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Congruences and conjectures for the partition function

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1. Introduction

The topic of congruences for the partition function $p(n)$ has been widely studied. The purpose of this paper is threefold. In the first part we give an account of some of the contributions which the two authors have made to the area in the past several years. In the second part we present a new construction of certain modular forms related to the partition function; this gives a new (and particularly simple) framework in which to consider congruences for the partition function modulo primes $\ell \geq 5$. Finally, we will pose some conjectures which we hope will clarify some of the interesting remaining questions on the congruential distribution of values of $p(n)$.

2. Recent results

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n (we define $p(0) = 1$ and $p(\alpha) = 0$ if $\alpha \notin \mathbb{Z}_{\geq 0}$). We are concerned with the topic of linear congruences for the partition function; i.e. relations of the form

$$p(An + B) \equiv 0 \pmod{M} \quad \text{for all } n,$$

where A, B, and M are integers. Such congruences were, of course, first discovered by Ramanujan. Throughout the paper, $\ell \geq 5$ will denote a prime number, and δ_ℓ will denote the integer

$$\delta_\ell := \frac{\ell^2 - 1}{24}. \tag{2.1}$$

With this notation, Ramanujan's famous congruences take the form

$$p(\ell n - \delta_\ell) \equiv 0 \pmod{\ell} \quad \text{if } \ell = 5,\ 7, \text{ or } 11. \tag{2.2}$$

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Ramanujan conjectured (and in some cases proved) extensions of these congruences to powers of 5, 7, and 11. In fact, after his work [**Be-O, R1, R2, R3**] and subsequent work of Watson [**Wa**] and Atkin [**At1**], it is now known that if $24m \equiv 1 \pmod{5^a 7^b 11^c}$, then we have

$$p(m) \equiv 0 \pmod{5^a 7^{\lfloor (b+2)/2 \rfloor} 11^c}. \tag{2.3}$$

Since the work of Ramanujan, further examples of congruences involving primes $\ell \leq 31$ have been found (see the works of [**At2, At-O, At-SwD1, At-SwD2, N4, L-O, W**]). The first systematic treatment of such congruences came only recently, when the second author [**O2**] proved that for any prime $\ell \geq 5$, there exist infinitely many (non-nested) congruences of the form

$$p(An + B) \equiv 0 \pmod{\ell} \quad \text{for all } n. \tag{2.4}$$

The first author extended the method to prove that if $\ell \geq 5$ is prime and m is a positive integer, then there exist infinitely many congruences of the form

$$p(An + B) \equiv 0 \pmod{\ell^m} \quad \text{for all } n. \tag{2.5}$$

In fact, it is shown that such congruences exist for any modulus M which is coprime to 6.

The residue class $-\delta_\ell \pmod{\ell}$ has always played a distinguished role in the theory. Indeed, all of Ramanujan's congruences and their extensions (2.3) lie within the progressions $\ell n - \delta_\ell$. Further, all of the congruences (2.4) and (2.5) whose existence is proven in [**A2, O2**] necessarily lie within this progression. To further highlight the importance of this class, we mention work of Kiming and Olsson [**K-O**], who proved that if $\ell \geq 5$ is prime and

$$p(\ell n + \beta) \equiv 0 \pmod{\ell} \quad \text{for all } n,$$

then $\beta \equiv -\delta_\ell \pmod{\ell}$.

However, some of the examples alluded to above–in particular those given in [**At2, N4**]–illustrate that congruences may indeed lie outside of this class. Atkin [**At2**], for example, proved that

$$p(17303n + 237) \equiv 0 \pmod{13}.$$

These examples call into question the true importance of the class $-\delta_\ell \pmod{\ell}$.

In a recent paper [**A-O**], the two authors have shown that congruences for $p(n)$ are much more widespread than was previously known. In fact, they show that the class $-\delta_\ell \pmod{\ell}$ is just one of $(\ell+1)/2$ residue classes modulo ℓ in which the partition function has similar congruence properties.

