THE CHEBOTAREV DENSITY THEOREM IN SHORT INTERVALS AND SOME QUESTIONS OF SERRE

ANTAL BALOG AND KEN ONO

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1. Introduction and Statement of Results.

As usual, let \( \tau(n) \) denote the coefficient of \( q^n \) (\( q := e^{2\pi iz} \) throughout) in the series for

\[
\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n := q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \ldots,
\]

the unique normalized cusp form of weight 12 with respect to the full modular group. Although Lehmer’s speculation that \( \tau(n) \neq 0 \) for every positive \( n \) remains open, Serre [S] has made substantial progress on the basic question regarding the number of Fourier coefficients of a modular form which can be zero. He shows (see [p. 179, S]) that \( \tau(n) \) is non-zero for the vast majority of \( n \).

In the same paper, Serre proposes the study of the nonvanishing of Fourier coefficients in short intervals. In particular, if

\[
(1.1) \quad f(z) = \sum_{n=1}^{\infty} a_f(n)q^n
\]
is in \( S_k(\Gamma_0(N), \chi) \), then he suggests the problem of finding upper bounds for the function \( i_f(n) \) defined by

\[
(1.2) \quad i_f(n) := \max \{ i : a_f(n+j) = 0 \text{ for all } 0 \leq j \leq i \}.
\]

Serre proved [p. 183, S] that if \( f(z) \) is a cusp form with integer weight \( k \geq 2 \) which is not a linear combination of forms with complex multiplication, then

\[
(1.3) \quad i_f(n) \ll n.
\]

In view of this estimate, he poses the following questions [p. 183, S]:

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Serre’s Questions. Assume the notation above.

1. Suppose that \( f(z) \) is a non-zero cusp form with integer weight \( \geq 2 \) which is not a linear combination of forms with complex multiplication. Can estimate (1.3) be improved to an estimate of the form

\[
i_f(n) \ll f, m \frac{n}{\log(n)^m} \quad \text{for all } m \geq 0,
\]

or one of the form

\[
i_f(n) \ll f n^\delta \quad \text{where } 0 < \delta < 1?
\]

2. More generally, are there analogous results for forms with non-integral weights, or forms with respect to other Fuchsian groups?

Such questions are directly related to some examples found by Knopp and Lehner [K-L]. Although these questions have not been addressed directly in the literature (to the best of our knowledge), quite a bit is known. For example, the first question follows from the classical result, due to Rankin and Selberg, that there is a positive constant \( A_f \) for which

\[
\sum_{n \leq X} |a_f(n)|^2 n^{1-k} = A_f X + O(X^{3/5}).
\]

It then follows that

\[
i_f(n) \ll f n^{3/5}.
\]

In the present paper we consider stronger forms of Serre’s questions. We seek similar short interval results with the additional property that a proportion or a ‘near proportion’ of the coefficients are non-zero. The first result, which pertains to Serre’s first question, follows from a recent sieve result of Wu [Wu] which is based on deep analytic estimates for exponential sums by Fouvry and Iwaniec [F-I].

**Theorem 1.** Suppose that \( f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k(\Gamma_0(N), \chi) \) is a non-zero form with integer weight \( k \geq 2 \) that is not a linear combination of forms with complex multiplication. For every \( \epsilon > 0 \) and \( X^{\frac{2}{3} + \epsilon} \leq Y \) we have

\[
\# \{ X < n < X + Y : a_f(n) \neq 0 \} \gg_{f, \epsilon} Y.
\]

In particular, we have that \( i_f(n) \ll_{f, \epsilon} n^{\frac{2}{3} + \epsilon} \).

In the direction of Serre’s second question, we consider weight 1 forms, half-integral weight forms, and forms which are linear combinations of forms with complex multiplication. We begin by proving a short interval version of the Chebotarev Density Theorem. This result is of independent interest.
First we fix notation. Let $K$ be a number field and let $L/K$ be a normal extension with Galois group $\text{Gal}(L/K)$ and let $[L : \mathbb{Q}] := n_L$ and $[K : \mathbb{Q}] := n_K$. Moreover, define the constant $c(L)$ by

$$c(L) := \begin{cases} n_L & \text{if } n_L \geq 3, \\ 8/3 & \text{if } n_L = 2, \\ 12/5 & \text{if } n_L = 1. \end{cases}$$

