EXTENSION OF RAMANUJAN'S CONGRUENCES FOR THE PARTITION FUNCTION MODULO POWERS OF 5

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1. Introduction and Statement of Results

A *partition* of a positive integer n is any non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n. (As usual, we adopt the convention that $p(0) = 1$ and $p(\alpha) = 0$ if $\alpha \notin \mathbb{N}$). Ramanujan's famous congruences, which were proved by Atkin, Ramanujan and Watson [2, 3, 13], assert that if j is a positive integer, then

$$p(5^j N + \beta_5(j)) \equiv 0 \pmod{5^j}, \tag{1}$$

$$p(7^j N + \beta_7(j)) \equiv 0 \pmod{7^{[j/2]+1}}, \tag{2}$$

$$p(11^j N + \beta_{11}(j)) \equiv 0 \pmod{11^j} \tag{3}$$

for every non-negative integer N where $\beta_m(j) := 1/24 \pmod{m^j}$.

These congruences are quite striking since a cursory examination of values of the partition function fails to reveal further congruences. The mere question as to whether there are infinitely many other congruences of the form

$$p(AN + B) \equiv 0 \pmod{M}$$

had remained open for some time. Although works by the second author [7, 8] have gone some way toward quantifying the rarity of these congruences, it is now known that there

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are indeed infinitely many such congruences. In fact, Ahlgren and the second author [1, 9] have shown that there are such congruences for every modulus M which is coprime to 6. Unfortunately, these results are not constructive. In fact, to our knowledge no explicit examples of such congruences are known with prime modulus $M > 31$.

In a similar direction, it is natural to ask whether the moduli in Ramanujan's congruences (1-3) are optimal. In particular, are there subprogressions, besides those found by Ramanujan, where the known congruence modulo m^j is a congruence modulo m^{j+1}? In this paper we revisit (1) and answer this question in the affirmative by explicitly exhibiting infinitely many such progressions for each j. Using the ideas found in [9], one can obtain similar extensions of (2) and (3). Unfortunately, these extensions are not palatable.

For convenience, define rational numbers $\beta(j,\ell)$ by

$$\beta(j,\ell) := \begin{cases} \frac{19\cdot 5^j\cdot \ell^2+1}{24} & \text{if } j \text{ is odd,} \\ \frac{23\cdot 5^j\cdot \ell^2+1}{24} & \text{if } j \text{ is even.} \end{cases} \tag{4}$$

Notice that $\beta(j,1) = \beta_5(j)$. In this notation, we obtain the following systematic families of multiplicative congruences.

Theorem 1. *Let $\ell \geq 7$ be prime.*
1) If $j \geq 1$ is odd, then for every non-negative integer n we have

$$p(5^j\ell^2 n + \beta(j,\ell)) \equiv$$

$$\equiv \left(\frac{15}{\ell}\right)\left(1+\ell-\ell^2\left(\frac{-24n-19}{\ell}\right)\right)p(5^j n+\beta_5(j)) - \ell p\left(\frac{5^j n}{\ell^2}+\beta(j,\ell^{-1})\right) \pmod{5^{j+1}}.$$

2) If $j \geq 2$ is even, then for every non-negative integer n we have

$$p(5^j\ell^2 n + \beta(j,\ell)) \equiv$$

$$\equiv \left(\frac{15}{\ell}\right)\left(1+\ell-\left(\frac{-24n-23}{\ell}\right)\right)p(5^j n+\beta_5(j)) - \ell p\left(\frac{5^j n}{\ell^2}+\beta(j,\ell^{-1})\right) \pmod{5^{j+1}}.$$

Using Theorem 1, we obtain two corollaries which reveal extensions of all of Ramanujan's congruences modulo powers of 5. In both cases, for every positive integer j we construct infinitely many distinct non-trivial subprogressions of the arithmetic progression

$$5^j N + \beta_5(j)$$

for which Ramanujan's congruence modulo 5^j is a congruence modulo 5^{j+1}.