To state the main result in [**A-O**] requires some notation. For each prime $\ell \geq 5$, define the integer $\epsilon_\ell \in \{\pm 1\}$ by

$$\epsilon_\ell := \left(\frac{-6}{\ell}\right), \tag{2.6}$$

and let S_ℓ denote the set of $(\ell+1)/2$ integers

$$S_\ell := \left\{ \beta \in \{0, 1, \dots, \ell - 1\} \ : \ \left(\frac{\beta + \delta_\ell}{\ell}\right) = 0 \text{ or } -\epsilon_\ell \right\}. \tag{2.7}$$

Then we have

THEOREM 1. *If $\ell \geq 5$ is prime, m is a positive integer, and $\beta \in S_\ell$, then a positive proportion of the primes $Q \equiv -1 \pmod{24\ell}$ have the property that*

$$p\left(\frac{Q^3n+1}{24}\right) \equiv 0 \pmod{\ell^m}$$

for all $n \equiv 1 - 24\beta \pmod{24\ell}$ with $\gcd(Q, n) = 1$.

We remark that the case when $\beta \equiv -\delta_\ell \pmod{\ell}$ contains the main results in [**O2**] and [**A2**]. Further, we note that given $\beta \in S_\ell$ and a prime Q as in Theorem 1, fixing n in an appropriate residue class modulo $24\ell Q$ gives a Ramanujan-type congruence within the progression $\ell n + \beta$. This yields the following

THEOREM 2. *If $\ell \geq 5$ is prime, m is a positive integer, and $\beta \in S_\ell$, then there are infinitely many non-nested arithmetic progressions $\{An + B\} \subseteq \{\ell n + \beta\}$ such that*

$$p(An + B) \equiv 0 \pmod{\ell^m}$$

for every integer n.

Finally, we note that if M is an integer coprime to 6, then Theorem 2 and the Chinese Remainder Theorem guarantee the existence of infinitely many congruences modulo M. The results in [**A-O**] provide a theoretical framework which (to our knowledge) explains every known partition function congruence.

3. A new construction

All of the results in [**A-O**] rely on the construction of half-integral weight modular forms whose coefficients capture values of $p(n)$ modulo ℓ^m. In this section we present an alternate construction in the case $m = 1$ using the theory of modular forms modulo ℓ as developed by Serre and Swinnerton-Dyer [**SwD**]. This approach yields an elegant proof of Theorem 1 in the case when $m = 1$, and is particularly convenient for constructing examples. The main advantage of this approach is that it allows us to work with modular forms on $\mathrm{SL}_2(\mathbb{Z})$; although the construction in [**A-O**] is more general, it requires the use of modular forms of much higher level. Throughout we use the notation $q := e^{2\pi i z}$, and we adopt standard notation from the theory of modular forms. Define the character χ_{12} by $\chi_{12}(d) := \left(\frac{12}{d}\right)$. Our aim in this section is to prove the following:

THEOREM 3.1. *Suppose that $\ell \geq 5$ is prime. Then there exists a cusp form $P_\ell(z)$ in $S_{(\ell^2-2)/2}(\Gamma_0(576), \chi_{12}) \cap \mathbb{Z}[[q]]$ such that*

$$P_\ell(z) \equiv \sum_{n \equiv 0 \pmod{\ell}} p(n - \delta_\ell) q^{24n-\ell^2} + 2 \sum_{\left(\frac{n}{\ell}\right) = -\epsilon_\ell} p(n - \delta_\ell) q^{24n-\ell^2} \pmod{\ell}.$$

We recall (see, for example [**S-St**]) that if $f(z) = \sum_{n=1}^{\infty} a(n) q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_1(N))$, and r and t are positive integers, then

$$\sum_{n \equiv r \pmod{t}} a(n) q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_1(Nt^2)).$$

Let $P_\ell(z)$ be as in Theorem 3.1, and suppose that $\beta \in S_\ell$, where S_ℓ is defined in (2.7). Extracting those terms from $P_\ell(z)$ whose exponents are congruent to $24\beta - 1 \pmod{\ell}$, we obtain the following corollary, which implies Theorem 2.1 of [**A-O**] in the case $m = 1$.

