Z}[\sqrt{6}]$ defined by

$$\chi(\mathfrak{a}) := \begin{cases} i^{yx}\left(\frac{12}{x}\right) & \text{if } y \text{ is even,} \\ i^{yx+1}\left(\frac{12}{x}\right) & \text{if } y \text{ is odd,} \end{cases} \tag{1.17}$$

when $\mathfrak{a} = (x + y\sqrt{6})$. If $1 \leq r < 48$ is an integer, then let $L_r(\rho, s)$ be the partial L-function defined by

$$L_r(\rho, s) := \sum_{n \equiv r \pmod{48}} \frac{a(n)}{n^s}. \tag{1.18}$$

By the orthogonality of the Dirichlet characters modulo 48 and the analytic continuation of the associated twists of $L(\rho, s)$, each $L_r(\rho, s)$ has an analytic continuation to $\mathbb{C}$.

Theorem 6. *As a power series in t, we have*

$$-2e^{t/24} \sum_{n=0}^{\infty} (1-e^{-t})(1-e^{-3t}) \cdots (1-e^{-(2n-1)t}) = \sum_{n=0}^{\infty} (-1/24)^n \cdot (L_{23}(\rho, -n) + L_{47}(\rho, -n)) \cdot \frac{t^n}{n!}.$$

Theorem 7. *As a power series in t, we have*

$$-2e^{-t/24} \sum_{n=0}^{\infty} (1-e^{-t})(1+e^{-2t}) \cdots (1+(-1)^n e^{-nt})$$
$$= \sum_{n=0}^{\infty} (-1/24)^n \cdot (L_1(\rho, -n) - L_{25}(\rho, -n)) \cdot \frac{t^n}{n!}.$$

In §2 we recall certain facts about q-series and basic hypergeometric series, and we prove Theorems 1 and 2. In §3 we prove Theorem 3 and in §4 we prove Theorems 4, 5, 6, and 7. In §5 we examine the partition theoretic consequences of the identities for $F_1(z)$ and $F_8(z)$. In §6 we give a few more identities which are related to eta-products. The most interesting of these is

$$\sum_{n=0}^{\infty} \left(\frac{1}{\eta^2(24z)} - \frac{1}{q^2(1-q^{24})^2(1-q^{48})^2 \cdots (1-q^{24n})^2} \right) = \frac{1}{\eta^2(24z)} \sum_{n=1}^{\infty} (d(n) + m(n))\, q^{24n},$$

where $m(n)$ denotes the number of *middle divisors* of n. A divisor is a middle divisor if it lies in the interval $[\sqrt{n/2}, \sqrt{2n})$.

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2. Preliminaries and important facts.

The ten identities in Theorem 3 rely on Theorems 1 and 2. Here we prove Theorems 1 and 2, and we begin with the following observation which follows from Abel's Theorem.

Proposition 2.1. *Suppose that* $f(z) = \sum_{n=0}^{\infty} \alpha(n) z^n$ *is analytic for* $|z| < 1$*. If* α *is a complex number for which*

(1) $\sum_{n=0}^{\infty} (\alpha - \alpha(n)) < +\infty$,

(2) $\lim_{n\to+\infty} n(\alpha - \alpha(n)) = 0$,

then

$$\lim_{z\to 1^-} \frac{d}{dz}(1-z)f(z) = \sum_{n=0}^{\infty} (\alpha - \alpha(n)). \tag{2.1}$$

In all of our applications, $f(z)$ will have a pole of order 1 at $z = 1$, so the $\lim_{z\to 1^-}$ can be replaced by a simple evaluation. We shall employ the standard notation of basic hypergeometric series [G-R; pp. 3-4, 125]:

$$ {}_{r+1}\phi_r \left(\begin{matrix} a_0, a_1, \ldots, a_r; \\ b_1, \ldots, b_r \end{matrix} \; q, z \right) = \sum_{j=0}^{\infty} \frac{(a_0;q)_j (a_1;q)_j \cdots (a_r;q)_j z^j}{(q;q)_j (b_1;q)_j \cdots (b_r;q)_j}, \tag{2.2}$$

$$ {}_{r}\psi_r \left(\begin{matrix} a_1, a_2, \ldots, a_r; \\ b_1, b_2, \ldots, b_r \end{matrix} \; q, z \right) = \sum_{j=-\infty}^{\infty} \frac{(a_1;q)_j (a_2;q)_j \cdots (a_r;q)_j z^j}{(b_1;q)_j (b_2;q)_j \cdots (b_r;q)_j}. \tag{2.3}$$

We shall also require Heine's transformation [G-R; p. 9]:

$$ {}_{2}\phi_1 \left(\begin{matrix} a, b; \\ c \end{matrix} \; q, z \right) = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty} \times {}_{2}\phi_1 \left(\begin{matrix} c/b, z; \\ az \end{matrix} \; q, b \right); \tag{2.4}$$

Ramanujan's summation [G-R; p. 126]:

$$ {}_{1}\psi_1 \left(\begin{matrix} a; \\ b \end{matrix} \; q, z \right) = \frac{(q;q)_\infty (b/a,q)_\infty (az;q)_\infty (q/az;q)_\infty}{(b;q)_\infty (q/a;q)_\infty (z;q)_\infty (b/az;q)_\infty}, \tag{2.5}$$

and the Rogers-Fine identity [F; p. 15]:

$$ (1-t) \sum_{n=0}^{\infty} \frac{(a;q)_n t^n}{(b;q)_n} = \sum_{n=0}^{\infty} \frac{(a;q)_n (atq/b;q)_n (1-atq^{2n}) b^n t^n q^{n^2-n}}{(b;q)_n (tq;q)_n}. \tag{2.6}$$