Corollary 2. *Let $\ell \equiv 4 \pmod 5$ be prime.*
1) If $j \geq 1$ is odd, let $0 \leq r, s \leq \ell - 1$ be integers such that

$$\begin{aligned} &(i) \quad 24r + 19 \equiv 0 \pmod{\ell}, \\ &(ii) \quad 24s\ell + 24r + 19 \not\equiv 0 \pmod{\ell^2}. \end{aligned}$$

2) If $j \geq 2$ is even, let $0 \leq r, s \leq \ell - 1$ be integers such that

$$\begin{aligned} &(i) \quad 24r + 23 \equiv 0 \pmod{\ell}, \\ &(ii) \quad 24s\ell + 24r + 23 \not\equiv 0 \pmod{\ell^2}. \end{aligned}$$

Then for every non-negative integer N we have

$$p(5^j\ell^4 N + 5^j\ell^3 s + 5^j\ell^2 r + \beta(j,\ell)) \equiv 0 \pmod{5^{j+1}}.$$

Corollary 3. *Let $7 \leq \ell \equiv 3 \pmod 5$ be prime. If $j \geq 1$ is odd (resp. even) and $0 \leq r \leq \ell - 1$ is an integer for which $\left(\frac{-24r-19}{\ell}\right) = 1$ (resp. $\left(\frac{-24r-23}{\ell}\right) = -1$), then for every non-negative integer N we have*

$$p(5^j\ell^3 N + 5^j\ell^2 r + \beta(j,\ell)) \equiv 0 \pmod{5^{j+1}}.$$

It is easy to see that all of the arithmetic progressions in Corollaries 2 and 3 are not subprogressions of $5^{j+1}N + \beta_5(j+1)$. For example, if j is odd in Corollary 3 then

$$\begin{aligned} &5^j\ell^3 N + 5^j\ell^2 r + \beta(j,\ell) - \beta(j+1,1) \equiv 0 \pmod{5^{j+1}} \\ \iff\ &5^j\ell^3 N + 5^j\ell^2 r + \frac{19 \cdot 5^j\ell^2 + 1}{24} - \frac{23 \cdot 5^{j+1} + 1}{24} \equiv 0 \pmod{5^{j+1}} \\ \iff\ &2N - r - 1 \equiv 0 \pmod 5 \end{aligned}$$

which is obviously not true for all N.

Examples. Here we illustrate the utility of Corollaries 2 and 3. If $j = 1$, $\ell = 19$, and $r = s = 0$ in Corollary 2, then we have

$$p(651605N + 1429) \equiv 0 \pmod{25}.$$

Similarly, if $j = 1$, $\ell = 13$ and $r = 1$ in Corollary 3, then we have

$$p(10985N + 1514) \equiv 0 \pmod{25}.$$

If we let $\ell = 13$ and $r = 1$ in Corollary 3, then for every $j \geq 1$ we have

$$p(5^j \cdot 13^3 N + 5^j \cdot 13^2 + \beta(j,13)) \equiv 0 \pmod{5^{j+1}}.$$

2. The important observations

As usual, let $\eta(z)$ denote Dedekind's eta-function given by the infinite product

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$$

where $q := e^{2\pi i z}$. If χ is the quadratic character

$$\chi(n) := \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5, \pmod{12}, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

then Euler's classical Pentagonal Number Theorem asserts that

$$\eta(24z) = \sum_{n=1}^{\infty} \chi(n) q^{n^2}.$$

This fact is very useful for computing the coefficients of all the modular forms in this paper. We shall study the two modular cusp forms

$$F(z) := \eta^{19}(24z) = \sum_{n=1}^{\infty} a(n)q^n = q^{19} - 19q^{43} + 152q^{67} - \dots, \tag{6}$$

$$G(z) := \eta^{23}(24z) := \sum_{n=1}^{\infty} b(n)q^n = q^{23} - 23q^{47} + 230q^{71} - \dots. \tag{7}$$

It is easy to deduce (see [5]) that $F(z)$ is a cusp form in $S_{19/2}(\Gamma_0(576), \chi)$ and that $G(z) \in S_{23/2}(\Gamma_0(576), \chi)$.

The following theorem was proved by Newman.

