ZETA FUNCTIONS OF AN INFINITE FAMILY OF $K3$ SURFACES

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Abstract. We identify a parameterized family of $K3$ surfaces with generic Picard number 19, and we employ elementary methods to determine their local zeta functions. In addition, we explicitly determine those surfaces which are modular.

1. Introduction

Given a projective variety defined over a number field, a central problem in number theory is to write down all of its local zeta functions. The case of elliptic curves over $\mathbb{Q}$ has recently been settled, and here we study the problem for $K3$ surfaces, a natural two-dimensional analog of elliptic curves. This is in general a difficult problem, since at any prime, the nontrivial factor of the local zeta function of a $K3$ surface is a polynomial of degree as high as 22.

In theory, one can compute the zeta function of varieties whose cohomology splits into 1-dimensional pieces. For instance, using p-adic Hodge theory, Livné [?] computed the zeta-function of a certain $K3$ surface which arose in the work of Peters, Top, and van der Vlugt [?]. Another important work in this area is due to Shioda and Inose. If a $K3$ surface has Picard number 20 (the largest it can be), they show in [?] that its L-function is essentially the symmetric square of the L-function of an elliptic curve with complex multiplication. Generally, we call a surface defined over $\mathbb{Q}$ *modular* if its L-function is, up to simple factors, the L-function of a weight 3 Hecke eigenform.

Although the Shioda and Inose result is general, the problem of computing explicit examples is nontrivial. In fact, only a few examples exist in the literature (for other examples see [?], [?], [?]), and they were obtained by a wide variety of methods, both geometric and transcendental. There are several natural difficulties. For instance, there is the problem of identifying suitable candidates for such surfaces. Then there is the issue of demonstrating that the L-function of an alleged modular surface agrees with the L-function of an explicit weight 3 modular form.

Work of Morrison [?] and van Geemen and Top [?] suggests that one can obtain similar arithmetic results for $K3$ surfaces with Picard number ≥ 19. In this vein we show, for a certain family of $K3$ surfaces, that the general problem of computing zeta functions can be quite simple, and involve little more than elementary character sums. To clarify the merit of such an approach, we note that our family includes, up to isogeny and quadratic twist, all but one of the modular examples alluded to above. We consider the one parameter family

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$\{X_\lambda\}$, for $\lambda \in \mathbb{Q} \setminus \{0, -1\}$, of $K3$ surfaces whose function fields are given by

$$X_\lambda \quad : \quad s^2 = xy(x+1)(y+1)(x+\lambda y) \tag{1}$$

in relation to the family of elliptic curves

$$E_\lambda \quad : \quad y^2 = (x-1)\left(x^2 - \tfrac{1}{\lambda+1}\right). \tag{2}$$

For convenience, if p is an odd prime, then let $\phi_p(x)$ denote the Legendre symbol $\left(\frac{x}{p}\right)$.

Theorem 1.1. *If $\lambda \in \mathbb{Q} \setminus \{0, -1\}$ and p is an odd prime at which E_λ has good reduction, then the local zeta function of X_λ at p is*

$$Z(X_\lambda/\mathbb{F}_p, T) = \frac{1}{(1-T)(1-p^2T)(1-pT)^{19}(1-\gamma pT)(1-\gamma\pi_{\lambda,p}^2 T)(1-\gamma\overline{\pi}_{\lambda,p}^2 T)},$$

where $\pi_{\lambda,p}$ and $\overline{\pi}_{\lambda,p}$ are the eigenvalues of the Frobenius at p on E_λ, and $\gamma = \phi_p(\lambda+1)$.

In view of Theorem **??**, it is easy to characterize those λ for which X_λ is modular.

Theorem 1.2. *If $\lambda \in \mathbb{Q} \setminus \{0, -1\}$, then the surface X_λ is modular if and only if*

$$\lambda \in \{1,\ 8,\ 1/8,\ -4,\ -1/4,\ -64,\ -1/64\}.$$

For completeness, we note that Beukers and Stienstra [**?**] show that X_{-1} is also a modular $K3$ surface. This follows easily from Theorem 1.2 since X_{-1} is a quadratic twist of both X_8 and $X_{1/8}$. Even though E_{-1} is singular, the modularity of X_{-1} can be proved using a minor modification of the arguments contained here. We prefer to omit them for brevity.

In §5 we shall show that all of the surfaces in Theorem **??** are related to weight 3 newforms that are products of Dedekind's eta-function. It is interesting to note that these forms constitute the complete list of weight 3 newforms which are eta-products. Proving Theorem **??** is tantamount to computing the Picard number of all of the X_λ. Finding the Picard number of a $K3$-surface is in general a difficult problem, and is often equivalent to calculating the rank of an elliptic curve over a function field. In our situation, the computation follows easily from a well known theorem of Ribet [**?**].