Throughout we assume that $|q| < 1$ and that the other parameters are restricted to domains that do not contain any singularities of the series or products under consideration. For succinctness of notation, we define the differential operator ϵ by

$$\epsilon f(z) = f'(1). \tag{2.7}$$

Proof of Theorem 1. By Proposition 2.1, we have that

$$\sum_{n=0}^{\infty}\left(\frac{(t;q)_\infty}{(a;q)_\infty} - \frac{(t;q)_n}{(a;q)_n}\right) = \epsilon(1-z)\sum_{n=0}^{\infty}\frac{(t;q)_n z^n}{(a;q)_n}.$$

By (2.2-6), this equals

$$= \epsilon(1-z)\left({}_1\psi_1\left(\begin{matrix} t; \\ a \end{matrix}\ q, z\right) - \sum_{n=-\infty}^{-1}\frac{(t;q)_n z^n}{(a;q)_n}\right)$$
$$= \epsilon\left(\frac{(q;q)_\infty(a/t;q)_\infty(tz;q)_\infty(q/tz;q)_\infty}{(a;q)_\infty(q/t;q)_\infty(zq;q)_\infty(a/tz;q)_\infty} - (1-z)\sum_{n=1}^{\infty}\frac{(q/a;q)_n(a/(tz))^n}{(q/t;q)_n}\right).$$

Differentiating this last expression with respect to z and then setting $z = 1$ yields the result.

Q.E.D.

Proof of Theorem 2. By Proposition 2.1, (2.2) and (2.4), we have that

$$\sum_{n=0}^{\infty}\left(\frac{(a;q)_\infty(b;q)_\infty}{(q;q)_\infty(c;q)_\infty} - \frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}\right)$$
$$= \epsilon(1-z)\quad {}_2\phi_1\left(\begin{matrix} a, b; \\ c \end{matrix}\ q, z\right)$$
$$= \epsilon\left(\frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(zq;q)_\infty}\left(1 + \sum_{n=1}^{\infty}\frac{(c/b;q)_n(z;q)_n b^n}{(q;q)_n(az;q)_n}\right)\right).$$

Noting that $\epsilon(z;q)_n = -(q;q)_{n-1}$ when $n > 0$, we differentiate this last expression with respect to z and then set $z = 1$. This yields the result.

Q.E.D.

3. Proof of Theorem 3.

In this section we prove each of the ten identities using the facts in §2.

Case of $F_1(z)$: This appears implicitly in [F; p. 14]. It is the instance of Theorem 2 where $a = 0$ and $b = c$.

Case of $F_2(z)$: This is the instance of $a=-q$ and $b=c$ in Theorem 2.

Case of $F_3(z)$: In Theorem 2, replace q by q^2, then set $b=c$ and $a=q$.

Case of $F_4(z)$: This result was proved in [A2; eq. (1.4)]. It follows from Theorem 1 with $t=-q$ with $a\to 0$.

Case of $F_5(z)$: This result was proved in [A2; eq. (1.5)]. In Theorem 2 replace q by q^2, then set $a=q^2$, $b=0$ and $c=q^3$. This yields, after multiplication by $(1-q)^{-1}$,

$$\begin{aligned}\sum_{n=0}^{\infty}&\left(\frac{1}{(q;q^2)_\infty}-\frac{1}{(q;q^2)_{n+1}}\right)\\&=\frac{1}{(q;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2+2n}}{(q^2;q^2)_n(1-q^{2n})}\\&=\frac{1}{(q;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2+2n}((1-q^{-n})+q^{-n})}{(q^2;q^2)_n(1-q^{2n})}\\&=\frac{1}{(q;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+n}}{(q^2;q^2)_n(1+q^n)}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2+n}}{(q^2;q^2)_n(1-q^{2n})}\right).\end{aligned}$$

By [F; p. 14, eq. (12.42)], we find that this equals

$$=\frac{1}{2(q;q^2)_\infty}\lim_{\tau\to 0}{}_3\phi_2\left(\begin{matrix}q/\tau,q/\tau,-1;\\-q,-q\end{matrix}\;q,-\tau^2\right)+\frac{1}{(q;q^2)_\infty}\left(-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right).$$

By [G-R; p. 241, eq. (III.9)] with $a=b=q/\tau$, $c=-1$ and $d=e=-q$, this equals

$$\begin{aligned}&=\frac{1}{2(q;q^2)_\infty(-q;q)_\infty}\lim_{\tau\to 0}{}_3\phi_2\left(\begin{matrix}q/\tau,-\tau,q;\\-q,-q\tau\end{matrix}\;q,-\tau\right)+\frac{1}{(q;q^2)_\infty}\left(-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right)\\&=\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q)_n}+\frac{1}{(q;q^2)}\left(-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right).\end{aligned}$$

This is the identity.