Lemma 2.1. (Newman [Th. 1, 6]) *If $\ell \geq 5$ is prime, then define $\lambda_a(\ell)$ and $\lambda_b(\ell)$ by*

$$\lambda_a(\ell) := a(19\ell^2) + \ell^8 \left(\frac{-57}{\ell}\right),$$

$$\lambda_b(\ell) := b(23\ell^2) + \ell^{10} \left(\frac{-69}{\ell}\right).$$

For every positive integer n we have

$$\lambda_a(\ell)a(n) = a(\ell^2 n) + \ell^8 \left(\frac{-3n}{\ell}\right) a(n) + \ell^{17} a(n/\ell^2),$$

$$\lambda_b(\ell)b(n) = b(\ell^2 n) + \ell^{10} \left(\frac{-3n}{\ell}\right) b(n) + \ell^{21} b(n/\ell^2).$$

Lemma 2.1 states that $F(z)$ and $G(z)$ are eigenforms of the half integer weight Hecke operators. Recall that if $g(z) = \sum_{n=0}^{\infty} c(n)q^n \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \psi)$ is a half integer weight modular form and $p \nmid 4N$ is prime, then the Hecke operator $T_\lambda(p^2)$ is given by

$$g|T_\lambda(p^2) := \sum_{n=0}^{\infty} \left(c(p^2 n) + \psi(p) \left(\frac{(-1)^\lambda n}{p} \right) p^{\lambda-1} c(n) + \psi(p^2) p^{2\lambda-1} c(n/p^2) \right) q^n.$$

Moreover, g is an eigenform if for every prime $p \nmid 4N$ there is a complex number $\lambda_g(p)$ for which

$$g|T_\lambda(p^2) = \lambda_g(p) g.$$

It turns out that the eigenvalues $\lambda_a(\ell)$ and $\lambda_b(\ell)$ satisfy the following convenient congruences. Without such congruences, it seems imposssible to obtain clean extensions of (1).

Theorem 2.2. *If $\ell \geq 5$ is prime, then*

$$\lambda_a(\ell) \equiv \lambda_b(\ell) \equiv \left(\frac{15}{\ell} \right) (1 + \ell) \pmod 5.$$

To prove this theorem we shall employ some well known facts about modular forms modulo ℓ and the Shimura correspondence [10]. This correspondence is a family of maps which send modular of forms of half-integral weight to those of integer weight. Suppose that $f(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \psi)$ is an eigenform with $\lambda \geq 2$. If t is any square-free integer, then define $A_t(n)$ by

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} := L(s - \lambda + 1, \psi \chi_{-1}^{\lambda} \chi_t) \cdot \sum_{n=1}^{\infty} \frac{b(tn^2)}{n^s}.$$

Here χ_{-1} (resp. $\chi_t = \left(\frac{t}{\bullet}\right)$) is the Kronecker character for $\mathbb{Q}(i)$ (resp. $\mathbb{Q}(\sqrt{t})$). These numbers $A_t(n)$ define the Fourier expansion of $S_{t,\lambda}(f(z))$, a cusp form

$$S_{t,\lambda}(f(z)) := \sum_{n=1}^{\infty} A_t(n) q^n$$

in $S_{2\lambda}(\Gamma_0(4N), \psi^2)$. For us, the important feature of the Shimura correspondence $S_{t,\lambda}$ is the fact that it commutes with the Hecke algebra. In other words, if $p \nmid 4N$ is prime, then

$$S_{t,\lambda}(f|T_\lambda(p^2)) = S_{t,\lambda}(f)|T_p^\lambda. \tag{8}$$

Here T_p^{λ} (resp. $T_{\lambda}(p^2)$) denotes the usual Hecke operator on the space $S_{2\lambda}(\Gamma_0(4N), \psi^2)$ (resp. $S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \psi)$).

Proof of Theorem 2.2. Lemma 2.1 implies that $F(z) = \eta^{19}(24z) \in S_{19/2}(\Gamma_0(576), \chi)$ and $G(z) = \eta^{23}(24z)$ in $S_{23/2}(\Gamma_0(576), \chi)$ are eigenforms of the half-integer weight Hecke operators on $M_{19/2}(\Gamma_0(576), \chi)$ and $M_{23/2}(\Gamma_0(576), \chi)$ respectively.

Now let $\mathfrak{F}(z)$ be the eigenform which is the image of $F(z)$ under $S_{19,9}$, and let $\mathfrak{G}(z)$ be the image of $G(z)$ under $S_{23,11}$.