2. A theorem on character sums

In this section we prove a theorem about character sums which is essential for all of the results in this paper. Suppose that $\lambda \in \mathbb{Q} \setminus \{0, -1\}$ and that p is an odd prime such that $\lambda \not\equiv 0, -1 \pmod{p}$. If $q = p^r$, then let ϕ_q be the extended Legendre symbol on $\mathbb{F}_q$ (we will often write $\phi = \phi_q$ for simplicity). Further, define quantities $a(\lambda, q)$ and $A(\lambda, q)$ by

$$\begin{aligned} a(\lambda, q) &:= -\sum_{x \in \mathbb{F}_q} \phi_q\left((x-1)\left(x^2 - \tfrac{1}{\lambda+1}\right)\right), \\ A(\lambda, q) &:= \sum_{x,y \in \mathbb{F}_q} \phi_q(xy(x+1)(y+1)(x+\lambda y)). \end{aligned} \tag{3}$$

Theorem 2.1. *If $\lambda \in \mathbb{Q}$, and $\lambda \not\equiv 0, -1 \pmod{p}$, then $A(\lambda, q) = \phi_q(\lambda+1)(a(\lambda, q)^2 - q)$.*

Before starting the proof, we mention that results of this type have been obtained by Greene [**?**] in the context of finite field analogues of hypergeometric functions. We will denote by χ, ψ, and A multiplicative characters on $\mathbb{F}_q$, and by ϵ the trivial character on $\mathbb{F}_q$ (we agree that $\epsilon(0) = 0$). Given characters χ and ψ, we define the usual Jacobi sum $J(\chi, \psi) := \sum_{x \in \mathbb{F}_q} \chi(x)\psi(1-x)$. Throughout this section, all sums (unless otherwise noted) will run over all elements or over all characters of $\mathbb{F}_q$.

Throughout, we will repeatedly use the following facts, which are valid for any characters. For the first, see [**?**] (recall our convention that $\epsilon(0) = 0$). The second and third are easy consequences of orthogonality.

$$J(\chi, \psi) = \chi(-1)J(\chi, \overline{\chi\psi}). \tag{4}$$

$$\sum_{\chi} \chi(x) = \begin{cases} q-1 & \text{if } x = 1 \\ 0 & \text{else.} \end{cases} \tag{5}$$

$$\psi(1-x) = \tfrac{1}{q-1} \sum_{\chi} J(\psi, \overline{\chi})\chi(x) + \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{else.} \end{cases} \tag{6}$$

We begin with an easy lemma. If $\lambda \in \mathbb{Q}$ and $\lambda \not\equiv 0, -1 \pmod{p}$, then define

$$g(\lambda) := \tfrac{1}{q-1} \sum_{\chi} J(\overline{\chi}, \phi)J(\overline{\chi}, \phi\overline{\chi})\chi(\tfrac{\lambda}{4(\lambda+1)}).$$

Lemma 2.2. *If $\lambda \in \mathbb{Q}$, $\lambda \not\equiv 0, -1 \pmod{p}$, then $-\phi(2)a(q, \lambda) = g(\lambda) + \phi(\frac{-\lambda}{\lambda+1})$.*

Proof. Expanding the Jacobi sums and using (**??**) gives

$$g(\lambda) = \tfrac{1}{q-1} \sum_{x,y} \phi(1-x)\phi(1-y) \sum_{\chi} \overline{\chi}(\tfrac{4(\lambda+1)xy(1-y)}{\lambda}) = \sum_{y \neq 0,1} \phi(1-y)\phi(1 - \tfrac{\lambda}{4(\lambda+1)y(1-y)}).$$

Simplifying the last sum (and accounting for the term $y = 1$), we find that

$$g(\lambda) = \sum_{y} \phi(y)\phi\left(4y(1-y) - \tfrac{\lambda}{\lambda+1}\right) - \phi(\tfrac{-\lambda}{\lambda+1}).$$

The lemma follows after the change of variables $y \to \frac{1-y}{2}$. □

To prove Theorem 2.1, we must expand the quantity $g(\lambda)^2$. In the course of the computation, several error terms will arise. Since it is straightforward, we will delay evaluation of these terms (which will be denoted by E_1, E_2, etc.) until the end.

Expanding $g(\lambda)^2$ as a sum over χ and ψ and making the change $\psi \to \psi\overline{\chi}$, we find that

$$g(\lambda)^2 = \tfrac{1}{(q-1)^2} \sum_{\chi,\psi} \psi\left(\tfrac{\lambda}{4(\lambda+1)}\right) J(\overline{\chi}, \phi)J(\overline{\chi}, \phi\overline{\chi})J(\overline{\psi}\chi, \phi)J(\overline{\psi}\chi, \phi\overline{\psi}\chi).$$

Expanding the final Jacobi sum gives

$$g(\lambda)^2 = \tfrac{1}{(q-1)^2} \sum_{\psi,x} \psi(\tfrac{\lambda}{4(\lambda+1)})\overline{\psi}(x)\phi\overline{\psi}(1-x) \sum_{\chi} J(\overline{\chi}, \phi)J(\overline{\psi}\chi, \phi)J(\overline{\chi}, \phi\overline{\chi})\chi(x)\chi(1-x). \tag{7}$$

Notice that (replacing w with y/w in the second step) we have

$$J(\overline{\chi},\phi)J(\overline{\psi}\chi,\phi)=\sum_{w,y}\overline{\chi}(w)\phi(1-w)\overline{\psi}\chi(y)\phi(1-y)=\sum_{w,y}\phi\chi(w)\phi(w-y)\overline{\psi}(y)\phi(1-y).$$

