

THE ARITHMETIC OF THE COEFFICIENTS OF HALF INTEGRAL WEIGHT EISENSTEIN SERIES

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1. INTRODUCTION AND STATEMENT OF RESULTS

If $H(-n)$ denotes the Hurwitz-Kronecker class number of positive definite binary quadratic forms with discriminant $-n$, then Zagier's weight $3/2$ Eisenstein series is given by ($q = e^{2\pi iz}$ throughout)

$$(1.1) \quad H_{3/2}(z) = -\frac{1}{12} + \sum_{n=1}^{\infty} H(1, n)q^n = -\frac{1}{12} + \sum_{0 < n \equiv 0, 3 \pmod{4}} H(-n)q^n.$$

This series fits into Cohen's general theory of weight $k + \frac{1}{2}$ Eisenstein series

$$(1.2) \quad H_{k+\frac{1}{2}}(z) = \sum_{n=0}^{\infty} H(k, n)q^n,$$

where $H(k, n) = H_1(k, n)$ is defined in (??). If $k \geq 2$, then $H_{k+\frac{1}{2}}(z)$ is a weight $k + \frac{1}{2}$ modular form on the congruence subgroup $\Gamma_0(4)$ (see Th. 3.1, [?]).

If $D \equiv 0, 1 \pmod{4}$, and $\chi_D(\bullet) = \left(\frac{D}{\bullet}\right)$ denotes the usual Kronecker character, then it is a classical fact ([?], Ch. 5, §4) that

$$H(1, -D) = H(D) = \sum_{a=0}^{|D|-1} \chi_D(a)a,$$

when $D < -4$ is a fundamental discriminant.

We generalize this classical result in a variety of ways. See, for example, Corollary ???. Moreover, we obtain an elegant uniform description of the $H(k, n)$ in terms of values of the Bernoulli polynomials $B_k(x)$, weighted against the mod n representation numbers of sums of squares.

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To state our result, if χ is a Dirichlet character, we let $L(s, \chi) = \sum \chi(n)n^{-s}$ its L -series. For positive integers r, n and k with r odd, let $D := (-1)^k n$ and set

$$(1.3) \quad h_r(k, n) = \begin{cases} \frac{(-1)^{\lfloor k/2 \rfloor} \chi_8(r) (k-1)! n^{k-\frac{r}{2}} L(k, \chi_D)}{2^{k-\frac{r+1}{2}} \pi^k} & \text{if } (-1)^k n \equiv 0 \pmod{4}, \\ \frac{(-1)^{\lfloor k/2 \rfloor} (k-1)! n^{k-\frac{r}{2}} L(k, \chi_D)}{2^{k-1} \pi^k} & \text{if } (-1)^k n \equiv 1 \pmod{4}, \\ 0 & \text{if } (-1)^k n \equiv 2, 3 \pmod{4}. \end{cases}$$

Define also

$$(1.4) \quad H_r(k, n) = \begin{cases} \sum_{d^2|n} h_r(k, n/d^2) & \text{if } (-1)^k n \equiv 0, 1 \pmod{4}, \\ \zeta(1-2k) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $h_r(k, n) \ll n^{k-\frac{r}{2}} \log(2n)$ (where the $\log(2n)$ is necessary only for $k = 1$). In particular,

$$(1.5) \quad H_r(k, n) \ll n^{\max(k-\frac{r}{2}, 0)+\varepsilon}.$$

Next, we define, for positive integers r and n ,

$$(1.6) \quad R_r(a, n) := \#\{(\nu_1, \nu_2, \dots, \nu_r) \in (\mathbb{Z}/n\mathbb{Z})^r : \nu_1^2 + \nu_2^2 + \dots + \nu_r^2 \equiv a \pmod{n}\}.$$

If k is a positive integer, then let B_k be the k th Bernoulli number and let $B_k(x)$ denote the usual k th Bernoulli function with the convention that

$$(1.7) \quad B_1(x) = \begin{cases} x - \frac{1}{2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Finally, define the two formal Dirichlet series $H_{r,k}(s)$ and $F_{r,k}(s)$ by

$$(1.8) \quad H_{r,k}(s) = \sum_{n=1}^{\infty} \frac{H_r(k, n)}{n^s}$$

and

$$(1.9) \quad F_{r,k}(s) = -\frac{\chi_{(-1)^k}(r)}{k} \cdot \sum_{n=1}^{\infty} \frac{\sum_{a=0}^{n-1} R_r(a, n) B_k(a/n)}{n^s}.$$