If $\mathfrak{P}$ is a prime ideal in $O_K$ which is unramified in $O_L$, then let $\left[\frac{L/K}{\mathfrak{P}}\right]$ denote the Artin symbol representing the conjugacy class of the Frobenius above $\mathfrak{P}$ in $\text{Gal}(L/K)$. With these assumptions, let $\pi_C(X; L/K)$ denote

$$\pi_C(X; L/K) := \# \left\{ \mathfrak{P} \in O_K : \mathfrak{P} \text{ unramified in } O_L, \left[\frac{L/K}{\mathfrak{P}}\right] = C, \text{ and } N_{K/\mathbb{Q}}(\mathfrak{P}) \leq X \right\}. $$

The Chebotarev Density Theorem asserts that as $X \to +\infty$ we have

$$\pi_C(X; L/K) \sim \frac{\#C \#\text{Gal}(L/K) \cdot X}{\log X}. $$

To obtain a short interval version, we follow the successful method for bounding the distance between consecutive prime numbers invented by Hoheisel, and generalized by Sokolovskií [So] for prime ideals in number fields. The main ingredients of our proof are formulas for the ‘prime ideal counting function’ due to Lagarias and Odlyzko [L-O], and zero density estimates and zero-free regions for Dedekind zeta-functions due to Heath-Brown [HB] and Mitsui [Mi].

**Theorem 2.** If $\epsilon > 0$ and $X^{1-\frac{1}{n_L}+\epsilon} \leq Y \leq X$, then as $X \to +\infty$ we have

$$\pi_C(X + Y; L/K) - \pi_C(X; L/K) \sim \frac{\#C \#\text{Gal}(L/K) \cdot Y}{\log X}. $$

We use Theorem 2 to obtain a result for coefficients in short intervals for every integer or half-integral weight cusp form with weight $\geq 3/2$ which is not a combination of the weight $3/2$ theta functions

$$\theta_{\delta,r,t}(z) = \sum_{n \equiv r \pmod{t}} n q^{\delta n^2}. $$

Since the celebrated Serre-Stark [S-S] basis theorem asserts that all weight $1/2$ modular forms are linear combinations of theta series of the form

$$\Theta_{\delta,r,t}(z) = \sum_{n \equiv r \pmod{t}} q^{\delta n^2},$$

we shall concentrate only on those half-integral weight cusp forms with weight $\geq 3/2$ which are not linear combinations of forms as in (1.6).
Theorem 3. Suppose that $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(\Gamma_0(N), \chi)$ is a non-zero cusp form with weight $1/2 < k \in \frac{1}{2}\mathbb{N}$. If $f(z)$ is not a linear combination of weight $3/2$ theta functions, then there is a positive integer $k_f$ such that for every $\epsilon > 0$ and $X^{1-k/f} + \epsilon \leq Y \leq X$ we have

$$\# \{ X < n < X + Y : a_f(n) \neq 0 \} \gg_{f,\epsilon} \frac{Y}{\log X}.$$  

In particular, we have that $i_f(n) \ll_{f,\epsilon} n^{1-k/f} + \epsilon$.

We shall discuss a few corollaries regarding critical values of modular $L$-functions and elliptic curves. Suppose that $F(z) = \sum_{n=1}^{\infty} A(n)q^n \in S_2(\Gamma_0(N), \chi_{\text{triv}})$ is an even weight newform and let $L(F, s) = \sum_{n=1}^{\infty} A(n)n^{-s}$ denote its $L$-function. For the remainder of this paper $D$ shall denote the fundamental discriminant of a quadratic field. Let $L(F_D, s)$ denote the $L$-function given by

$$L(F_D, s) = \sum_{n=1}^{\infty} \frac{A(n)\chi_D(n)}{n^s}$$

where $\chi_D$ is the Kronecker character for $\mathbb{Q}(\sqrt{D})$. A well known conjecture due to Goldfeld [Go] asserts that $L(F_D, k) \neq 0$ for ‘half’ the $D$, and at present, the best general result in this direction (see [O-Sk]) is

$$\# \{ |D| < X : L(F_D, k) \neq 0 \} \gg_{F} \frac{X}{\log X}. \tag{1.7}$$

We obtain the following refinement of (1.7) indicating some regularity in the distribution of non-zero $L$-values in a family of quadratic twists.