Rearranging and using (**??**), this becomes

$$\begin{aligned}\sum_{\substack{w,y\\ y\neq 1}}\phi\chi(w)\phi(1-\tfrac{1-w}{1-y})\overline{\psi}(y)&=\tfrac{1}{q-1}\sum_{\substack{w,y\\ y\neq 1}}\phi\chi(w)\overline{\psi}(y)\sum_{A}J(\overline{A},\phi)A(\tfrac{1-w}{1-y})+\sum_{y\neq 1}\overline{\psi}(y)\\ &=\tfrac{1}{q-1}\sum_{A}J(\overline{A},\phi)J(\phi\chi,A)J(\overline{\psi},\overline{A})+\sum_{y\neq 1}\overline{\psi}(y).\end{aligned}$$

This together with (**??**) shows that $g(\lambda)^2$ is given by the expression

$$\tfrac{1}{(q-1)^3}\sum_{\psi,x}\psi(\tfrac{\lambda}{4(\lambda+1)})\overline{\psi}(x)\phi\overline{\psi}(1-x)\sum_{A}J(\overline{A},\phi)J(\overline{\psi},\overline{A})\sum_{\chi}J(\phi\chi,A)J(\overline{\chi},\phi\overline{\chi})\chi(x)\chi(1-x)+E_1, \tag{8}$$

where

$$E_1:=\tfrac{1}{(q-1)^2}\sum_{\psi,x}\psi(\tfrac{\lambda}{4(\lambda+1)})\overline{\psi}(x)\phi\overline{\psi}(1-x)\sum_{\chi}J(\overline{\chi},\phi\overline{\chi})\chi(x)\chi(1-x)\sum_{y\neq 1}\overline{\psi}(y). \tag{9}$$

By (**??**), we may assume that $x\neq 0,1$. Therefore the sum on χ in (**??**) can be written

$$\phi(\tfrac{x}{1-x})\sum_{\substack{w,y\\ w\neq 0,1}}A(1-y)\phi(w)\sum_{\chi}\phi\chi\left(\tfrac{x(1-x)y}{w(1-w)}\right)=(q-1)\phi(\tfrac{x}{1-x})\sum_{w\neq 1}A\left(1-\tfrac{w(1-w)}{x(1-x)}\right)\phi(w).$$

When $w=1$, the summand above equals $(q-1)\phi(\frac{x}{1-x})$. Accounting for this term, making the change of variables $w\to wx$ and factoring, the last displayed equation becomes

$$(q-1)\phi(1-x)\sum_{w}\phi(w)A(1-w)A(1-\tfrac{wx}{1-x})-(q-1)\phi(\tfrac{x}{1-x}).$$

Together with (**??**) this yields

$$\begin{aligned}&g(\lambda)^2=E_1+E_2+\\ &\quad\tfrac{1}{(q-1)^2}\sum_{\psi,x}\psi(\tfrac{\lambda}{4(\lambda+1)})\overline{\psi}(x)\overline{\psi}(1-x)\sum_{A}J(\overline{A},\phi)J(\overline{\psi},\overline{A})\sum_{w}\phi(w)A(1-w)A(1-\tfrac{wx}{1-x}),\end{aligned} \tag{10}$$

where

$$E_2:=\tfrac{-1}{(q-1)^2}\sum_{\psi,x}\psi(\tfrac{\lambda}{4(\lambda+1)})\overline{\psi}\phi(x)\overline{\psi}(1-x)\sum_{A}J(\overline{A},\phi)J(\overline{\psi},\overline{A}). \tag{11}$$

Using (**??**), the sum on A in (**??**) can be written

$$\begin{aligned}&\sum_{\substack{w,z\\ w\neq 1,z\neq 0}}\phi(1-z)\phi(w)\sum_{A}J(\overline{\psi},\overline{A})A(\tfrac{1}{z}(1-w)(1-\tfrac{wx}{1-x}))\\ &\quad=(q-1)\sum_{\substack{w,z\\ w\neq 1,z\neq 0}}\phi(1-z)\phi(w)\overline{\psi}\left(1-\tfrac{1}{z}(1-w)(1-\tfrac{wx}{1-x})\right)-(q-1)\sum_{z\neq 0}\phi(1-z)\phi(\tfrac{1-x}{x}).\end{aligned}$$

Replacing z by $(1-w)/z$ and simplifying, this becomes

$$(q-1)\sum_{\substack{w,z\\ w\neq 1}}\phi(z)\phi(z-1+w)\phi(w)\overline{\psi}\left(1-z(1-\tfrac{wx}{1-x})\right)+(q-1)\phi(\tfrac{1-x}{x})$$
$$=(q-1)\sum_{\substack{w,z\\ z\neq 1}}\phi(z)\phi(z-1+w)\phi(w)\overline{\psi}\left(1-z(1-\tfrac{wx}{1-x})\right)+F(\psi,x), \qquad (12)$$

where $F(\psi,x)$ contains the error arising from the terms $w=1$ and $z=1$; i.e.

$$F(\psi,x)=(q-1)\phi(\tfrac{1-x}{x})-(q-1)\sum_{z\neq 0}\overline{\psi}\left(1-z(1-\tfrac{x}{1-x})\right)+(q-1)\sum_{w}\overline{\psi}\left(\tfrac{wx}{1-x}\right). \qquad (13)$$