By (??), $H_{r,k}(s)$ converges for $\operatorname{Re}(s) > \max(1, k+1-\frac{r}{2})$. For $F_{r,k}(s)$, we note that $B_k(x)$ is bounded in the interval $[0, 1]$ and $\sum R_r(a, n) = n^r$. This implies that $F_{r,k}(s)$ converges for $\operatorname{Re}(s) > r+1$. As a consequence of the following theorem, we can extend the region of convergence for $F_{r,k}(s)$ to $\operatorname{Re}(s) > \max(\frac{r}{2}+1, r-k+1)$.

By modifying an argument of J. Bruinier (see Th. 11 [?]) in the case where $k = 1$, for positive odd integers r , we obtain the following description of $H_{r,k}(s)$ as a formal

product of $F_{r,k}(s)$ and a simple quotient of the Riemann zeta function $\zeta(s)$. In particular, if $r = 1$, we obtain a convenient description of the Mellin transform of Cohen's half integral weight Eisenstein series.

Theorem 1.1. *If r is a positive odd integer and $k \geq 1$, then*

$$H_{r,k}(s) = \frac{\zeta(2s)}{\zeta(s)} \cdot F_{r,k}(s - k + r).$$

As is well-known, $\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \lambda(n)n^{-s}$, where λ is the Liouville function, the totally multiplicative function on positive integers defined on primes p by $\lambda(p) = -1$. Then Theorem ?? gives an immediate formula for $H_r(k, n)$.

Corollary 1.2. *For $r, n, k \in \mathbb{Z}$ with r odd,*

$$H_r(k, n) = -\frac{\chi_{(-1)^k}(r)}{k} \sum_{d|n} \lambda(n/d) d^{k-r} \sum_{a=0}^{d-1} R_r(a, d) B_k(a/d).$$

We cannot resist interpreting the $r = 1$ case of Theorem ?? in terms of the *logarithmic derivatives* of the infinite products

$$(1.10) \quad F_k(q) = \prod_{n=1}^{\infty} (1 - q^n)^{H(k,n)/n}.$$

Corollary 1.3. *If k is a positive integer, then*

$$\frac{q \frac{d}{dq} (F_k(q))}{F_k(q)} + \sum_{\substack{a, b \geq 1 \\ c \geq 2 \\ b \text{ square-free}}} H(k, a) q^{abc^2} = \frac{1}{k} \sum_{n=1}^{\infty} \sum_{a=0}^{n-1} R_1(a, n) B_k(a/n) n^{k-1} q^n.$$

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2. PROOFS

We begin with the following elementary lemmas. The first lemma is standard and can be found in ([?], p. 522).

Lemma 2.1. *The Fourier expansion of $B_k(x)$ is given by:*

$$(2.1) \quad B_k(x) = \frac{-k!}{(2\pi i)^k} \sum'_{n \in \mathbb{Z}} \frac{e(nx)}{n^k}.$$

where \sum' indicates that we are summing over all non-zero integers, and $e(x) = e^{2\pi i x}$.

The proof of the following lemma is a straightforward analogue of the proof of Möbius inversion and will be left to the reader.

Lemma 2.2. *Let $f : \mathbb{Z}^+ \rightarrow \mathbb{C}$. Define $F, G : \mathbb{Z}^+ \rightarrow \mathbb{C}$ by*

$$(2.2) \quad \begin{aligned} F(N) &= \sum_{d|N} f\left(\frac{N}{d}\right) \quad \text{and} \\ G(N) &= \sum_{d^2|N} f\left(\frac{N}{d^2}\right). \end{aligned}$$

Then

$$(2.3) \quad \begin{aligned} (i) \quad F(N) &= \sum_{\substack{a|N \\ N/a \text{ square-free}}} G(a) \quad \text{and} \\ (ii) \quad G(N) &= \sum_{a|N} \lambda\left(\frac{N}{a}\right) F(a). \end{aligned}$$

Fixing notation, we define, for integers m and a ,

$$(2.4) \quad G(m, a) = \sum_{\nu \in \mathbb{Z}/a\mathbb{Z}} e(m\nu^2).$$

In the following lemma, we recall some basic facts of the Gauss sums in ([?]). For their proofs, we refer the reader to ([?], IV.3).