Case of $F_6(z)$: In Theorem 2 let $a=q$, $b=0$ and $c=-q$. This yields

$$\begin{aligned}\sum_{n=0}^{\infty}&\left(\frac{1}{(-q;q)_\infty}-\frac{1}{(-q;q)_n}\right)\\&=\frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{(n^2+n)/2}}{(q;q)_n(1-q^n)}\\&=\frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^{(n^2+n)/2}(1-(-1)^{n-1}+(-1)^{n-1})}{(q;q)_n(1-q^n)}.\end{aligned}$$

By [F; p. 14, eq. (12.42)], this equals

$$= \frac{2}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+3n+1}}{(q;q)_{2n+1}(1-q^{2n+1})} + \frac{1}{(-q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}$$

$$= \frac{2q}{(-q;q)_\infty (1-q)^2} \lim_{\tau\to 0} {}_3\phi_2 \left(\begin{matrix} q/\tau, q/\tau, q; \\ q^3, q^3 \end{matrix} \; q^2, \tau^2 q^3 \right) + \frac{1}{(-q;q)_\infty} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}$$

By [G-R; p. 241, eq. (III.10)] with q replaced by q^2 followed by letting $a = c = -q/\tau$, $b = q$ and $d = e = q^3$, this equals

$$= \frac{2q(q;q^2)_\infty}{(-q;q)_\infty (q^3;q^2)_\infty^2 (1-q)^2} \sum_{m=0}^{\infty} (q^2;q^2)_m q^m + \frac{1}{(q;q^2)_\infty} \left(-\frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \right).$$

$$= 2 \sum_{m=0}^{\infty} (q^2;q^2)_m q^m + \frac{1}{(q;q^2)_\infty} \left(-\frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \right)$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n^2}}{(q;q^2)_n} + \frac{1}{(q;q^2)_\infty} \left(-\frac{1}{2} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \right).$$

This is the identity for $F_6(z)$.

Case of $F_7(z)$: In Theorem 1 replace q by q^2, and then set $a = 0$ and $t = q$.

Case of $F_8(z)$: This is identity (1.2), and it is Theorem 2 in [Z]. We include a proof for completeness. In Proposition 2.1 set $\alpha = (q;q)_\infty$ and let $\alpha(n) := (q;q)_n$. This yields

$$\sum_{n=0}^{\infty} ((q;q)_\infty - (q;q)_n) = \epsilon(1-z) \sum_{n=0}^{\infty} (q;q)_n z^n$$

$$= -\epsilon \sum_{n=1}^{\infty} (q;q)_{n-1} q^n z^n$$

(3.1)
$$= -\sum_{n=1}^{\infty} (q;q)_{n-1} n q^n.$$

Now it is immediate (because the partial sums equal the partial products) that

(3.2,)
$$1 - \sum_{n=1}^{\infty} z q^n (zq;q)_{n-1} = (zq;q)_\infty$$

and applying ϵ to (3.2) we find that

(3.3)
$$-\sum_{n=1}^{\infty} q^n \left(\epsilon(zq;q)_{n-1}\right) = -1 - (q;q)_\infty \left(-1 + \sum_{j=1}^{\infty} \frac{q^j}{1-q^j} \right).$$

Now set $b = 0$, $a = z^2q$ and $t = z^2$ in (2.6), and after simplification we obtain upon multiplication by z

$$z - z^3 \sum_{n=1}^{\infty}(z^2q;q)_{n-1}(z^2q)^n = z + \sum_{n=1}^{\infty}(-1)^n \left(q^{(3n^2-3n)/2}z^{6n-1} + q^{(3n^2+n)/2}z^{6n+1}\right). \tag{3.4}$$

Noting that $\epsilon\left(zf(z^2)\right) = f(1) + 2\epsilon f(z)$ we apply ϵ to (3.4) and find that

$$\begin{aligned}
1+&\sum_{n=1}^{\infty}(-1)^n(6n-1)q^{(3n^2-n)/2} + \sum_{n=1}^{\infty}(-1)^n(6n+1)q^{(3n^2+n)/2} \\
&= (q;q)_\infty + 2\epsilon\left(1 - z\sum_{n=1}^{\infty}(zq;q)_{n-1}z^nq^n\right) \\
&= (q;q)_\infty - 2\left((q;q)_\infty - 1\right) - 2\sum_{n=1}^{\infty}(q;q)_{n-1}nq^n - 2\sum_{n=1}^{\infty} q^n\left(\epsilon(zq;q)_{n-1}\right) \\
&= 2 - (q;q)_\infty + 2\sum_{n=0}^{\infty}\left((q;q)_\infty - (q;q)_n\right) + 2\left(-1 - (q;q)_\infty\left(-1 + \sum_{j=1}^{\infty}\frac{q^j}{1-q^j}\right)\right),
\end{aligned} \tag{3.5}$$

by (3.1) and (3.3). This is the identity.