Making the change of variables $w\to(1-z)w$ and applying (**??**), the sum in (**??**) becomes

$$(q-1)\sum_{w,z}\phi(z)\phi(-1+w)\phi(w)\overline{\psi}(1-z)\overline{\psi}\left(1+\tfrac{wzx}{1-x})\right)+F(\psi,x)$$
$$=\sum_{w,z}\phi(z)\phi(-1+w)\phi(w)\overline{\psi}(1-z)\sum_{\chi}J(\overline{\chi},\overline{\psi})\chi(\tfrac{-wzx}{1-x})+F(\psi,x)$$
$$=\sum_{\chi}J(\overline{\chi},\phi)J(\overline{\psi},\overline{\chi})J(\overline{\psi},\phi\chi)\chi(\tfrac{x}{x-1})+F(\psi,x).$$

With (**??**) this yields

$$g(\lambda)^2=\tfrac{1}{(q-1)^2}\sum_{\psi,\chi}\psi(\tfrac{\lambda}{4(\lambda+1)})\chi(-1)J(\overline{\psi}\chi,\overline{\psi}\overline{\chi})J(\overline{\psi},\overline{\chi})J(\overline{\chi},\phi)J(\overline{\psi},\phi\chi)+E_1+E_2+E_3, \qquad (14)$$

where

$$E_3:=\tfrac{1}{(q-1)^2}\sum_{\psi,x}\psi(\tfrac{\lambda}{4(\lambda+1)})\overline{\psi}(x)\overline{\psi}(1-x)F(\psi,x). \qquad (15)$$

To simplify the main term in (**??**), we apply (**??**) to find that

$$\chi(-1)J(\overline{\psi}\chi,\overline{\psi}\overline{\chi})J(\overline{\psi},\overline{\chi})=J(\overline{\psi}\chi,\psi^2)J(\overline{\psi},\chi\psi)=\sum_{w,y}\overline{\psi}\chi(w)\psi^2(1-w)\overline{\psi}(1-y)\psi\chi(y).$$

Making the change $w\to\frac{w}{y}$ and simplifying, this becomes

$$\sum_{\substack{w,y\\ y\neq 0}}\overline{\psi}\chi(w)\psi^2(y-w)\overline{\psi}(1-y)=\sum_{\substack{w,y\\ w\neq 1}}\overline{\psi}\chi(w)\psi^2(y-w)\overline{\psi}(1-y)-\sum_{w}\psi\chi(w)+\sum_{y}\psi(1-y). \qquad (16)$$

For ease of notation, define a function δ by

$$\delta(\chi):=\begin{cases}0 & \text{if } \chi\neq\epsilon\\ 1 & \text{if } \chi=\epsilon.\end{cases}$$

Making the change $y\to 1-(1-w)y$ and simplifying, the expression in (**??**) becomes

$$\psi(-1)J(\overline{\psi}\chi,\psi)J(\overline{\psi},\overline{\psi})-(q-1)\delta(\psi\chi)+(q-1)\delta(\psi).$$

It is well-known (see[?]) that $\overline{\psi}(4)J(\overline{\psi},\overline{\psi}) = J(\overline{\psi},\phi) + (q-1)\delta(\psi)$. This, together with (??) and the last displayed equation, shows that

$$g(\lambda)^2 = \tfrac{1}{(q-1)^2}\sum_{\chi} J(\overline{\chi},\phi)\sum_{\psi}\psi(\tfrac{-\lambda}{\lambda+1})J(\overline{\psi},\phi)J(\overline{\psi}\chi,\psi)J(\overline{\psi},\phi\chi) + E_1 + E_2 + E_3 + E_4, \tag{17}$$

where

$$E_4 := \tfrac{-1}{q-1}\sum_{\psi}\psi(\tfrac{\lambda}{4(\lambda+1)})J(\psi,\phi)J(\overline{\psi},\phi\overline{\psi}) + \tfrac{1}{q-1}\sum_{\chi}J(\overline{\chi},\phi)J(\epsilon,\phi\chi) + \tfrac{1}{q-1}\sum_{\chi}J(\overline{\chi},\phi)J(\chi,\epsilon)J(\epsilon,\phi\chi). \tag{18}$$

The main term in (??) is

$$\tfrac{1}{(q-1)^2}\sum_{\chi} J(\overline{\chi},\phi)\sum_{\substack{w,y,z\\ wyz\neq 0}}\phi(1-w)\phi\chi(1-y)\chi(z)\sum_{\psi}\overline{\psi}\left(\tfrac{(\lambda+1)wyz}{-\lambda(1-z)}\right). \tag{19}$$