COROLLARY 3.2. *Suppose that $\ell \geq 5$ is prime and that $\beta \in S_\ell$. Then there exists a cusp form $F_{\ell,\beta}(z) \in S_{(\ell^2-2)/2}(\Gamma_1(576\ell^2)) \cap \mathbb{Z}[[q]]$ for which*

$$F_{\ell,\beta}(z) \equiv \sum_{n=0}^{\infty} p(\ell n+\beta) q^{24\ell n+24\beta-1} \pmod{\ell}.$$

Applying the arguments in Section 3 of [**A-O**] (which rely on certain facts arising from the theory of Galois representations associated to modular forms and Shimura's theory of half integral weight modular forms), the forms given in Corollary 3.2 yield a proof of Theorem 1 in the case when $m = 1$.

Before beginning the proof of Theorem 3.1, we briefly recall certain facts about the theory of modular forms modulo ℓ (see [**Sw-D**] for details). If k is an even integer, then let M_k (resp. S_k) denote the $\mathbb{C}$-vector space of weight k modular (resp. cusp) forms with respect to $\mathrm{SL}_2(\mathbb{Z})$. Let $M_{k,\ell}$ and $S_{k,\ell}$ denote the $\mathbb{F}_\ell$-vector spaces given by

$$M_{k,\ell} := \left\{ f(z) = \sum_{n=0}^{\infty} a(n)q^n \pmod{\ell} \ : \ f(z) \in M_k \cap \mathbb{Z}[[q]] \right\},$$

$$S_{k,\ell} := \left\{ f(z) = \sum_{n=1}^{\infty} a(n)q^n \pmod{\ell} \ : \ f(z) \in S_k \cap \mathbb{Z}[[q]] \right\}.$$

As usual, let $E_k(z)$ denote the normalized weight k Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$. Using the fact that

$$E_{\ell-1}(z) \equiv 1 \pmod{\ell},$$

one sees that the set of modular forms modulo ℓ forms a graded algebra.

We recall that if $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ is a modular form with integral coefficients, then $\omega_\ell(f)$, the filtration of f modulo ℓ, is defined by

$$\omega_\ell(f) := \min\{k \ : \ f \pmod{\ell} \in M_{k,\ell}\}.$$

Also, if $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ has integral coefficients, then the Ramanujan theta operator is defined by

$$\Theta_\ell(f) :\equiv \sum_{n=0}^{\infty} na(n)q^n \pmod{\ell}.$$

Finally, we record the following

PROPOSITION 3.3. [**Sw-D, §3, Lemma 5**] *Suppose that $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ is a modular form with integral coefficients and that $\omega_\ell(f) \not\equiv 0 \pmod{\ell}$. Then*

$$\omega_\ell(\Theta_\ell(f)) = \omega_\ell(f) + \ell + 1.$$

PROOF OF THEOREM 3.1. We begin by recalling Dedekind's eta function

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n). \tag{3.1}$$

If $\ell \geq 5$ is prime, then define $f_\ell(z) = \sum_{n=1}^{\infty} a_\ell(n)q^n$ by

$$f_\ell(z) = \sum_{n=1}^{\infty} a_\ell(n)q^n := \frac{\eta^\ell(\ell z)}{\eta(z)}. \tag{3.2}$$

Using classical facts (see, for example [**G-H, N1, N2**], we see that $f_\ell(z)$ is a modular form in $M_{(\ell-1)/2}\left(\Gamma_0(\ell), \left(\frac{\bullet}{\ell}\right)\right)$ with integral coefficients. We have (see, for example, [**An, Th. 1.1**]), the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Using this with (3.1) and (3.2), we find that

$$\sum_{n=1}^{\infty} a_\ell(n)q^n = \left(\sum_{n=0}^{\infty} p(n)q^{n+\delta_\ell}\right) \cdot \prod_{n=1}^{\infty}(1-q^{\ell n})^\ell. \tag{3.3}$$

Recall that $\delta_\ell = (\ell^2-1)/24$. We have $f_\ell(z) \equiv \Delta^{\delta_\ell}(z) \pmod{\ell}$, where

$$\Delta(z) = \eta^{24}(z)$$

is the unique normalized cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. By [**SwD, §3, Lemma 6**], we see that $\omega_\ell(\Delta^{\delta_\ell}) = (\ell^2-1)/2$; therefore Proposition 3.3 implies that

$$\omega_\ell\left(\Theta_\ell^{(\ell-1)/2}\left(\Delta^{\delta_\ell}(z)\right)\right) = \ell^2 - 1.$$