The first few terms of $\mathfrak{F}(z)$ and $\mathfrak{G}(z)$ are

$$\mathfrak{F}(z) = \sum_{n=1}^{\infty} A(n)q^n = q - 645150q^5 - 3974432q^7 - \ldots,$$

$$\mathfrak{G}(z) = \sum_{n=1}^{\infty} B(n)q^n = q + 23245050q^5 + 1322977768q^7 - \ldots$$

Since $A(1) = B(1) = 1$, for every prime $\ell \geq 5$ we have that $A(\ell)$ (resp. $B(\ell)$) is the eigenvalue of $\mathfrak{F}(z)$ (resp. $\mathfrak{G}(z)$) with respect to T_ℓ^9 (resp. T_ℓ^{11}). Therefore, by the commutativity of the Shimura correspondence (8) we have that $A(\ell) = \lambda_a(\ell)$ and $B(\ell) = \lambda_b(\ell)$ for every prime $\ell \geq 5$. Although Shimura's correspondence guarantees that $\mathfrak{F}(z) \in S_{18}(\Gamma_0(288), \chi_{triv})$ (resp. $\mathfrak{G}(z) \in S_{22}(\Gamma_0(288), \chi_{triv})$), it turns out that $\mathfrak{F}(z)$ is in $S_{18}(\Gamma_0(144), \chi_{triv})$ and that $\mathfrak{G}(z)$ is in $S_{22}(\Gamma_0(144), \chi_{triv})$ (see the Appendix).

If $\sigma_k(n)$ denotes the sum of the kth powers of the positive divisors of an integer n, then it suffices to prove that

$$\mathfrak{F}(z) \equiv \mathfrak{G}(z) \equiv \sum_{n=1}^{\infty} \left(\frac{60}{n}\right) \sigma_1(n) q^n \pmod{5}. \tag{9}$$

This follows from the fact that $A(n) = B(n) = 0$ if $\gcd(n, 6) \neq 1$. This fact is easily deduced from the definition of $S_{19,9}$ and $S_{23,9}$ and the fact that $a(n) = 0$ (resp. $b(n) = 0$) unless $n \equiv 19 \pmod{24}$ (resp. $n \equiv 23 \pmod{24}$).

Recall that the classical weight 6 Eisenstein series $E_6(z)$ on $SL_2(\mathbb{Z})$ is given by

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \equiv 1 + \sum_{n=1}^{\infty} \sigma_1(n) q^n \pmod{5}.$$

On the basis of Ramanujan's study of differential operators on modular forms, Swinnerton-Dyer [Lemma 5, 12] proved that if $\ell \geq 5$ is prime and $\sum_{n=0}^{\infty} c(n)q^n$ is a weight k modular form with integer coefficients, then there is a weight $k + l + 1$ modular form $\sum_{n=0}^{\infty} \alpha(n)q^n$ on $SL_2(\mathbb{Z})$ with integer coefficients whose Fourier expansion satisfies

$$\sum_{n=0}^{\infty} \alpha(n)q^n \equiv \sum_{n=0}^{\infty} nc(n)q^n \pmod{\ell}.$$

By applying this procedure twice to $E_6(z)$ with $\ell = 5$, we find that there is a weight 18 modular form $H_0(z) = \sum_{n=0}^{\infty} C(n)q^n$ with respect to $SL_2(\mathbb{Z})$ such that

$$H_0(z) = \sum_{n=0}^{\infty} C(n)q^n \equiv \sum_{n=1}^{\infty} n^2\sigma_1(n)q^n \equiv \sum_{n=1}^{\infty} \left(\frac{5}{n}\right)\sigma_1(n)q^n \pmod 5. \tag{10}$$

If $H_1(z)$ is the χ quadratic twist of $H_0(z)$, then we have

$$H_1(z) = \sum_{n=0}^{\infty} \chi(n)C(n)q^n \equiv \sum_{n=1}^{\infty} \left(\frac{60}{n}\right)\sigma_1(n)q^n \pmod 5.$$

By [III §3 Prop. 17, 5], $H_1(z)$ is in the space $M_{18}(\Gamma_0(144), \chi_{triv})$.

Therefore, congruence (9) for $\mathfrak{F}(z)$ is equivalent to the assertion that $H_1(z) \equiv \mathfrak{F}(z) \pmod 5$. By a theorem of Sturm [Th. 1, 11], it suffices to show that

$$A(n) \equiv \left(\frac{60}{n}\right)\sigma_1(n) \pmod 5$$

for every $n \leq 433$. A simple computation verifies the congruence (see Appendix for details on computing $\mathfrak{F}(z)$).