Corollary 4. Let $F(z) = \sum_{n=1}^{\infty} A(n)q^n \in S_2(\Gamma_0(N), \chi_{\text{triv}})$ be an even weight newform. There is a positive integer $k_F$ such that for each $\epsilon > 0$ and $X^{1-k_F} + \epsilon \leq Y \leq X$ we have

$$\# \{ X < |D| < X + Y : L(F_D, k) \neq 0 \} \gg_{F,\epsilon} \frac{Y}{\log X}.$$  

Let $E/\mathbb{Q}$ be a modular elliptic curve and let $E_D$ denote its $D$-quadratic twist. Moreover, let $rk(E_D)$ denote the Mordell-Weil rank of $E_D$ over $\mathbb{Q}$. We obtain:

Corollary 5. If $E/\mathbb{Q}$ is a modular elliptic curve, then there is a positive integer $k_E$ such that for every $\epsilon > 0$ and $X^{1-k_E} + \epsilon \leq Y \leq X$ we have

$$\# \{ X < |D| < X + Y : rk(E_D) = 0 \} \gg_{E,\epsilon} \frac{Y}{\log X}.$$  

V. K. Murty, M. R. Murty, Saradha, Serre and Wan have obtained estimates in the direction of the Lang-Trotter Conjecture regarding the distribution of $a_E(p)$, the traces of the
Frobenius endomorphisms of an elliptic curve $E/\mathbb{Q}$. Here we obtain estimates regarding the short interval distribution of $a_E(p) \mod m$ for any given integer $m$. First we mention an immediate consequence of a striking result of Shiu [Shi] on consecutive primes in arithmetic progressions. Let $p_1 = 2 < p_2 = 3 < \ldots$ be the primes in increasing order, and let $E/\mathbb{Q}$ be an elliptic curve with a rational point of prime order $\ell$. Shiu’s theorem implies for every positive integer $k$ and each $1 \not\equiv i \mod \ell$ that there is an $n$ for which

$$a_E(p_n) \equiv a_E(p_{n+1}) \equiv a_E(p_{n+2}) \equiv a_E(p_{n+k}) \equiv i \mod \ell.$$  

Here we obtain a short interval result for $a_E(p) \mod m$ for any integer $m$ which indicates that strings as in (1.8), and more generally strings for any $E$, require that $k$ be small compared to $p_n$.

**Corollary 6.** Let $E/\mathbb{Q}$ be an elliptic curve, $m$ a positive integer, and $i \mod m$ a residue class for which there is a prime of good reduction $p_0$ with $a_E(p_0) \equiv i \mod m$. There is a positive integer $k_{E,m}$ such that for every $\epsilon > 0$ and $X^{1-\frac{1}{k_{E,m}}+\epsilon} \leq Y \leq X$ we have

$$\#\{X < p < X + Y \text{ prime} : a_E(p) \equiv i \mod m\} \gg_{E,\epsilon} \frac{Y}{\log X}.$$  

In §2 we prove Theorem 1, and in §3 we prove Theorem 2 and in §4 we prove Theorem 3. Corollaries 4, 5, and 6 are proved in §5.

2. Proof of Theorem 1.

The general case of Theorem 1 follows from the special case for newforms, and so our first objective is to prove Theorem 1 for newforms. We begin by recalling an important fact about newforms (see [A-L, Li, M]).

**Proposition 2.1.** If $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(\Gamma_0(N),\chi)$ is an integer weight newform and $m$ and $n$ are coprime integers, then

$$a_f(mn) = a_f(m)a_f(n).$$

We shall require the following important result due to Serre [p. 174, Cor. 2, S].

**Lemma 2.2.** Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(\Gamma_0(N),\chi)$ be an integer weight newform with weight $k \geq 2$ which does not have complex multiplication. For every $\epsilon > 0$ we have

$$\#\{p < X \text{ prime} : a_f(p) = 0\} \ll_{f,\epsilon} \frac{X}{(\log X)^{\frac{k}{2}-\epsilon}}.$$  