Lemma 2.3. *Suppose that a, b and c are integers with $b, c \geq 1$.*

- (i) *If $c \mid (a, b)$, then $G(a, b) = c \cdot G\left(\frac{a}{c}, \frac{b}{c}\right)$*
- (ii) *If $(b, c) = 1$ and $(a, bc) = 1$, then $G(a, bc) = G(ab, c)G(ac, b)$*
- (iii) *If b is odd and $(a, b) = 1$, then $G(a, b) = \left(\frac{a}{b}\right)G(1, b)$*
- (iv) *If a is odd, then $G(a, 2^s) = \left(\frac{-2^s}{a}\right) \varepsilon((a-1)/2)G(1, 2^s)$*

$$(v) \quad G(1, b) = \begin{cases} (1+i)\sqrt{b} & \text{if } b \equiv 0 \pmod{4} \\ \sqrt{b} & \text{if } b \equiv 1 \pmod{4} \\ 0 & \text{if } b \equiv 2 \pmod{4} \\ i\sqrt{b} & \text{if } b \equiv 3 \pmod{4} \end{cases}$$

Here, $\varepsilon(b) = 1$ if b is even and $\varepsilon(b) = i$ if b is odd.

For positive integers k and r , let

$$(2.5) \quad \mathfrak{g}_{k,r}(m, a) = \begin{cases} \operatorname{Re}(G(m, a)^r) & \text{if } k \in 2\mathbb{Z} \\ \operatorname{Im}(G(m, a)^r) & \text{if } k \in 2\mathbb{Z} + 1. \end{cases}$$

Then we see immediately from the definition and (i) above that, for $b \mid (m, a)$,

$$(2.6) \quad \mathfrak{g}_{k,r}(m, a) = b^r \mathfrak{g}_{k,r}\left(\frac{m}{b}, \frac{a}{b}\right).$$

Then we have the following proposition.

Proposition 2.4. *For integers m and a such that $(m, a) = 1$,*

$$(2.7) \quad \mathfrak{g}_{k,r}(m, a) = a^{\frac{r}{2}} \left(\frac{(-1)^k a}{m} \right) \cdot \begin{cases} 2^{\frac{r-1}{2}} \chi_{(-1)^k 8}(r) & \text{if } (-1)^k a \equiv 0 \pmod{4} \\ \chi_{(-1)^k}(r) & \text{if } (-1)^k a \equiv 1 \pmod{4} \\ 0 & \text{if } (-1)^k a \equiv 2, 3 \pmod{4}. \end{cases}$$

Proof. If $(-1)^k a \equiv 1 \pmod{4}$, then $G(m, a)^r = \left(\frac{m}{a}\right)^r \varepsilon(k)^r a^{r/2}$. Therefore,

$$(2.8) \quad \mathfrak{g}_{k,r}(m, a) = a^{\frac{r}{2}} \left(\frac{m}{a}\right) \varepsilon(k)^{r-1} = a^{\frac{r}{2}} \left(\frac{m}{a}\right) \chi_{(-1)^k}(r).$$

Clearly, by Lemma ??, if $(-1)^k a \equiv 2 \pmod{4}$, $\mathfrak{g}_{k,r}(m, a) = 0$. If $(-1)^k a \equiv 3 \pmod{4}$, then $G(m, a)^r = \left(\frac{m}{a}\right)^r \varepsilon(k-1)^r a^{r/2}$ and $\mathfrak{g}_{k,r}(m, a) = 0$.