Further, notice that

$$\Theta_\ell^{(\ell-1)/2}\left(\Delta^{\delta_\ell}(z)\right) \equiv \sum_{n=0}^{\infty}\left(\frac{n}{\ell}\right)a_\ell(n)q^n \pmod{\ell}.$$

Therefore, there is a cusp form $P_{0,\ell}(z)$ in $S_{\ell^2-1} \cap \mathbb{Z}[[q]]$ for which

$$P_{0,\ell}(z) \equiv \sum_{n=0}^{\infty}\left(\frac{n}{\ell}\right)a_\ell(n)q^n \pmod{\ell}.$$

Let $P_{1,\ell}(z) \in S_{\ell^2-1} \cap \mathbb{Z}[[q]]$ be the cusp form defined by

$$P_{1,\ell}(z) := \Delta^{\delta_\ell}(z) \cdot E_{\ell-1}^{(\ell+1)/2}(z).$$

Define the cusp form $P_{2,\ell}(z)$ in $S_{\ell^2-1} \cap \mathbb{Z}[[q]]$ by

$$P_{2,\ell}(z) := P_{1,\ell}(z) - \epsilon_\ell P_{0,\ell}(z).$$

Using (3.3), we obtain

(3.4)

$$P_{2,\ell} \equiv \left\{ \sum_{n\equiv 0 \pmod{\ell}} p(n-\delta_\ell)q^n + 2\sum_{\left(\frac{n}{\ell}\right)=-\epsilon_\ell} p(n-\delta_\ell)q^n \right\} \prod_{n=1}^{\infty}(1-q^{\ell n})^\ell \pmod{\ell}.$$

By construction, the first exponent in $P_{2,\ell}(z) \in S_{\ell^2-1}$ whose coefficient could be non-zero is $\delta_\ell + 1$. Since S_{ℓ^2-1} has a basis of the form

$$\left\{\Delta(z)^j E_4(z)^{\frac{\ell^2-1}{4}-3j} \;:\; 1 \le j \le \tfrac{\ell^2-1}{12}\right\},$$

we see that there exists $C(z) \in M_{(\ell^2-25)/2} \cap \mathbb{Z}[[q]]$ such that

$$P_{2,\ell}(z) = \Delta^{\delta_\ell+1}(z) \cdot C(z).$$

It follows that

$$P_{2,\ell}(z)/\eta^{\ell^2}(z) = \eta^{23}(z)C(z).$$

Recall that $\eta(24z) \in S_{1/2}(\Gamma_0(576), \chi_{12})$, and that $\eta^{\ell}(\ell z) \equiv \eta^{\ell^2}(z) \pmod{\ell}$. Using these facts together with (3.4), we conclude that there exists a cusp form $P_\ell(z)$ in $S_{(\ell^2-2)/2}(\Gamma_0(576), \chi_{12}) \cap \mathbb{Z}[[q]]$ for which

$$P_\ell(z) \equiv \sum_{n\equiv 0 \pmod{\ell}} p(n-\delta_\ell)q^{24n-\ell^2} + 2\sum_{\left(\frac{n}{\ell}\right)=-\epsilon_\ell} p(n-\delta_\ell)q^{24n-\ell^2} \pmod{\ell}.$$

This gives Theorem 3.1. □

4. Examples

Here we present examples of Theorem 3.1 for the primes $\ell = 5$, 7, and 11. Using [**SwD, §3, Lemma 6**], one can verify that

$$P_{2,5} \equiv 2\Delta^2 \pmod 5,$$

$$P_{2,7} \equiv 2\Delta^3 E_4^3 + 6\Delta^4 \pmod 7,$$

$$P_{2,11} \equiv 2\Delta^6 E_4^{12} + 10\Delta^7 E_4^9 + 8\Delta^8 E_4^6 + 2\Delta^9 E_4^3 + 5\Delta^{10} \pmod{11}.$$