Congruence (9) for $\mathfrak{G}(z)$ can be handled similarly. Using the fact that the classical Eisenstein series

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n \equiv 1 \pmod 5,$$

it suffices to check that the weight 22 modular form $H_1(z)E_4(z)$ in $M_{22}(\Gamma_0(144), \chi_{triv})$ satisfies the congruence

$$H_1(z)E_4(z) \equiv \mathfrak{G}(z) \pmod 5.$$

Using Sturm's theorem again, this congruence is easily verified by checking that

$$B(n) \equiv \left(\frac{60}{n}\right)\sigma_1(n) \pmod 5$$

for every $n \leq 529$ (see Appendix for details on computing $\mathfrak{G}(z)$).

□

3. Proof of Theorem 1 and Corollaries 2 and 3

We begin by recalling the following classical fact (see [p. 111, 13]).

Theorem 3.1. *If $j \geq 1$, then the generating function for the numbers $p(5^j n + \beta_5(j))$ is of the form*

$$\sum_{n=0}^{\infty} p(5^j n + \beta_5(j))q^n = \begin{cases} \sum_{i\geq 1}\left(x_{j,i}q^{i-1}\prod_{n=1}^{\infty}\frac{(1-q^{5n})^{6i-1}}{(1-q^n)^{6i}}\right), & \text{if } j \text{ is odd,} \\ \sum_{i\geq 1}\left(x_{j,i}q^{i-1}\prod_{n=1}^{\infty}\frac{(1-q^{5n})^{6i}}{(1-q^n)^{6i+1}}\right), & \text{if } j \text{ is even,} \end{cases} \tag{11}$$

where

$$x_{j,i} = \begin{cases} 3^{j-1}5^j \pmod{5^{j+1}} & \text{if } i = 1, \\ 0 \pmod{5^{j+1}} & \text{if } i \geq 2. \end{cases}$$

The following corollary clarifies the importance of Lemma 2.1.

Corollary 3.2. *If $j \geq 1$, then for every non-negative integer n we have*

$$p(5^j n + \beta_5(j)) \equiv \begin{cases} 3^{j-1}5^j a(24n+19) \pmod{5^{j+1}} & \text{if } j \text{ is odd,} \\ 3^{j-1}5^j b(24n+23) \pmod{5^{j+1}} & \text{if } j \text{ is even.} \end{cases}$$

Proof. ¿From Theorem 3.1, if $j \geq 1$ is odd, then

$$\frac{1}{3^{j-1}5^j}\sum_{n=0}^{\infty} p(5^j n + \beta_5(j))q^{24n+19} \equiv q^{19}\prod_{n=1}^{\infty}\frac{(1-q^{120n})^5}{(1-q^{24n})^6} \equiv \sum_{n=0}^{\infty} a(n)q^n \pmod{5}.$$

Similarly, if $j \geq 2$ is even, then

$$\frac{1}{3^{j-1}5^j}\sum_{n=0}^{\infty} p(5^j n + \beta(5,j))q^{24n+23} \equiv q^{23}\prod_{n=1}^{\infty}\frac{(1-q^{120n})^6}{(1-q^{24n})^7} \equiv \sum_{n=0}^{\infty} b(n)q^n \pmod{5}.$$

The result follows immediately.

□

Proof of Theorem 1. Lemma 2.1 implies that

$$a(\ell^2 n) \equiv \left\{\left(\frac{15}{\ell}\right)\left(1+\ell-\ell^2\left(\frac{-n}{\ell}\right)\right)\right\} a(n) - \ell a(n/\ell^2) \pmod{5} \tag{12}$$

and

$$b(\ell^2 n) \equiv \left\{ \left(\frac{15}{\ell}\right) \left(1 + \ell - \left(\frac{-n}{\ell}\right)\right) \right\} b(n) - \ell b(n/\ell^2) \pmod{5}. \tag{13}$$

Congruence (12) follows from the simple observation that $\left(\frac{5}{\ell}\right) \equiv \ell^2 \pmod{5}$.

Replacing n by $24n+19$ and $24n+23$ in (12) and (13), respectively, and applying Corollary 3.2 immediately establishes the result.

□

We conclude with the proofs of Corollaries 2 and 3.