In view of Proposition 2.1 and Lemma 2.2, proving Theorem 1 in the case of newforms follows from the next result which is a special case of a theorem due to Wu [Wu].
Lemma 2.3. Let $S$ be a set of primes for which

$$\#\{p \in S \text{ and } p \leq X\} \ll \frac{X}{(\log X)^{1+\delta}}$$

where $\delta > 0$. If $N_S$ denotes the set of square-free positive integers with no prime factors in $S$, then for every $\epsilon > 0$ and $X^{\frac{17}{41}+\epsilon} < Y$ we have

$$\#\{X < n < X + Y : n \in N_S\} \gg_{S, \epsilon} Y.$$

Proof of Lemma 2.3. Let $\mathcal{B}$ denote a sequence of increasing integers $b_1 < b_2 < \ldots$ of mutually coprime positive integers for which $\sum_{i=1}^{\infty} \frac{1}{b_i} < +\infty$. Let $N_{\mathcal{B}}$ denote the set of positive integers which contain none of the $b_i$ as divisors. If $\epsilon > 0$ and $X^{\frac{17}{41}+\epsilon} < Y$, then a theorem of Wu [Wu] states that

$$\#\{X < n < X + Y : n \in N_{\mathcal{B}}\} \gg_{\mathcal{B}, \epsilon} Y.$$

Lemma 2.3 follows immediately by defining $\mathcal{B}$ by $\mathcal{B} := \{p \in S\} \cup \{q^2 : q \not\in S \text{ prime}\}$.

Q.E.D.

Now we use Lemma 2.3 to prove Theorem 1.

Proof of Theorem 1. Assume that $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n$ is a newform with weight $k \geq 2$ without complex multiplication. Let $S$ denote the set of primes

$$S := \{p \mid N \text{ prime}\} \cup \{p \text{ prime} : a_f(p) = 0\}.$$

By Lemma 2.2 and Lemma 2.3, we have for every $\epsilon > 0$ and $X^{\frac{17}{41}+\epsilon} < Y$ that

$$\#\{X < n < X + Y : n \in N_S\} \gg_{S, \epsilon} Y.$$

However, by Proposition 2.1 it follows immediately that all such $n$ have the property that $a_f(n) \neq 0$. This proves Theorem 1 for newforms.

Now suppose that $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n$ is an integer weight cusp form in $S_k(\Gamma_0(N), \chi)$ with weight $k \geq 2$ which is not a linear combination of forms with complex multiplication. We shall reduce the claim of Theorem 1 for $f(z)$ to the case of a single newform.

First we recall the definition of the integer weight Hecke operators. If $p$ is prime, then the Hecke operators $T_p$ are defined by

$$T_p | f(z) := \sum_{n=1}^{\infty} (a_f(pn) + \chi(p)p^{k-1}a_f(n/p)) q^n.$$ (2.1)
If \( g_1(z), g_2(z), \ldots g_s(z) \) are the weight \( k \) newforms with level dividing \( N \), then let their Fourier expansions be given by

\[
g_i(z) = \sum_{n=1}^{\infty} b_i(n)q^n.
\]

Therefore, if \( p \nmid N \) is prime, then we have

\[
T_p \mid g_i(z) = b_i(p)g_i(z).
\]

Moreover by “multiplicity one”, if \( i \neq j \), then there are infinitely many primes \( p \) for which \( b_i(p) \neq b_j(p) \).

By the theory of newforms, \( f(z) \) has a unique decomposition of the form

\[
f(z) = \sum_{i=1}^{s} \sum_{\delta \mid N} \alpha_{i,\delta} g_i(\delta z)
\]

where \( \alpha_{i,\delta} \) are complex numbers. By hypothesis, we may without loss of generality assume that \( \alpha_{1,\delta} \neq 0 \) for some \( \delta \mid N \) where \( g_1(z) \) is a newform without complex multiplication. Moreover, let \( \delta_1 \) be the smallest divisor of \( N \) for which \( \alpha_{1,\delta_1} \neq 0 \). Let \( p_1 \nmid N \) be any prime for which \( b_1(p_1) \neq b_2(p_1) \). Then consider the form

\[
f_1(z) = \sum_{n=1}^{\infty} a_1(n)q^n \quad := \quad T_{p_1} \mid f(z) - b_2(p_1)f(z) = \sum_{i=1}^{s} (b_i(p_1) - b_2(p_1)) \sum_{\delta \mid N} \alpha_{i,\delta} g_i(\delta z).
\]

It is clear that the cusp forms \( g_2(\delta z) \) do not occur in the newform decomposition of \( f_1(z) \) but \( g_1(\delta_1 z) \) does appear. Moreover, by (2.1) it is easy to see that

\[
a_1(n) = a_f(pn) + \chi(p)p^{k-1}a_f(n/p) - b_2(p_1)a_f(n).
\]