Consequently, we have

$$\sum_{n=0}^{\infty} p(5n+1)q^{120n+23} + \sum_{n=0}^{\infty} p(5n+2)q^{120n+47} \equiv \eta^{23}(24z) \pmod 5, \tag{4.1}$$

$$\begin{aligned}\sum_{n=0}^{\infty} p(7n+1)q^{168n+23} + \sum_{n=0}^{\infty} p(7n+3)q^{168n+71} + \sum_{n=0}^{\infty} p(7n+4)q^{168n+95}\\ \equiv \eta^{23}(24z)E_4^3(24z) + 3\eta^{47}(24z) \pmod 7,\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\sum_{n=0}^{\infty} p(11n+1)q^{264n+23} + \sum_{n=0}^{\infty} p(11n+2)q^{264n+47} + \sum_{n=0}^{\infty} p(11n+3)q^{264n+71}\\ &\quad + \sum_{n=0}^{\infty} p(11n+5)q^{264n+119} + \sum_{n=0}^{\infty} p(11n+8)q^{264n+191}\\ &\quad \equiv \eta^{23}(24z)E_4^{12}(24z) + 5\eta^{47}(24z)E_4^9(24z) + 4\eta^{71}(24z)E_4^6(24z)\\ &\quad + \eta^{95}(24z)E_4^3(24z) + 8\eta^{119}(24z) \pmod{11}.\end{aligned} \tag{4.3}$$

5. Conjectures

We conclude with a variety of conjectures and open problems. We begin with questions related to the existence of further Ramanujan-type congruences.

CONJECTURE 5.1. (Subbarao [**Su**]) *If A and B are integers with $0 \leq B < A$, then there are infinitely many integers n for which*

$$p(An + B) \not\equiv 0 \pmod{2}.$$

This conjecture is known for every arithmetic progression $B \pmod{A}$ for which there is at least one n with $p(An + B) \equiv 1 \pmod{2}$ [**Th. 2, O1**]. The conjecture is also known for every arithmetic progression $B \pmod{A}$ where A is a power of 2 [**Th. 1, B-O**].

Unfortunately, very little is known about the partition function modulo 3. For example, it is not even known that there are infinitely many n for which $3 \mid p(n)$. As an analogue to Conjecture 5.1, it seems reasonable to make the following conjecture.

CONJECTURE 5.2. *If A and B are integers with $0 \leq B < A$, then there are infinitely many integers n for which*

$$p(An + B) \not\equiv 0 \pmod{3}.$$

Apart from Ramanujan's original congruences (2.2), (2.3), no others are known where the modulus of the congruence equals the modulus of the arithmetic progression. In view of this and the work of Kiming and Olsson mentioned in the introduction, we pose the following.

CONJECTURE 5.3. *If $\ell \geq 13$ is prime, and β is an integer, then there are infinitely many integers n for which*

$$p(\ell n + \beta) \not\equiv 0 \pmod{\ell}.$$

Based on the results in [**A-O**], it seems reasonable to make the following conjecture.

CONJECTURE 5.4. *Suppose that $\ell \geq 5$ is prime and that*

$$p(An + B) \equiv 0 \pmod{\ell}$$

for every integer n. Then there exists $\beta \in S_\ell$ such that $\{An + B\} \subseteq \{\ell n + \beta\}$.

We recall the following important conjecture of Newman [**N3**].

CONJECTURE 5.5. *If M is a positive integer, then for every residue class $r \pmod{M}$ there are infinitely many integers n for which $p(n) \equiv r \pmod{M}$.*

Although the results in [**A2, O1**] provide a simple criterion for deducing Conjecture 5.5 for any M coprime to 6, it remains open. In fact, Conjecture 5.5 has not been proven for infinitely many M.

The remaining conjectures and problems are devoted to questions involving the distribution of $p(n)$ modulo integers M.