Case of $F_9(z)$: In Theorem 2, let $a = b = q$ and $c = -q$. This yields

$$\sum_{n=0}^{\infty}\left(\frac{(q;q)_\infty}{(-q;q)_\infty} - \frac{(q;q)_n}{(-q;q)_n}\right) = -2\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-q;q)_{n-1}q^n}{(q;q)_n(1-q^n)}. \tag{3.6}$$

Now in (2.6) set $t = z$, $a = zq$, $b = -zq$ (cf. [F; p. 15, eq. (14.31)]). This yields after simplification

$$(1-z)\sum_{n=0}^{\infty}\frac{(zq;q)_nz^n}{(-zq;q)_n} = 1 + 2\sum_{n=1}^{\infty}(-z^2)^nq^{n^2}. \tag{3.7}$$

Applying ϵ to (3.7) yields

$$\begin{aligned}
4\sum_{n=1}^{\infty}(-1)^nnq^{n^2} &= \\
&= \epsilon(1-z){}_2\phi_1\left(\begin{matrix} q, zq; \\ -zq \end{matrix}\; q, z\right).
\end{aligned}$$

By (2.4) with $a = q$, $b = zq$ and $c = -zq$, and also (3.6), we find that this equals

$$(3.8)\qquad = \epsilon \frac{(zq;q)_\infty}{(-zq;q)_\infty}\ {}_2\phi_1\left(\begin{matrix} -1, z; \\ zq \end{matrix}\ q, zq\right).$$

$$= \frac{(q;q)_\infty}{(-q;q)_\infty}\left(\sum_{j=1}^{\infty}\frac{-q^j}{1-q^j} - \sum_{j=1}^{\infty}\frac{q^j}{1+q^j}\right) - 2\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-q;q)_{n-1}q^n}{(q;q)_n(1-q^n)}$$

$$= -2\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{j=1}^{\infty}\frac{q^j}{1-q^{2j}} + \sum_{n=0}^{\infty}\left(\frac{(q;q)_\infty}{(-q;q)_\infty} - \frac{(q;q)_n}{(-q;q)_n}\right).$$

Case of $F_{10}(z)$: In Theorem 2 replace q by q^2, and then set $a = b = q^2$ and $c = q^3$. This yields, after multiplication by $(1-q)^{-1}$

$$(3.9)\qquad \sum_{n=0}^{\infty}\left(\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} - \frac{(q^2;q^2)_n}{(q;q^2)_{n+1}}\right) = -\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(q;q^2)_n q^{2n}}{(q^2;q^2)_n(1-q^{2n})}.$$

Now in (2.6) replace q by q^2, then set $a = zq$ and $b = zq^2$ (cf. [F; p. 16, eq. (14.4)]). This yields, after simplification,

$$(3.10)\qquad \sum_{n=0}^{\infty}\frac{(zq;q^2)_n z^n q^n}{(zq^2;q^2)_n} = \sum_{n=0}^{\infty} z^n q^{(n^2+n)/2}.$$

Applying ϵ to (3.10), we find that

$$\sum_{n=1}^{\infty} n q^{(n^2+n)/2} = \epsilon\ {}_2\phi_1\left(\begin{matrix} zq, q^2; \\ zq^2 \end{matrix}\ q^2, zq\right).$$

This equals, by (2.4) with q replaced by q^2 and z replaced by zq with $a = zq$, $b = q^2$ and $c = zq^2$,

$$(3.11)\qquad = \epsilon\frac{(q^2;q^2)_\infty(z^2q^2;q^2)_\infty}{(zq^2;q^2)_\infty(zq;q^2)_\infty}\ {}_2\phi_1\left(\begin{matrix} z, zq; \\ z^2q^2 \end{matrix}\ q^2, q^2\right)$$

$$= \epsilon(q^2;q^2)_\infty(-zq;q)_\infty\ {}_2\phi_1\left(\begin{matrix} z, zq; \\ z^2q^2 \end{matrix}\ q^2, q^2\right)$$

$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{j=1}^{\infty}\frac{q^j}{1+q^j} - \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(q;q^2)_n q^{2n}}{(q^2;q^2)_n(1-q^{2n})}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{j=1}^{\infty}\frac{(-1)^j q^j}{1-q^j} + \sum_{n=0}^{\infty}\left(\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} - \frac{(q^2;q^2)_n}{(q;q^2)_{n+1}}\right).$$

We use (3.9) in the last step above.

4. Proof of Theorems 4, 5, 6, and 7.

In this section we prove Theorems 4, 5, 6, and 7. The proofs are similar to the proof of [Th. 3, Z], and so we give a brief proof of Theorem 4 and we give sketches of the remaining cases. In each case, it is well known that the relevant L-function has an analytic continuation to $\mathbb{C}$ (with the exception of a simple pole at $s = 1$ for $\zeta(s)$) and a functional equation via a Mellin transformation.

Proof of Theorems 4, 5, 6, and 7.

Case of Theorem 4: The identity for $F_9(z)$ in Theorem 3 is

$$\sum_{n=0}^{\infty}\left(F_9(z) - \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1+q)(1+q^2)\cdots(1+q^n)}\right) = 2F_9(z)\left(\sum_{n=1}^{\infty} d_1(n)q^n\right) + 2\sqrt{\Theta}\left(F_9(z)\right). \tag{4.1}$$

It is well known that

$$F_9(z) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}. \tag{4.2}$$

Notice that $F_9(z)$ vanishes to infinite order as $q \to 1$. Therefore, we replace q by e^{-t} with $t \searrow 0$. Now define coefficients $c_9(n)$ and $b_9(n)$ by the asymptotic expansions

$$-\sum_{n=0}^{\infty}\frac{(1-e^{-t})(1-e^{-2t})\cdots(1-e^{-nt})}{(1+e^{-t})(1+e^{-2t})\cdots(1+e^{-nt})} = \sum_{n=0}^{\infty} c_9(n)t^n \tag{4.3}$$

and

$$H_9\left(e^{-t}\right) = 4\sum_{n=1}^{\infty}(-1)^n n e^{-n^2 t} \sim \sum_{n=0}^{\infty} b_9(n)t^n \qquad \text{as } t \searrow 0.$$