Therefore, the only case left is $(-1)^k a \equiv 0 \pmod{4}$. Suppose $a = 2^s a_0$ with a_0 odd; then, by Lemma ??, we see that

$$(2.9) \quad G(m, a) = \left(\frac{-m}{a_0}\right) \left(\frac{-2^s}{m}\right) \varepsilon\left(\frac{ma_0 - 1}{2}\right) G(1, 2^s) G(1, a_0).$$

Therefore, if $a_0 \equiv 1 \pmod{4}$, then

$$(2.10) \quad G(m, a) = \left(\frac{-2^s a_0}{m}\right) \varepsilon\left(\frac{m-1}{2}\right) (1+i) 2^{s/2} \sqrt{a_0} = \left(\frac{-a}{m}\right) \varepsilon\left(\frac{m-1}{2}\right) \sqrt{a} (1+i),$$

and if $a_0 \equiv 3 \pmod{4}$, then

$$(2.11) \quad \begin{aligned} G(m, a) &= -\left(\frac{-a_0}{m}\right) \left(\frac{-2^s}{m}\right) \varepsilon\left(\frac{m+1}{2}\right) i \sqrt{a_0} (1+i) 2^{s/2} \\ &= -\left(\frac{a}{m}\right) \varepsilon\left(\frac{m+1}{2}\right) i \sqrt{a} (1+i) \\ &= \left(\frac{-a}{m}\right) \varepsilon\left(\frac{m-1}{2}\right) \sqrt{a} (1+i). \end{aligned}$$

Then we see that

$$(2.12) \quad G(m, a)^r = 2^{(r-1)/2} a^{r/2} \left(\frac{-a}{m}\right)^r \varepsilon\left(\frac{m-1}{2}\right)^r i^{(r-1)/2}.$$

After a lengthy verification, we obtain

$$(2.13) \quad \mathfrak{g}_{k,r}(m, a) = \left(\frac{(-1)^k a}{m}\right) 2^{\frac{r-1}{2}} a^{\frac{r}{2}} \chi_{(-1)^k 8}(r),$$

as desired. □

The following proposition provides the foundation for Theorem ??.

Proposition 2.5. *For $r, k, n \in \mathbb{Z}^+$ with r odd,*

$$\sum_{a=0}^{n-1} R_r(a, n) B_k(a/n) = -\frac{\chi_{(-1)^k}(r) k}{n^{k-r}} \sum_{d|n} h_r(k, d).$$

Proof. Let N be a positive integer; then by Lemma ??,

$$\begin{aligned}
(2.14) \quad \sum_{a=0}^{N-1} R_r(a, N) B_k(a/N) &= \sum_{\nu_1, \dots, \nu_r \pmod{N}} B_k \left(\frac{\nu_1^2}{N} + \dots + \frac{\nu_r^2}{N} \right) \\
&= \frac{-k!}{(2\pi i)^k} \sum'_{n \in \mathbb{Z}} n^{-k} \sum_{\nu_1, \dots, \nu_r \pmod{N}} e \left(n \left(\frac{\nu_1^2}{N} + \dots + \frac{\nu_r^2}{N} \right) \right) \\
&= \frac{-k!}{(2\pi i)^k} \sum'_{n \in \mathbb{Z}} n^{-k} G(n, N)^r \\
&= \frac{-k!}{(2\pi i)^k} \sum'_{n \in \mathbb{Z}} n^{-k} (\operatorname{Re}(G(n, N)^r) + i \operatorname{Im}(G(n, N)^r)) \\
&= \frac{-2\varepsilon(k)k!}{(2\pi i)^k} \sum_{n \geq 1} n^{-k} \mathfrak{g}_{k,r}(n, N).
\end{aligned}$$

Reindexing this sum on the divisors of N , we find

$$\begin{aligned}
(2.15) \quad \sum_{a=0}^{N-1} R_r(a, N) B_k(a/N) &= \frac{-2\varepsilon(k)k!}{(2\pi i)^k} \sum_{a|N} \sum_{\substack{n \geq 1 \\ (n, N) = \frac{N}{a}}} n^{-k} \mathfrak{g}_{k,r}(n, N) \\
&= \frac{-2\varepsilon(k)k!}{(2\pi i)^k} N^{r-k} \sum_{a|N} a^{k-r} \sum_{\substack{m \geq 1 \\ (m, a) = 1}} m^{-k} \mathfrak{g}_{k,r}(m, a).
\end{aligned}$$