Notice that $\frac{z}{1-z} = \frac{-\lambda}{(\lambda+1)wy}$ if and only if $z = \frac{\frac{-\lambda}{\lambda+1}}{wy-\frac{\lambda}{\lambda+1}}$. Using this fact, making the changes $w \to \frac{w\lambda}{\lambda+1}$, $y \to \frac{y}{y-1}$, and then applying (??), the main term becomes

$$\begin{aligned}
&\tfrac{1}{(q-1)}\sum_{\chi}J(\overline{\chi},\phi)\sum_{\substack{w,y\\ wy\neq 0}}\phi(1-\tfrac{w\lambda}{\lambda+1})\phi\chi(1-y)\overline{\chi}(1-wy)\\
&\quad= \tfrac{1}{(q-1)}\sum_{\chi}J(\overline{\chi},\phi)\sum_{\substack{w,y\\ wy\neq 0}}\phi(1-\tfrac{w\lambda}{\lambda+1})\phi\overline{\chi}(1-y)\overline{\chi}(1-\tfrac{wy}{y-1})\\
&\quad= \tfrac{1}{(q-1)}\sum_{\substack{w,y\\ wy\neq 0}}\phi(1-\tfrac{w\lambda}{\lambda+1})\phi(1-y)\sum_{\chi}J(\overline{\chi},\phi)\chi(\tfrac{1}{1-y+wy})\\
&\quad= \sum_{\substack{w,y\\ w\neq 0}}\phi(1-\tfrac{w\lambda}{\lambda+1})\phi(1-y)\phi(1-y(1-w))\phi(-y)\phi(1-w)\\
&\quad= \sum_{w,y}\phi(1-\tfrac{w\lambda}{\lambda+1})\phi(1-y)\phi(1-y(1-w))\phi(-y)\phi(1-w) + \phi(-1).
\end{aligned}$$

Taking w to $1-\frac{w}{y}$ and then replacing w, y by $-w$, $-y$, this becomes

$$\phi(1+\lambda)\sum_{w,y}\phi(1+\tfrac{w\lambda}{y})\phi(1-y)\phi(1-w)\phi(-w) + \phi(-1) = \phi(1+\lambda)A(\lambda) + \phi(-1).$$

Together with (??), this yields

$$g(\lambda)^2 = \phi(1+\lambda)A(\lambda) + E_1 + E_2 + E_3 + E_4 + \phi(-1).$$

A calculation (see below) shows that

$$E_1 + E_2 + E_3 + E_4 = q - 1 - \phi(-1) - 2\phi(\tfrac{-\lambda}{\lambda+1})g(\lambda). \tag{20}$$

Together with Lemma 2.2, the last two formulae give the statement in Theorem 2.1; i.e.

$$a(\lambda,q)^2 = q + \phi(1+\lambda)A(\lambda).$$

To finish the proof, it is necessary to verify (**??**); we will only provide details for the computation of E_1 (which is given by (**??**)). We have

$$\sum_{\chi} J(\overline{\chi}, \phi\overline{\chi})\chi(x)\chi(1-x) = \sum_{w\neq 1}\phi(w)\sum_{\chi}\overline{\chi}\left(\tfrac{w(1-w)}{x(1-x)}\right).$$

The argument of χ equals one when $w = x$ or $w = 1 - x$. Taking into account the case when $x = \frac{1}{2}$, we find that

$$\sum_{\chi} J(\overline{\chi}, \phi\overline{\chi})\chi(x)\chi(1-x) = (q-1)\left[\phi(x) + \phi(1-x)\epsilon(x - \tfrac{1}{2})\right].$$

Returning to (**??**), we note that $\sum_{y\neq 1}\overline{\psi}(y) = -1 + (q-1)\delta(\psi)$. Therefore,

$$E_1 := \tfrac{-1}{q-1}\sum_{\psi}\psi(\tfrac{\lambda}{4(\lambda+1)})J(\phi\overline{\psi}, \phi\overline{\psi}) - \tfrac{1}{q-1}\sum_{\substack{\psi,x\\ x\neq\frac{1}{2}}}\psi(\tfrac{\lambda}{4(\lambda+1)})\overline{\psi}(x)\overline{\psi}(1-x) + \sum_{x\neq\frac{1}{2},0,1} 1 + \sum_{x}\phi(x)\phi(1-x)$$
$$= \tfrac{-1}{q-1}\sum_{\psi}\psi\left(\tfrac{\lambda}{4(\lambda+1)}\right)J(\phi\overline{\psi}, \phi\overline{\psi}) - \tfrac{1}{q-1}\sum_{\psi}\psi\left(\tfrac{\lambda}{4(\lambda+1)}\right)J(\overline{\psi}, \overline{\psi}) + q - 3 - \phi(-1).$$

In the second sum we have used the fact that the summand is zero when $x = \frac{1}{2}$.

For the remaining error terms (which are given by (**??**), (**??**), (**??**)), we use similar arguments to find that

$$E_2 = -\phi(\tfrac{-\lambda}{\lambda+1})g(\lambda),$$
$$E_3 = 1 + \tfrac{1}{q-1}\sum_{\psi}\psi\left(\tfrac{\lambda}{4(\lambda+1)}\right)J(\phi\overline{\psi}, \phi\overline{\psi}) + \tfrac{1}{q-1}\sum_{\psi}\psi\left(\tfrac{\lambda}{4(\lambda+1)}\right)J(\overline{\psi}, \overline{\psi}),$$
$$E_4 = 1 - \phi(\tfrac{-\lambda}{\lambda+1})g(\lambda).$$

Equation (**??**) follows, and the proof of Theorem 2.1 is complete.