Now we observe, by (4.1) and (4.2), that $c_9(n) = b_9(n)$ for all n. On the other hand

$$\int_0^{\infty} H_9\left(e^{-t}\right) t^{s-1}dt = 4\sum_{n=1}^{\infty}(-1)^n n\int_0^{\infty} e^{-n^2 t}t^{s-1}dt.$$

By replacing T by n^2t, we find that

$$\int_0^{\infty} H_9\left(e^{-t}\right) t^{s-1}dt = 4\Gamma(s)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2s-1}} = 4\Gamma(s)(4^{1-s}-1)\zeta(2s-1). \tag{4.4}$$

To compute the coefficients $b_9(n)$, notice that

$$\int_0^{\infty} H_9\left(e^{-t}\right) t^{s-1}dt = \int_0^{\infty}\left(\sum_{n=0}^{N-1} b_9(n)t^n + O(t^N)\right)t^{s-1}dt = \sum_{n=0}^{N-1}\frac{c_9(n)}{s+n} + \mathfrak{F}_9(s), \tag{4.5}$$

where $\mathfrak{F}_9(s)$ is analytic for $Re(s) > -N$. The residue at $s = -n$ of (4.4) is

$$b_9(n) = \frac{(-1)^n}{n!} \cdot 4(4^{1+n} - 1) \cdot \zeta(-2n-1).$$

This completes the proof of Theorem 4.

Case of Theorem 5: It is well known that

$$F_{10}(z) = \sum_{n=0}^{\infty} q^{(2n+1)^2}. \tag{4.6}$$

It is easy to see that the identity in Theorem 3 for $F_{10}(z)$ is equivalent to

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(F_{10}(z/8) - \frac{q^{1/8}}{1-q} \times \frac{(1-q^2)(1-q^4)\cdots(1-q^{2n})}{(1-q^3)(1-q^5)\cdots(1-q^{2n+1})} \right) \\ &\qquad = F_{10}(z/8)\left(-\frac{1}{2} + \sum_{n=1}^{\infty} d_8(n) q^n\right) + \frac{1}{2}\sum_{n=0}^{\infty}(2n+1) q^{(2n+1)^2/8}. \end{aligned} \tag{4.7}$$

Observe that $F_{10}(z/8)$ vanishes to infinite order as $q \to -1$. Define the coefficients $c_{10}(n)$ and $b_{10}(n)$ by the asymptotic expansions

$$-e^{-t/8} \sum_{n=0}^{\infty} \frac{(1-e^{-2t})(1-e^{-4t})\cdots(1-e^{-2nt})}{(1+e^{-t})(1+e^{-3t})\cdots(1+e^{-(2n+1)t})} = \sum_{n=0}^{\infty} c_{10}(n) t^n$$

and

$$H_{10}\left(e^{-t}\right) = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)(-1)^{n(n+1)/2} e^{-t(2n+1)^2/8} \sim \sum_{n=0}^{\infty} b_{10}(n) t^n \qquad \text{as } t \searrow 0.$$

Therefore, by replacing q by $-e^{-t}$ with $t \searrow 0$ and $\zeta_8 := e^{\pi i/8}$, (4.6) and (4.7) imply that $c_{10}(n) = b_{10}(n)$ for all n and, on the other hand

$$\begin{aligned} \int_0^{\infty} H_{10}\left(e^{-t}\right) t^{s-1} dt &= \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)(-1)^{n(n+1)/2} \int_0^{\infty} e^{-(2n+1)^2 t/8} t^{s-1} dt \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \chi_2(n) n \int_0^{\infty} e^{-n^2 t/8} t^{s-1} dt \\ &= \frac{1}{2} 8^s \Gamma(s) L(\chi_2, 2s-1). \end{aligned} \tag{4.8}$$

The rest of the proof is identical to the proof of Theorem 4.

Case of Theorem 6: By [C], it is known that the coefficients $a(n)$ defining $L(\rho, s)$ in (1.16) are defined by

$$\sum_{n=1}^{\infty} a(n)q^n = M_1(q) + 2M_2(q). \tag{4.9}$$

In this case we use the identity for $F_7(z)$ in Theorem 3

(4.10)

$$\sum_{n=0}^{\infty}\left(F_7(z) - q^{-1}(1-q^{24})(1-q^{72})\cdots(1-q^{24(2n-1)})\right) = F_7(z)\left(\sum_{n=1}^{\infty} d(n)q^{24n}\right) + M_2(q).$$

Moreover, notice in (1.8) that the non-zero coefficients of $M_2(q)$ are supported on those exponents $n \equiv 23 \pmod{24}$. Now observe that $F_7(z/24)$ vanishes to infinite order as $q \rightarrow 1$. The rest of the argument is virtually identical to those above.