Arguing in this way, one may inductively remove all the non-zero newform components \( g_i(\delta z) \) for all \( 2 \leq i \leq s \) to obtain a cusp form \( F(z) \) (after by dividing by the obvious non-zero scalar) in \( S_k(\Gamma_0(N), \chi) \)

\[
F(z) = \sum_{n=1}^{\infty} A(n)q^n := \sum_{\delta \mid N} \alpha_{1,\delta} g_1(\delta z).
\]

Moreover, by iterating (2.5) and (2.6) this form has the property that there are finitely many algebraic numbers \( \beta_j \) and positive rational numbers \( \gamma_j \) for which

\[
A(n) = \sum_{\delta \mid N} \alpha_{1,\delta} b_1(n/\delta) = \sum_j \beta_j a_f(\gamma_j n)
\]
for all $n$. By applying the $U(\delta_1)$ operator which acts by

$$U(\delta_1) \mid \sum_{n=0}^{\infty} c(n)q^n := \sum_{n=0}^{\infty} c(\delta_1 n)q^n$$

we obtain a cusp form $F^*(z) = \sum_{n=1}^{\infty} A^*(n)q^n$ in $S_k(\Gamma_0(N), \chi)$ with the property that $A^*(n) = A(\delta_1 n)$ for all $n$.

Let $S_1$ denote the set of primes

$$S_1 := \{ p \mid N \text{ prime} \} \cup \{ p \text{ prime} : b_1(p) = 0 \},$$

and let $N_{S_1}$ denote the set of square-free positive integers with no prime factors in $S_1$. By (2.8) and the minimality of $\delta_1$, for every integer $n \in N_{S_1}$ we have that

$$A^*(n) = \alpha_{1, \delta_1}(n) = \sum_j \beta_j a_f(\gamma_j \delta_1 n).$$

Hence, for every integer $n \in N_{S_1}$ we have that $b_1(n) \not= 0$ implies that $a_f(\gamma_j \delta_1 n) \not= 0$ for at least one $j$. The conclusion of Theorem 1 for $f(z)$ follows immediately from the result for the newform $g_1(z)$.

Q.E.D.

3. Proof of Theorem 2.

In this section we prove Theorem 2. We begin by fixing notation. We fix an arbitrary element $g \in C$ of the conjugacy class $C$, and let $H = \langle g \rangle$ be the cyclic subgroup of $\text{Gal}(L/K)$ generated by $g$. Moreover, recall the definition of $c(L)$ from (1.4). Throughout, $\sum_\chi$ (resp. $\prod_\chi$) shall denote the sum (resp. product) over all irreducible characters $\chi$ of $H$. Moreover, if $\chi$ is a character of $H$, then let $L(s, \chi)$ be its associated Hecke $L$-function. Instead of studying $\pi_C(X; L/K)$ directly, we study $\Psi_C(X; L/K)$, the analog of Chebyshev’s function, given by

$$(3.1) \quad \Psi_C(X; L/k) := \sum_{\substack{N_{K/Q}(\mathfrak{P}) \leq X, \\ \mathfrak{P} \text{ unramified}, \\ [L/\mathcal{O}]^{\mathfrak{m}} = C}} \log(N_{K/Q}(\mathfrak{P})).$$

Using the explicit formula for $\Psi_C(X; L/K)$ due to Lagarias and Odlyzko [Th. 7.1, L-O], it is easy to obtain the following result.

**Lemma 3.1.** If $2 \leq T \leq X$, then

$$\Psi_C(X; L/K) = \frac{\#C}{\#\text{Gal}(L/K)} \left( X - \sum_\chi \tilde{\chi}(g) \left( \sum_{|\gamma| \leq T} \frac{X^\rho}{\rho} - \sum_{|\rho| \leq \frac{1}{2}} \frac{1}{\rho} \right) \right) + O \left( \frac{X \log^2 X}{T} \right),$$
where the inner sums extend over the nontrivial zeros \( \rho = \beta + i\gamma \) of \( L(s, \chi) \), and the implied constant may depend on \( K \) and \( L \).

If \( 2 \leq T \leq Y \leq X \) are fixed, then it is easy to see that

\[
\left| \frac{(X + Y)^\rho - X^\rho}{\rho} \right| = \left| \int_X^{X+Y} t^{\rho-1} \, dt \right| \leq \int_X^{X+Y} t^{\beta-1} \, dt \leq YX^{\beta-1}.
\]

Therefore, by Lemma 3.1 we find that

\[
(3.2) \quad \Psi_C(X + Y; L/K) - \Psi_C(X; L/K) = \sum_{\chi} \sum_{\rho, |\gamma| \leq T} (X + Y)^\rho - X^\rho \rho + O \left( \frac{X \log^2 X}{T} \right) = \sum_{\rho, |\gamma| \leq T} YX^{\beta-1} + O \left( \frac{X \log^2 X}{T} \right),
\]

where the last sum extends to zeros of all Hecke \( L \)-functions associated to irreducible characters of \( H \).