Proof of Corollary 2. Replace n by $N\ell^2 + s\ell + r$ in Theorem 1 and note that

$$\frac{5^j(\ell^2 N + s\ell + r)}{\ell^2} + \beta(j, \ell^{-1})$$

is not an integer. Therefore, since $\left(\frac{a}{\ell}\right) = 0$ if $\ell \mid a$ and $p(5^j n + \beta_5(j)) \equiv 0 \pmod{5^j}$, we have that

$$p(5^j \ell^4 N + 5^j \ell^3 s + 5^j \ell^2 r + \beta(j, \ell)) \equiv 0 \pmod{5^{j+1}}.$$

□

Proof of Corollary 3. Replace n by $\ell N + r$ and it is easy to see that

$$\frac{5^j(\ell N + r)}{\ell^2} + \beta(j, \ell^{-1})$$

cannot be an integer. Therefore, since $p(5^j n + \beta_5(j)) \equiv 0 \pmod{5^j}$ we have that

$$p(5^j \ell^3 N + 5^j \ell^2 r + \beta(j, \ell)) \equiv 0 \pmod{5^{j+1}}.$$

□

Appendix

Here we obtain a "closed formula" for the coefficients of the newform $\mathfrak{F}(z)$. Computing $\mathfrak{G}(z)$ is handled similarly. Let $\mathfrak{F}_0(z) = \sum_{n=1}^{\infty} A_0(n) q^n$ (resp. $\mathfrak{G}_0(z) = \sum_{n=1}^{\infty} B_0(n) q^n$) be the unique newform in the space $S_{18}(\Gamma_0(6), \chi_{triv})$ (resp. $S_{22}(\Gamma_0(6), \chi_{triv})$ whose Fourier expansion are

$$\mathfrak{F}_0(z) = q - 256q^2 - 6561q^3 + 65536q^4 + 645150q^5 + 1679616q^6 + \ldots$$
$$\mathfrak{G}_0(z) = q + 1024q^2 + 59049q^3 + 1048576q^4 - 23245050q^5 + 60466176q^6 - \ldots.$$

The newform $\mathfrak{F}(z)$ (resp. $\mathfrak{G}(z)$) is the χ quadratic twist of $\mathfrak{F}_0(z)$ (resp. $\mathfrak{G}_0(z)$). In particular, we have that

$$\mathfrak{F}(z) = \sum_{n=1}^{\infty} A(n)q^n = \sum_{\gcd(n,6)=1} \left(\frac{12}{n}\right) A_0(n)q^n, \tag{14}$$

$$\mathfrak{G}(z) = \sum_{n=1}^{\infty} B(n)q^n = \sum_{\gcd(n,6)=1} \left(\frac{12}{n}\right) B_0(n)q^n. \tag{15}$$

The proof of Theorem 2.2 requires the first 865 (resp. 1057) terms of $\mathfrak{F}(z)$.

Define rational numbers $D(n)$ by

$$\begin{aligned}\sum_{n=1}^{\infty} D(n)q^n = {} & \frac{-2132029}{4734}\eta^{29}(z)\eta^5(2z)\eta(3z)\eta(6z) - \frac{45171755}{4734}\eta^{25}(z)\eta(2z)\eta^5(3z)\eta^5(6z) \\ & - \frac{49149076}{7101}\eta^{24}(z)\eta^6(2z)\eta^6(6z) - \frac{14062152}{263}\eta^{20}(z)\eta^2(2z)\eta^4(3z)\eta^{10}(6z) \\ & + \frac{204636}{263}\eta^{18}(z)\eta^{18}(3z) + \left(\eta^{29}(z)\eta^5(2z)\eta(3z)\eta(6z) \mid U(2)\right) \\ & + \left(\frac{4539931}{113616}\eta^{24}(z)\eta^6(2z)\eta^6(6z) - \frac{25504}{2367}\eta^{25}(z)\eta(2z)\eta^5(3z)\eta^5(6z)\right) \mid U(2).\end{aligned} \tag{16}$$

As usual, the U-operator is defined by

$$\left(\sum_{n=0}^{\infty} b(n)q^n\right) \mid U(M) := \sum_{n=0}^{\infty} b(Mn)q^n.$$

It turns out that if n is coprime to 6, then

$$D(n) = A_0(n) = \chi(n)A(n). \tag{17}$$

One may use (14) and (17) to compute the first 865 coefficients of the newform $\mathfrak{F}(z)$.

Similarly, it is staightforward to obtain $\mathfrak{G}(z)$ in terms of eta-products and their images under certain Hecke operators. For brevity we omit the details.

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