3. The Geometry of these $K3$ surfaces

In this section we follow closely the exposition in [**?**]. Let K denote either $\mathbb{Q}$ or $\mathbb{F}_p$ (p an odd prime). For each $\lambda \in K \setminus \{0, -1\}$, we define the surface X_λ/K to be the smooth complete model of the double cover of $\mathbb{P}^2$ branched over the union U_λ of six lines given by

$$U_\lambda \quad : \quad XYZ(X + \lambda Y)(Y + Z)(Z + X) = 0.$$

To see that X_λ is a $K3$ surface, one can refer to [**?**], or check that the elliptic fibration of X_λ we give below satisfies the conditions given in [**?**], §4.

We now give a description of the natural map $\psi : X_\lambda \to \mathbb{P}^2$. Let B_λ be the surface obtained in the following way: first blow up $\mathbb{P}^2$ at the triple points of U_λ, then blow up the resulting surface at each of the double points of the total transform of U_λ. The total transform $T_\lambda \subset B_\lambda$ of U_λ consists of a connected union of rational curves whose only singularities are ordinary double points, and at each of these points a component of odd multiplicity and one of even multiplicity meet. The branch locus U_λ and the dual graph of T_λ are depicted here (• denotes a component of odd multiplicity, and ▪ denotes a component of even multiplicity):

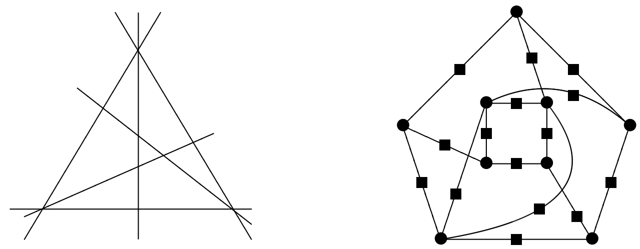

Figure 1

Then X_λ is the double cover of B_λ branched above the components of T_λ having odd multiplicity. Moreover, $\psi^{-1}(U_\lambda)$ is a union of rational curves whose dual graph is exactly that of T_λ, and $X_\lambda \setminus \psi^{-1}(U_\lambda)$ is a smooth affine surface given by the equation

$$s^2 = xy(x+1)(y+1)(x+\lambda y). \tag{21}$$

Now let us show that the X_λ are elliptic surfaces over $\mathbb{P}^1$. For a fixed λ, consider the family of cubics (with parameter τ) in $\mathbb{P}^2$ given by

$$X(Y+Z)(Z+X) - \tau(X+\lambda Y)YZ = 0. \tag{22}$$

One can check that the generic cubic in this family intersects U_λ at all of its triple points, and at the three double points $(1,0,-1)$, $(\lambda,-1,1)$ and $(\lambda,-1,-\lambda)$. At the double points, the cubic is transversal to both of the components of U_λ meeting there, and at the triple points is transversal to two of the components meeting there and tangential to the third. It follows that the proper transform of the generic cubic in B_λ intersects only components of T_λ of even multiplicity, and therefore the family (**??**) lifts through ψ to X_λ.

To see the elliptic fibration explicitly, the change of variables

$$X = \lambda t^2 VW, \quad Y = U^2 + \lambda t^2 UW, \quad Z = UV$$

(where $\tau = t^2$, reflecting the double cover) transforms (**??**) into

$$C_\lambda \quad : \quad V^2W + (1-t^2)UVW + \lambda t^2 VW^2 = U^3 + \lambda t^2 U^2 W. \tag{23}$$

Note that X_λ is a two to one base change of a rational semistable elliptic surface, where the ramification of the base change occurs at points where the rational surface has a bad fiber (compare to examples 4.4 and 4.7 in [**?**]). The discriminant of C_λ is

$$-\lambda^3 t^8 \left[t^6 + (8\lambda - 3)t^4 + (16\lambda^2 + 20\lambda + 3)t^2 - (\lambda+1)\right].$$

Using Tate's algorithm, we find that the singular fibers are generically of the following types:

$$I_{10}\ I_8\ I_1\ I_1\ I_1\ I_1\ I_1\ I_1. \tag{24}$$

The singular fibers above $t = \infty$ and $t = 0$ are visible in the dual graph of T_λ (see Figure 1) as the outer and inner cycles, respectively.

The Néron-Severi group of $X_\lambda/\overline{K}$ is the group of algebraic equivalence classes of divisors on X_λ. Shioda [**?**] showed that the rank r_λ of the Néron-Severi group of $X_\lambda/\overline{K}$ is given by

$$r_\lambda = \sum (m_j - 1) + 2 + \operatorname{rank}(C_\lambda(\overline{K}(t))),$$

where the sum is over the singular fibers, and m_j is the number of components of the singular fiber j. In particular, he showed that the equivalence classes of the generic fiber, the zero section and the components of the singular fibers not meeting the zero section are linearly independent elements of the Néron-Severi group modulo torsion. From (**??**), then, these give 18 linearly independent elements. Also note that $(0,0,1)$ is an element of $C_\lambda(\overline{K}(t))$ (indeed, of $C_\lambda(K(t))$), and one can check that it has infinite order. Therefore $r_\lambda \geq 19$.

4. Proof of Theorem 1.1

Given a prime p, the zeta function of a projective variety $Y/\mathbb{F}_p$ is

$$Z(Y/\mathbb{F}_p, T) = \exp\left(\sum_{n=1}^{\infty} \#Y(\mathbb{F}_{p^n}) \cdot T^n/n\right).$$

The following proposition is our first step in proving Theorem **??**.