Case of Theorem 7: In this case we consider the identity for $F_4(z)$ in Theorem 3

(4.11)

$$\sum_{n=0}^{\infty}\left(F_4(z) - q(1+q^{24})(1+q^{48})\cdots(1+q^{24n})\right) = F_4(z)\left(-\frac{1}{2} + \sum_{n=1}^{\infty} d(n)q^{24n}\right) + \frac{1}{2}M_1(q).$$

Notice that $F_4(z/24)$ vanishes to infinite order as $q \rightarrow -1$. Arguing as before, we consider the asymptotic t-series expansion of

$$-\zeta_{24}e^{-t/24}\sum_{n=0}^{\infty}(1-e^{-t})(1+e^{-2t})\cdots(1+(-1)^n e^{-nt})$$

where $\zeta_{24} = e^{\pi i/24}$. The rest of the proof is a routine exercise using ζ_{24}, the Mellin transform, and the fact that the non-zero coefficients of $M_1(q)$ are supported on those exponents $n \equiv 1 \pmod{24}$.

Q.E.D.

5. Partition theoretic consequences.

Recall that a *partition* of a non-negative integer N is any nonincreasing sequence of positive integers whose sum is N. If $p(N)$ denotes the number of partitions of N, then we have

$$\frac{1}{\eta(24z)} = \sum_{N=0}^{\infty} p(N)q^{24N-1} = q^{-1}\prod_{n=1}^{\infty}\frac{1}{1-q^{24n}} = q^{-1} + q^{23} + 2q^{47} + 3q^{71} + \cdots. \tag{5.1}$$

If $p_e(N)$ (resp. $p_o(N)$) denote the number of partitions of N into an even (resp. odd) number of distinct parts, then Euler's Pentagonal Number Theorem asserts that the Fourier expansion of $\eta(24z) \in S_{1/2}(\Gamma_0(576), \chi)$ is

$$\eta(24z) = \sum_{N=0}^{\infty}\left(p_e(N) - p_o(N)\right)q^{24N+1} = q - q^{25} - q^{49} + \cdots = \sum_{n=1}^{\infty}\chi(n)q^{n^2} \tag{5.2}$$

where

$$\chi(n) := \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

As a consequence for our identity for $F_1(z) = 1/\eta(24z)$, we obtain the following partition theoretic result which is equivalent to the observation made by Erdös [E] in the beginning of his study of the asymptotics of $p(N)$ by elementary means.

Theorem 5.1. *If n is a positive integer, then let $a_n(N)$ denote the number of partitions of N into parts not exceeding n. For every positive integer N we have*

$$(N+1)p(N) = \sum_{n=1}^{N} p(N-n)d(n) + \sum_{n=1}^{N} a_n(N).$$

Proof. Here we use the identity for $F_1(z) = 1/\eta(24z)$ in the form

$$\sum_{n=0}^{\infty} \left(P(q) - \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)} \right) = P(q)\mathcal{D}(q), \tag{5.4}$$

where

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$
$$\mathcal{D}(q) = \sum_{n=1}^{\infty} d(n)q^n.$$

If m is a non-negative integer, then the series

$$P(q) - \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} = q^{m+1} + \sum_{n=m+2}^{\infty} b_m(n)q^n \tag{5.5}$$

for some sequence of integers $b_m(n)$.

Now let N be a positive integer, then it is easy to see that the coefficient of N on the right hand side of (5.4) is

$$\sum_{n=1}^{N} p(N-n)d(n).$$

Therefore, to prove the result it suffices to show that the coefficient of q^N on the left hand side of of (5.4) is

$$(N+1)p(N) - \sum_{n=1}^{N} a_n(N). \tag{5.6}$$

By letting $m = N$ as in (5.5), it is easy to see that the coefficient of q^N on the left hand side of (5.4) is the coefficient of q^N in

$$\sum_{n=0}^{N}\left(P(q) - \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}\right).$$

Claim (5.6) follows from the obvious fact that

$$\sum_{N=0}^{\infty} a_n(N)q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

This completes the proof.

Q.E.D.

As with Theorem 5.1, Zagier's identity for $F_8(z) = \eta(24z)$ has an interesting partition theoretic consequence which is similar in flavor to Euler's Pentagonal Number Theorem (5.2). We require some notation. If π is a partition, then let m_π denote its largest part. Moreover, if N is an integer, then let $S_e(N)$ (resp. $S_o(N)$) denote the set of partitions of N into an even (resp. odd) number of distinct parts. Using this notation, define the two partition functions $A_e(N)$ and $A_o(N)$ by

$$A_e(N) := \sum_{\pi \in S_e(N)} m_\pi, \tag{5.7}$$

$$A_o(N) := \sum_{\pi \in S_o(N)} m_\pi \tag{5.8}$$

Theorem 5.2. *If N is a positive integer, then*

$$A_e(N) - A_o(N) =$$
$$= \sum_{k\in\mathbb{Z}} (-1)^k d\left(N - (3k^2+k)/2\right) + \begin{cases} (-1)^k(3k) & \text{if } N = \frac{3k^2+k}{2} \text{ with } k \geq 0, \\ (-1)^k(3k-1) & \text{if } N = \frac{3k^2-k}{2} \text{ with } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\mathcal{E}(q)$ is the q-series defined by

$$\mathcal{E}(q) = \prod_{n=1}^{\infty}(1-q^n) = 1 - q - q^2 + \cdots,$$

then we claim that

$$\sum_{n=0}^{\infty}\left(\mathcal{E}(q) - (1-q)(1-q^2)\cdots(1-q^n)\right) = -\sum_{n=1}^{\infty} nq^n(1-q)(1-q^2)\cdots(1-q^{n-1}). \tag{5.9}$$

As described above, we have that

$$\mathcal{E}(q) = \prod_{n=1}^{\infty}(1-q^n) = \sum_{n=0}^{\infty}(p_e(n)-p_o(n))\,q^n.$$

Since $\mathcal{E}(q)$ is the generating function for the number of partitions into an even number of distinct parts minus the number into an odd number of distinct parts, it is easy to see that

$$\mathcal{E}(q) - (1-q)(1-q^2)\cdots(1-q^n)$$

is the generating function whose coefficient of q^N is the number of partitions of N into an even number of distinct parts where at least one part exceeds n minus the number of partitions of N into an odd number of distinct parts where at least one part exceeds n. Consequently, each partition π into distinct parts is counted with multiplicity $(-1)^{n_\pi} m_\pi$ where n_π is the number of parts in π. This implies (5.9).