To prove Theorem 2, it suffices to show that main term in (3.2) dominates the two error terms for the appropriate range of \( Y \). The first error term depends on the zeros of the Hecke \( L \)-functions, which are precisely the zeros of the Dedekind zeta-function \( \zeta_L(s) \). This follows from the following fundamental identity (see [Th. 6, He]):

\[
\zeta_L(s) = \prod_{\chi} L(s, \chi).
\]

Consequently, it is important to have some knowledge of the distribution of the zeros of Dedekind zeta-functions. We summarize the facts we require in the following two lemmas which are obtained from the works of Heath-Brown, Mitsui, and Sokolovskii (see [H-B] and [Mi] or [So]).

**Lemma 3.2.** If \( \epsilon > 0 \), then there is a positive number \( A = A(\epsilon, L) \) such that

\[
N_L(\sigma, T) = \# \{ \rho : \zeta_L(\rho) = 0, \sigma \leq \beta \leq 1, |\gamma| \leq T \} \ll T^{(c(L)+\epsilon)(1-\sigma)} \log^A T
\]

uniformly in \( \frac{1}{2} \leq \sigma \leq 1 \).
Lemma 3.3. There are positive numbers \( t_0 \) and \( B = B(L) \) such that

\[
\zeta_L(\sigma + it) \neq 0
\]

whenever

\[
t \geq t_0 \quad \text{and} \quad \sigma \geq 1 - \frac{B}{(\log t)^{2/3}(\log \log t)^{1/3}}.
\]

Proof of Theorem 2. If \( X^{1-\frac{1}{\sigma(T)}} < Y < X \), then let \( T \) be

\[
T := \frac{X \log^3 X}{Y^\epsilon}.
\]

For this \( T \) the second error term in (3.2) is dominated by the main term \( \frac{\#C}{\#\text{Gal}(L/K)} Y \).

Therefore, it suffices to examine the first error term which depends on the zeros of \( \zeta_L(s) \). By Lemmas 3.2 and 3.3 we have that

\[
\sum_{\rho} |\gamma| \leq \frac{X \log X}{\max_{1/2 \leq \sigma \leq \sigma_X} N_L(\sigma, T)} \ll Y \log A X^{1-\sigma} \left( \frac{T^{c(L)+\epsilon}}{X} \right)^{1-\sigma},
\]

where the maximums are taken over

\[
1/2 \leq \sigma \leq \sigma_X = 1 - \frac{B}{(\log X)^{2/3}(\log \log X)^{1/3}}.
\]

This follows from the fact that for every \( \sigma \geq \sigma_X \) we have \( N_L(\sigma, T) = 0 \) provided \( T \leq X \) is sufficiently large. Our choice of \( T \) and \( Y \) in (3.3) implies that \( \frac{T^{c(L)+\epsilon}}{X} \leq X^{-2\epsilon} \) and (3.4) is maximized at \( \sigma = \sigma_X \). Therefore, by (3.4) the first error term in (3.2) is bounded by

\[
Y \log A X^{1-\sigma} \ll Y e^{-\epsilon B \left( \frac{\log X}{\log \log X} \right)^{1/3}}.
\]

which is dominated by the main term. Thus we have proved that

\[
\Psi_C(X + Y; L/K) - \Psi_C(X; L/K) = \frac{\#C}{\#\text{Gal}(L/K)} Y + O \left( \frac{Y}{\log X} \right).
\]

Finally, by the Prime Ideal Theorem, the contribution of the proper powers of prime ideals is at most \( X^{1/2} \log X \leq Y \) and the contribution of the prime ideals is

\[
\log N_K/Q(\Psi) = \log X + O(1).
\]

The transition from (3.5) to the statement of Theorem 2 is now straightforward.

Q.E.D.
4. Proof of Theorem 3.

We begin by proving the following result about newforms.