Proposition 4.1. *If p is an odd prime, $\lambda \in \mathbb{F}_p \setminus \{0,-1\}$ and $q = p^r$, then*

$$\#X_\lambda(\mathbb{F}_q) = 1 + q^2 + 19q + \sum_{x,y \in \mathbb{F}_q} \phi_q(xy(x+1)(y+1)(x+\lambda y)).$$

Proof. Recall (see §3) that we have a map $\psi : X_\lambda \to \mathbb{P}^2$ ramified above U_λ, that $X_\lambda \setminus \psi^{-1}(U_\lambda)$ is a smooth affine surface given by (**??**) and that $\psi^{-1}(U_\lambda)$ is a union of rational curves whose dual graph is given in Figure 1. Since all the components of U_λ are lines defined over $\mathbb{F}_p$, we can read from Figure 1 that the number of $\mathbb{F}_q$-rational points on $\psi^{-1}(U_\lambda)$ is

$$9(q+1) + 15(q-1) = 24q - 6.$$

Moreover, the number of $\mathbb{F}_q$-rational points of $\mathbb{A}^3$ satisfying (**??**) with $s = 0$ is $5q - 7$. It follows that

$$\#X_\lambda(\mathbb{F}_q) = \sum_{x,y \in \mathbb{F}_q} (\phi_q(xy(x+1)(y+1)(x+\lambda y)) + 1) + (24q-6) - (5q-7).$$

□

Now let us conclude the proof of Theorem **??**. The Weil conjectures tell us that $Z(X_\lambda/\mathbb{F}_p, T) \in \mathbb{Q}(T)$. Indeed, since X_λ is a $K3$ surface, its zeta function has the form

$$Z(X_\lambda/\mathbb{F}_p, T) = \frac{1}{(1-T)(1-p^2T)P_{\lambda,p}(T)},$$

with

$$P_{\lambda,p}(T) = \prod_{i=1}^{22}(1-\alpha_i T), \quad \alpha_i \in \mathbb{C}, \quad |\alpha_i| = p.$$

Notice that with this notation, if $q = p^r$, then

$$\#X_\lambda(\mathbb{F}_q) = \frac{1}{(r-1)!}\frac{d^r}{dT^r}\log(Z(X_\lambda/\mathbb{F}_p, T))|_{T=0} = 1 + q^2 + \sum_{i=1}^{22} \alpha_i^r. \tag{25}$$

Recall that if E_λ ($\lambda \in \mathbb{Q}$) has good reduction at p, then the zeta function of E_λ at p is

$$Z(E_\lambda/\mathbb{F}_p, T) = \frac{1 - a(\lambda,p)T + pT^2}{(1-T)(1-pT)} = \frac{(1-\pi_{\lambda,p}T)(1-\overline{\pi}_{\lambda,p}T)}{(1-T)(1-pT)}. \tag{26}$$

We can now prove

Theorem 4.2. *If $\lambda \in \mathbb{Q} \setminus \{0, -1\}$, E_λ has good reduction at a prime p and $q = p^r$, then*

$$\#X_\lambda(\mathbb{F}_q) = 1 + q^2 + 19q + \phi_p(\lambda+1)^r \left(\pi_{\lambda,p}^{2r} + \overline{\pi}_{\lambda,p}^{2r} + q\right). \tag{27}$$

Proof. From (**??**) we have that

$$a(\lambda, q) = (q+1) - \#E(\mathbb{F}_q) = \pi_{\lambda,p}^r + \overline{\pi}_{\lambda,p}^r.$$

Since $\pi_{\lambda,p}\overline{\pi}_{\lambda,p} = p$, it follows that

$$a(\lambda, q)^2 - q = (\pi_{\lambda,p}^r + \overline{\pi}_{\lambda,p}^r)^2 - q = \pi_{\lambda,p}^{2r} + \overline{\pi}_{\lambda,p}^{2r} + q. \tag{28}$$

Therefore, since $\phi_q(\lambda+1) = \phi_p(\lambda+1)^r$, Theorem **??**, Proposition 4.1 and (**??**) imply our result. □

Now Theorem **??** follows from our two expressions for $\#X_\lambda(\mathbb{F}_q)$ (see (**??**) and (**??**)).

Remark 4.3. As in [**?**], §12, one can use the Néron-Severi group of $X_\lambda/\overline{\mathbb{F}}_p$ to find a large factor of $P_{\lambda,p}(T)$. Recall from §3 that on the elliptic surface X_λ, the generic fiber, zero section and the components of the singular fibers not meeting the zero section are linearly independent elements of the Néron-Severi group modulo torsion. By (**??**) these comprise 18 linearly independent elements, all of which are fixed by the action of Frobenius. Furthermore, we saw that $C_\lambda(K(t))$ has an element of infinite order. This shows that $(1-pT)^{19}$ is a factor of $P_{\lambda,p}(T)$. Given this, one can prove Theorem **??** using only the $r = 1$ and $r = 2$ cases of Theorem **??**.