CONJECTURE 5.6. *If $0 \leq r < M$, then define $\delta_r(M, X)$ by*

$$\delta_r(M,X) := \frac{\#\{0 \leq n < X \ : \ p(n) \equiv r \pmod{M}\}}{X}.$$

1. *If $0 \leq r < M$, then there is a real number $0 < d_r(M) < 1$ for which*

$$\lim_{X \to \infty} \delta_r(M, X) = d_r(M).$$

2. *If $s \geq 1$ and $M = 2^s$, then for every $0 \leq i < 2^s$ we have*

$$d_i(2^s) = \frac{1}{2^s}.$$

3. *If $s \geq 1$ and $M = 3^s$, then for every $0 \leq i < 3^s$ we have*

$$d_i(3^s) = \frac{1}{3^s}.$$

4. *If there is a prime $\ell \geq 5$ for which $\ell \mid M$, then for every $0 \leq r < M$ we have*

$$d_r(M) \neq \frac{1}{M}.$$

Virtually nothing is known about Conjecture 5.6. Part (1) is not known for any values of r and M. If M is coprime to 6, then Theorem 2 implies that

$$\liminf_{X \to \infty} \delta_0(M, X) > 0.$$

There are no other pairs of integers $0 < r < M$ for which it is known that

$$\liminf_{X \to \infty} \delta_r(M, X) > 0.$$

When $M = 2$, part (2) is the well known "folklore conjecture" studied by Parkin and Shanks in the 1960s [**P-S**]. In this direction, the best results are due to Serre [**N-R-S**] and the first author [**A1**]. It is now known that

$$\#\{n \leq X \ : \ p(n) \equiv 0 \pmod 2\} \gg \sqrt{X}$$
$$\#\{n \leq X \ : \ p(n) \equiv 1 \pmod 2\} \gg \sqrt{X}/\log X.$$

Obviously, this falls far short of Conjecture 5.6. The table below provides data supporting parts (2) and (3) of Conjecture 5.6 when $s = 1$.

X	$\delta_0(2;X)$	$\delta_1(2;X)$	$\delta_0(3;X)$	$\delta_1(3;X)$	$\delta_2(3;X)$
200,000	0.5012	0.4988	0.3332	0.3331	0.3337
400,000	0.5000	0.5000	0.3339	0.3324	0.3336
600,000	0.5000	0.5000	0.3337	0.3326	0.3337
800,000	0.5006	0.4994	0.3331	0.3333	0.3336
1,000,000	0.5004	0.4996	0.3330	0.3336	0.3334

Although we have insufficient data to conjecture a value for $d_r(M)$ for any $0 \leq r < M$ with M coprime to 6, it is natural to consider the following problem.

PROBLEM 5.7. *Find a lower bound for $d_0(M)$ when M is coprime to 6.*

Suppose that $\ell \geq 5$ is prime. In view of Theorem 1, Theorem 2, and Conjecture 5.4, it is natural to consider the distribution of $p(n) \pmod{\ell}$ for those $n \pmod{\ell}$ which do not belong to S_ℓ. Based on preliminary calculations, the following speculation does not appear to be too far-fetched.

SPECULATION 5.8. *If $\ell \geq 5$ is prime and $0 \leq r < \ell$, then define $\delta_r'(\ell, X)$ by*

$$\delta_r'(\ell, X) := \frac{\#\{n < X \ : \ p(n) \equiv r \pmod{\ell} \text{ and } n \pmod{\ell} \notin S_\ell\}}{\#\{n < X \ : \ n \pmod{\ell} \notin S_\ell\}}.$$

For every $0 \leq r < \ell$, is it true that $\lim_{X \to \infty} \delta_r'(\ell, X) = \frac{1}{\ell}$?

To support this speculation, we give data describing these functions when $\ell = 5$.

X	$\delta_0'(5;X)$	$\delta_1'(5;X)$	$\delta_2'(5;X)$	$\delta_3'(5;X)$	$\delta_4'(5;X)$
200,000	0.2006	0.1987	0.2011	0.2007	0.1988
400,000	0.2000	0.1996	0.2001	0.2007	0.1996
600,000	0.1999	0.1992	0.1995	0.2009	0.2006
800,000	0.1995	0.1996	0.1994	0.2010	0.2005
1,000,000	0.1999	0.1996	0.1997	0.2006	0.2002

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