It is an easy exercise to deduce Theorem 5.2 from the identity in Theorem 1 for $F_8(z) = \eta(24z)$.

Q.E.D.

6. Related Results

The uniform nature of the ten results proved in Theorem 3 makes it natural to group those identities together. There are, however, a variety of results that can be deduced from Theorems 1 and 2. Here we record a number of further identities.

Theorem 6.1. *Let $m(n)$ denote the number of divisors of n in the interval $[\sqrt{n/2}, \sqrt{2n})$. Moreover, let $d_e(n)$ (resp. $d_o(n)$) denote the number of even (resp. odd) divisors of n. The following identities are true:*

$$\sum_{n=0}^{\infty}\left(\frac{1}{\eta^2(24z)} - \frac{1}{q^2(q^{24};q^{24})_n^2}\right) = \frac{1}{\eta^2(24z)}\sum_{n=1}^{\infty}(d(n)+m(n))\,q^{24n}, \tag{6.1}$$

$$\sum_{n=0}^{\infty}\left(\frac{\eta^2(2z)}{\eta^4(z)} - \frac{(-q;q)_n^2}{(q;q)_n^2}\right) = \frac{4\eta^2(2z)}{\eta^4(z)}\sum_{m=0}^{\infty}d_o(2m+1)q^{2m+1}, \tag{6.2}$$

$$\sum_{n=0}^{\infty}\left(\frac{\eta^2(8z)}{\eta^4(16z)} - \frac{(q^8;q^{16})_n^2}{q^2(q^{16};q^{16})_n^2}\right) = \frac{\eta^2(8z)}{\eta^4(16z)}\sum_{n=1}^{\infty}(d_e(n)-2d_o(n))\,q^{8n}, \tag{6.3}$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{\eta(24z)} - \frac{1}{q(q^{24};q^{24})_{2n}}\right) = \frac{1}{\eta(24z)}\left(\sum_{n=1}^{\infty}d(n)q^{48n} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{24n^2}}{1-q^{48n}}\right), \tag{6.4}$$

$$\sum_{n=0}^{\infty}\left(\frac{\eta(2z)}{\eta^2(z)}-\frac{(-q;q)_{2n}}{(q;q)_{2n}}\right)=\frac{2\eta(2z)}{\eta^2(z)}\sum_{n=1}^{\infty}\frac{q^{2n}-(-q)^n}{1-q^{4n}}, \tag{6.5}$$

(6.6)

$$\sum_{n=0}^{\infty}\left(\frac{\eta(8z)}{\eta^2(16z)}-\frac{(q^8;q^{16})_{2n}}{q(q^{16};q^{16})_{2n}}\right)=\frac{\eta(8z)}{4\eta^2(16z)}\left(1-\sum_{n=-\infty}^{\infty}q^{8n^2}+2\sum_{n=1}^{\infty}\left(d_e(n)-d_o(n)\right)q^{8n}\right).$$

Proof. Here we prove these identities.

Case of (6.1): Let $a=b=0$ and $c=q$ in Theorem 2. This yields

$$\sum_{n=0}^{\infty}\left(\frac{1}{(q;q)_\infty^2}-\frac{1}{(q;q)_n^2}\right)=\frac{1}{(q;q)_\infty^2}\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{(n^2+n)/2}}{1-q^n}\right). \tag{6.7}$$

In order to complete the proof of (6.1), we must establish that

$$\sum_{n=1}^{\infty}m(n)q^n=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{(n^2+n)/2}}{1-q^n}. \tag{6.8}$$

To see (6.8), we first note that $m(n)$ equals $d(n)$ minus the number of divisors in the two intervals $[1,\sqrt{n/2})$ and $[\sqrt{2n},n]$. Hence, we have that

$$\begin{aligned}\sum_{n=1}^{\infty}m(n)q^n&=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}-\sum_{n=1}^{\infty}q^{(n)(2n)}\left(1+2q^n+2q^{2n}+2q^{3n}+\cdots\right)\\&=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{(1+q^n)q^{2n^2}}{1-q^n}. \qquad (6.9)\end{aligned}$$

Next in [G-R, eq. (III.17), p. 242] set $a=e=z$ and let b,c,d and $f\to+\infty$. This yields

$$\frac{(q;q)_\infty}{(zq;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(z;q)_n^2(1-zq^{2n})q^{2n^2}z^n}{(q;q)_n^2(1-z)}\right)=\sum_{n=0}^{\infty}\frac{(z;q)_n(-1)^nq^{(n^2+n)/2}}{(q;q)_n}.$$

Hence we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{(n^2+n)/2}}{1-q^n}&=\\&=\epsilon\sum_{n=0}^{\infty}\frac{(z;q)_n(-1)^nq^{(n^2+n)/2}}{(q;q)_n}\\&=\epsilon\frac{(q;q)_\infty}{(zq;q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(z;q)_n^2(1-zq^{2n})q^{2n^2}z^n}{(q;q)_n^2(1-z)}\right)\\&=\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{(1+q^n)q^{2n^2}}{1-q^n}\\&=\sum_{n=1}^{\infty}m(n)q^n, \qquad (6.10)\end{aligned}$$

which proves (6.8) and therefore (6.1).