**Theorem 4.1.** Let \( f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(\Gamma_0(N), \chi) \) be a non-zero integer weight newform where the coefficients are algebraic integers in a number field \( K_f \). Let \( \mathcal{L} \subseteq O_{K_f} \) be a prime ideal for which there is a prime \( p_0 \nmid N\ell \), where the characteristic of \( O_{K_f}/\mathcal{L} \) is \( \ell \), for which

\[
a_f(p_0) \not\equiv 0 \pmod{\mathcal{L}}.
\]

Then there is a positive integer \( k_{f,\mathcal{L}} \) such that for every \( \epsilon > 0 \) and \( X^{1-\frac{1}{k_{f,\mathcal{L}}}+\epsilon} \leq Y \leq X \) we have

\[
\#\{X < p < X + Y \text{ prime} : a_f(p) \not\equiv 0 \pmod{\mathcal{L}}\} \gg_{f,\mathcal{L},\epsilon} \frac{Y}{\log X}.
\]

**Proof of Theorem 4.1.** By the work of Eichler, Shimura, Deligne, and Serre (see [D], [D-S], [Sh]) there is a finite Galois extension \( L/\mathbb{Q} \) which is unramified outside \( \ell N \) and a semi-simple Galois representation

\[
\rho_{f,\mathcal{L}} : \text{Gal}(L/\mathbb{Q}) \to GL_2(O_{K_f}/\mathcal{L})
\]

for which

\[
(4.1) \quad \text{trace } \rho_{f,\mathcal{L}}(\text{frob}_p) \equiv a_f(p) \pmod{\mathcal{L}}
\]

for every prime \( p \nmid \ell N \). Here \( \text{frob}_p \) denotes any Frobenius element for the prime \( p \). By the Chebotarev Density Theorem and (4.1), the conjugacy class \( C \) in \( \text{Gal}(L/\mathbb{Q}) \) containing \( \text{frob}_{p_0} \) has the property that

\[
0 \not\equiv a_f(p) \equiv a_f(p_0) \pmod{\mathcal{L}}
\]

for every \( \text{frob}_p \in C \). The result now follows immediately from Theorem 2.

Q.E.D.

**Proof of Theorem 3.** We shall prove Theorem 3 by considering the following two cases:

I. The case where \( f(z) \) is an integer weight cusp form.
II. The case where \( f(z) \) is a half-integral weight cusp form with weight \( \geq 3/2 \) which is not a finite linear combination of weight 3/2 theta functions.

**Case I.** Suppose that \( f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \) is an integer weight cusp form in \( S_k(\Gamma_0(N), \chi) \). Arguing as in the proof of Theorem 1, we may reduce this case to the result for newforms (i.e. Theorem 4.1). In particular, there is a newform \( g_1(z) = \sum_{n=1}^{\infty} b_1(n)q^n \) and finitely many positive rational numbers \( \gamma_j \) and a fixed positive integer \( \delta_1 \mid N \) for which every sufficiently large prime \( p \) with \( b_1(p) \not= 0 \) has the property that \( a_f(\gamma_j\delta_1 p) \not= 0 \) for at least one \( j \). The conclusion of Theorem 3 for \( f(z) \) follows immediately from Theorem 4.1.
Case II. Suppose that \( f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \) is a cusp form in \( S_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi) \) with \( \lambda \geq 1 \) which is not a linear combination of weight 3/2 theta functions. If \( p \nmid N \) is prime, then the Hecke operator \( T(p^2) \) on this space is defined by

\[
T(p^2) | f(z) := \sum_{n=0}^{\infty} (a_f(p^2n) + \chi(p)\left(\frac{(-1)^\lambda n}{p}\right)p^{\lambda-1}a_f(n) + \chi(p^2)p^{2\lambda-1}a_f(n/p^2))q^n.
\]

Generalizing [p. 82, Sh], there is a set of eigenforms \( g_1(z), g_2(z) \ldots g_s(z) \) in \( S_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi) \) of the Hecke operators \( T(p^2) \) for primes \( p \nmid N \) with Fourier expansions

\[
g_i(z) = \sum_{n=1}^{\infty} b_i(n)q^n
\]

that satisfy the following two properties:

\[
(4.4) \quad \text{For each } 1 \leq i \leq s \text{ there is a square-free } n \text{ coprime to } N \text{ with } b_i(n) \neq 0.
\]

\[
(4.5) \quad S_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi) \text{ is spanned by the forms } g_i(\delta z) \text{ where } \delta \mid N.
\]

By combining the theory of modular symbols, the Shimura correspondence, and a theorem of Waldspurger (see [G-S, M-T-T, Sh, W]), we may assume that the coefficients of each of \( g_i(z) \) are algebraic integers in a number field \( K \). Moreover, by Shimura’s correspondence each \( g_i(z) \) which is not a weight 3/2 theta function has the property that its eigenvalue of \( T(p^2) \), for \( p \nmid N \), is an eigenvalue of \( T_p \) of a fixed newform \( G_i(z) \) of weight \( 2\lambda \) and level dividing \( N \).