Remark 4.4. The form of the local zeta functions of X_λ given by Theorem **??** suggests a close relationship between X_λ and the Kummer surface K_λ associated to $E_\lambda \times E_\lambda$. Indeed, the local zeta function of K_λ at p has sixteen factors of the form $(1 \pm pT)$ coming from the lines on K_λ lying above the 2-division points of $E_\lambda \times E_\lambda$; combining these with the remaining factor coming from $E_\lambda \times E_\lambda$ gives an expression very much parallel to that in Theorem **??**. Another approach to proving this result would be to seek an explicit finite rational map between X_λ and K_λ (see [**?**]).

5. Proof of Theorem 1.2

Recall that X_λ is modular if there is a weight 3 cusp form $f_\lambda(z) = \sum_{n=1}^\infty a_\lambda(n)q^n$ ($q := e^{2\pi i z}$ throughout) which is an eigenform of the Hecke operators and has the property that

$$L(X_\lambda, s) := \prod_p Z^*(X_\lambda/\mathbb{F}_p, p^{-s}) = \sum_{n=1}^\infty \frac{a_\lambda(n)}{n^s}. \tag{29}$$

Here the product is over the primes p at which E_λ has good reduction, and $Z^*(X_\lambda/\mathbb{F}_p, T)$ denotes the product of the two nontrivial factors of $Z(X_\lambda/\mathbb{F}_p, T)$. In view of Theorem 1.1 and our formulation above, care must be taken in defining $Z^*(X_\lambda/\mathbb{F}_p, T)$ for the primes p at which E_λ has supersingular reduction.

If $\eta(z)$ is Dedekind's eta-function

$$\eta(z) := q^{1/24} \prod_{n=1}^\infty (1-q^n), \tag{30}$$

then the four weight 3 newforms we require are:

$$A(z) := \eta^6(4z) = q - 6q^5 + 9q^9 + \cdots \in S_3(\Gamma_0(16), \chi_{-1}), \tag{31}$$
$$B(z) := \eta^2(z)\eta(2z)\eta(4z)\eta^2(8z) = q - 2q^2 - 2q^3 + \cdots \in S_3(\Gamma_0(8), \chi_{-2}), \tag{32}$$
$$C(z) := \eta^3(2z)\eta^3(6z) = q - 3q^3 + 2q^7 + 9q^9 - \cdots \in S_3(\Gamma_0(12), \chi_{-3}), \tag{33}$$
$$D(z) := \eta^3(z)\eta^3(7z) = q - 3q^2 + 5q^4 - 7q^7 - \cdots \in S_3(\Gamma_0(7), \chi_{-7}). \tag{34}$$

These forms constitute the complete list of weight 3 newforms that are eta-products (e.g. see [?]).

We begin by considering the "if" direction of Theorem 1.2 (i.e. that X_λ is modular for the seven exceptional values of λ). If $\lambda \in \mathbb{Q} \setminus \{0, -1\}$, then the j-invariant of E_λ is

$$j(E_\lambda) = \frac{64(\lambda+4)^3}{\lambda^2}. \tag{35}$$

Using the fact that an elliptic curve $E/\mathbb{Q}$ has complex multiplication if and only if

$$j(E) \in \{12^3, 66^3, 20^3, 0, 2\cdot 30^3, -3\cdot 160^3, -15^3, 255^3, -32^3, -96^3, -960^3, -5280^3, -640320^3\},$$

(i.e. these correspond to the CM orders with class number 1 [?]) (??) implies that E_λ has complex multiplication if and only if λ is one of the seven exceptional values. Arguing as in [?], one can show that $f_1 = B \otimes \chi_{-4}$, $f_8 = A$, $f_{1/8} = A \otimes \chi_8$, $f_{-4} = C$, $f_{-1/4} = C \otimes \chi_{-4}$, $f_{-64} = D$, and $f_{-1/64} = D \otimes \chi_{-4}$. Here $f \otimes \chi$ denotes the χ-twist of the modular form f, and χ_D denotes the Kronecker character for the quadratic field $\mathbb{Q}(\sqrt{D})$.

Now we prove the "only if" direction in Theorem 1.2. If $\lambda \in \mathbb{Q} \setminus \{0, -1\}$ is not one of these exceptional values, then it suffices to show that $L(X_\lambda, s)$ cannot be a modular L-function. For these λ, it is well known that the set of primes p at which E_λ has supersingular reduction has density zero. Moreover, if p is a prime of good reduction which is not supersingular for E_λ, then the local Euler factor at p for $L(X_\lambda, s)$ is

$$\frac{1}{1 - \phi_p(\lambda+1)\,(a(\lambda,p)^2 - 2p)\,p^{-s} + p^{2-2s}}. \tag{36}$$

Therefore, for such p the coefficient $a_\lambda(p)$ in $L(X_\lambda, s)$ is $\phi_p(\lambda+1)(a(\lambda,p)^2 - 2p)$. If X_λ is modular, then for every prime ℓ there is a Galois representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell)$$

with the property that

$$\mathrm{Tr}(\rho_\ell(\mathrm{Frob}(p)) \equiv \phi_p(\lambda+1)(a(\lambda,p)^2 - 2p) \pmod{\ell} \tag{37}$$

for a set of primes with density one. This contradicts a well known theorem due to Ribet which implies that all but finitely many such ρ_ℓ are large for eigenforms without complex multiplication (for example see [?], Lemma, p. 459).

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