Case of (6.2): To prove this identity let $a = b = -q$ and $c = q$ in Theorem 2. The result now follows easily by combining the resulting Lambert series.

Case of (6.3): To prove this identity replace q by q^2 and set $a = b = q$ and $c = q^2$ in Theorem 2. The result now follows easily by combining the resulting Lambert series.

Case of (6.4): To prove this identity replace q by q^2 and set $a = b = 0$ and $c = q^2$ in Theorem 2.

Case of (6.5): This identity follows from Theorem 2 by replacing q by q^2 and the setting $a = -q, b = -q^2$ and $c = q$.

Case of (6.6): Surprisingly, (6.6) is more intricate. Here we use Proposition 2.1 with

$$\alpha(n) = (q;q^2)_{2n}/(q^2;q^2)_{2n}.$$

Consequently (noting that $\epsilon f(z) = \frac{1}{2}\epsilon f(z^2)$), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} - \frac{(q;q^2)_{2n}}{(q^2;q^2)_{2n}}\right)\\
&\quad = \epsilon(1-z)\sum_{n=0}^{\infty}\frac{(q;q^2)_{2n}z^n}{(q^2;q^2)_{2n}}\\
&\quad = \frac{1}{2}\epsilon(1-z^2)\sum_{n=0}^{\infty}\frac{z^n(q;q^2)_n(1+(-1)^n)}{2(q^2;q^2)_n}\\
&\quad = \frac{1}{4}\epsilon\left\{(1+z)\frac{(qz;q^2)_\infty}{(zq^2;q^2)_\infty} + (1-z)\cdot\frac{(-zq;q^2)_\infty}{(-zq^2;q^2)_\infty}\right\}\\
&\quad = \frac{1}{4}\left\{\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} - \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty} + \frac{2(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\left(\frac{q^{2n}}{1-q^{2n}} - \frac{q^{2n-1}}{1-q^{2n-1}}\right)\right\}.
\end{aligned}$$

Identity (6.6) now follows by recalling [A1; p. 21, eq. (2.2.10)], with $z = 1$, that

$$\frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty} = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Q.E.D.

Finally, we recall the following attractive and very elementary result on partitions into consecutive integers (cf. [L, p. 85, Problem 4]).

Proposition 6.2. *The number of partitions of n into consecutive integers equals the number of odd divisors of n.*

Proof. This result is easily deduced from the formula for the sum of an arithmetic progression. It is also directly deduced from the generating function identity [M, p. 28]

$$\sum_{m=1}^{\infty} \frac{q^{2m-1}}{1-q^{2m-1}} = \sum_{m=1}^{\infty} \frac{q^{(m^2+m)/2}}{1-q^m}.$$

Q.E.D.

From our proof of (6.1), we may easily deduce the following result for $c_e(n)$ (resp. $c_o(n)$) the number of partitions of n into an even (resp. odd) number of consecutive integers.

Theorem 6.3. *For every positive integer n we have*

$$c_o(n) - c_e(n) = m(n).$$

Proof. This is immediate from (6.10):

$$\sum_{n=1}^{\infty} m(n)q^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{(n^2+n)/2}}{1-q^n} = \sum_{n=1}^{\infty} \left(c_o(n) - c_e(n)\right) q^n.$$

Q.E.D.

References

[A1] G. E. Andrews, *The theory of partitions*, Cambridge University Press, 1984.

[A2] G. E. Andrews, *Ramanujan's "Lost" Notebook V: Euler's partition identity*, Advances in Mathematics **61** (1986), 156-164.

[A3] G. E. Andrews, *Mock theta functions*, Proc. Symp. Pure Math. **49** (1989), 283-298.

[A-D-H] G. E. Andrews, F. Dyson and D. Hickerson, *Partitions and indefinite quadratic forms*, Inventiones Mathematicae **91** (1988), 391-407.

[C] H. Cohen, *q-identities for Maass waveforms*, Inventiones Mathematicae **91** (1988), 409-422.

[E] P. Erdös, *On an elementary proof of some asymptotic formulas in the theory of partitions*, Annals of Mathematics **43** (1942), 437-450.

[F] N. J. Fine, *Basic hypergeometric series and applications*, Math. Sruveys and Monographs, Vol. 27, Amer. Math.Soc., Providence, 1988.

[G-R] G. Gasper and M. Rahman, *Basic hypergeometric series*, Ency. of Math. and its Appl., Vol. 2, Addison-Wesley, Reading, 1976.

[L] W. J. Leveque, *Topics in Number Theory, Vol. 1*, Addison-Wesley, Reading, 1956.

[M] P. A. MacMahon, *Combinatory Analysis, Vol. 2*, Cambridge Univ. Press, Cambridge, 1916 (reissued by Chelsea, New York, 1960).

[Z] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function*, to appear, Topology.

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