As in Case I, we have a decomposition of \( f(z) \) as

\[
f(z) = \sum_{i=1}^{s} \sum_{\delta \mid N} \alpha_{i,\delta}g_i(\delta z).
\]

Moreover, by hypothesis we may assume that \( \alpha_{1,\delta} \neq 0 \) for some \( \delta \mid N \) where \( g_1(z) \) is not a theta function. Chose \( \delta_1 \) minimally so that \( \alpha_{1,\delta_1} \neq 0 \).

Arguing as in Case I with the Hecke operators \( T_p \) being replaced by the operators \( T(p^2) \), we can conclude that there are finitely many positive rational numbers \( \gamma_j \) for which

\[
b_1(n) \neq 0 \text{ with } \gcd(n, N) = 1 \implies a_f(\gamma_j \delta_1 n) \neq 0 \text{ for some } j.
\]

Hence, it suffices to prove that \( g_1(z) \) satisfies the conclusion of Theorem 4 where one excludes those \( n \) which are not coprime to \( N \). By replacing \( g_1(z) \) by a suitable linear combination of its twists (possibly trivial), we may without loss of generality assume that \( b_1(n) = 0 \) for those \( n \) which are not coprime to \( N \) and those \( n \) which are perfect squares. Since \( g_1(z) \) is not a linear combination of weight 3/2 theta functions, by a theorem of
Vigneras [V] there are infinitely many square-free integers $t$ for which $b_1(tn^2) \neq 0$ for some $n$. Moreover, since $g_1(z)$ is an eigenform with coefficients which are algebraic integers in a number field, by (4.2) we have that $b_1(t) \mid b_1(tn^2)$ for every $n$. Therefore, the minimal 2-adic valuation of the coefficients $b_1(n)$ is attained by $b_1(t_0)$ for some square-free integer $t_0$.

By the proof of the [Fund. Lemma, O-Sk], the minimal 2-adic behavior of the coefficients $b_1(n)$ is controlled by the Fourier expansion of some weight $\lambda + 1$ cusp form. The trick is simply to multiply $g_1(z)$ by $\theta(z) = 1 + 2q + 2q^4 + \cdots \equiv 1 \pmod{2}$. Using $b(t_0)$ in [Fund. Lemma, O-Sk], we find that the conclusion of Theorem 3 for $g_1(z)$ follows immediately from Case I. The result for $f(z)$ follows easily from (4.7).

Q.E.D.

5. Proofs of Corollaries.

Here we prove the corollaries described in the introduction.

**Proofs of Corollaries 4 and 5.** Both results follow from Theorem 3 on the nonvanishing of the Fourier coefficients in the case of half-integral weight cusp forms. The works of Shimura and Waldspurger [Sh2, W] shows that the coefficients of a half-integral weight cusp form $g(z)$, which is an eigenform but not a theta function, interpolates many central critical values of the quadratic twists of the modular $L$-function associated to the Shimura correspondent of $g(z)$. Although the Shimura correspondence is not surjective, it is shown in [§2, O-Sk] that such critical values can be obtained in this way for every modular $L$ function of an even weight newform with trivial Nebentypus.

Corollary 5 is an immediate consequence of Corollary 4, the fact that the $L$-function of $E_D$ is the $D$-quadratic twist of $L(E, s)$ when $\gcd(D, 4N) = 1$, and the celebrated theorem of Kolyvagin that [Ko]

$$L(E, 1) \neq 0 \implies \text{rk}(E) = 0.$$  

Q.E.D.

**Proof of Corollary 6.** This result follows immediately from Theorem 2 and the mere definition of the action of Galois on the torsion points of an elliptic curve $E$.

Q.E.D.

**References**


14 ANTAL BALOG AND KEN ONO


Mathematical Institute of the Hungarian Academy of Sciences, P.O. Box 127, Budapest 1364, Hungary

E-mail address: balog@hexagon.renyi.hu

Dept. Math., University of Wisconsin, Madison, Wisconsin, 53706, USA.

E-mail address: ono@math.wisc.edu