Since $\chi_a(m) = 0$ if $(m, a) \neq 1$, Proposition ?? yields

$$\begin{aligned}
(2.16) \quad a^{k-r} \sum_{\substack{m \geq 1 \\ (m, a) = 1}} m^{-k} \mathfrak{g}_{k,r}(m, a) &= 2^{\frac{r-1}{2}} \chi_{(-1)^k 8}(r) a^{k-r/2} \sum_{m \geq 1} m^{-k} \chi_{(-1)^k a}(m) \\
&= 2^{\frac{r-1}{2}} \chi_{(-1)^k 8}(r) a^{k-r/2} L(k, \chi_{(-1)^k a})
\end{aligned}$$

if $(-1)^k a \equiv 0 \pmod{4}$, and

$$\begin{aligned}
(2.17) \quad a^{k-r} \sum_{\substack{m \geq 1 \\ (m, a) = 1}} m^{-k} \mathfrak{g}_{k,r}(m, a) &= \chi_{(-1)^k}(r) a^{k-r/2} \sum_{m \geq 1} m^{-k} \chi_{(-1)^k a}(m) \\
&= \chi_{(-1)^k}(r) a^{k-r/2} L(k, \chi_{(-1)^k a})
\end{aligned}$$

if $(-1)^k a \equiv 1 \pmod{4}$. If $(-1)^k a \equiv 2, 3 \pmod{4}$, the sum is zero. (??) yields

$$(2.18) \quad \sum_{a=0}^{N-1} R_r(a, N) B_k(a/N) = -\frac{\chi_{(-1)^k}(r)k}{N^{k-r}} \sum_{d|N} h_r(k, d)$$

as desired. \square

We are now prepared to prove the main theorem.

Proof of Theorem ??. Combining Proposition ?? with Lemma ??(ii) yields

$$(2.19) \quad H_r(k, n) = -\frac{\chi_{(-1)^k}(r)}{k} \sum_{d|n} \lambda(n/d) d^{k-r} \sum_{a=0}^{d-1} R_r(a, d) B_k(a/d).$$

A straightforward calculation involving formal Dirichlet series verifies that, for any series $\sum_{n \geq 1} b(n) n^{-s}$,

$$\frac{\zeta(2s)}{\zeta(s)} \sum_{n \geq 1} b(n) n^{-s} = \sum_{n \geq 1} \left(\sum_{d|n} \lambda(n/d) b(d) \right) n^{-s}.$$

Applying this to (??), we obtain the theorem. □

Proof of Corollary ??. This is (??) in the proof of Theorem ?? □

Proof of Corollary ??. If we define a formal product

$$(2.20) \quad F(q) = \prod_{n=1}^{\infty} (1 - q^n)^{c(n)},$$

then one obtains formally that

$$(2.21) \quad \frac{q \frac{d}{dq} (F(q))}{F(q)} = - \sum_{n=1}^{\infty} \left(\sum_{d|n} c(d) d \right) q^n.$$

Combining Proposition ?? with Lemma ??(i), we get

$$(2.22) \quad -\frac{\chi_{(-1)^k}(r)}{n^{r-k} k} \sum_{a=0}^{n-1} R_r(a, n) B_k(a/n) = \sum_{\substack{d|n \\ n/d \text{ square-free}}} H_r(k, d).$$

The result follows from this and (??) when $r = 1$. □

3. REMARKS

We remark, here, that these calculations can be carried out in similar settings with similar results. We state here some interesting formulas that are obtained when r is even in the above calculations. We obtain the following Theorem.

Theorem 3.1. *With notation as above,*

(i) *If $r \equiv 2 \pmod{4}$ and k is even,*

$$\frac{1}{B_k} \sum_{a=0}^{n-1} R_r(a, n) B_k(a/n) = n^{r-k} \sum_{\substack{d|n \\ 2 \nmid d}} \left(\frac{-1}{d} \right) d^{k-r/2} \prod_{p|d} \left(1 - \frac{1}{p^k} \right).$$

(ii) If $r \equiv 2 \pmod{4}$ and k is odd,

$$\frac{(-1)^{(r-2)/4}}{2^{r/2} B_k(\frac{1}{4})} \sum_{a=0}^{n-1} R_r(a, n) B_k(a/n) = n^{r-k} \sum_{\substack{d|n \\ 4|d}} d^{k-r/2} \prod_{\substack{p|d \\ p \neq 2}} \left(1 - \frac{(-1)}{p^k}\right).$$

(iii) If $r \equiv 0 \pmod{4}$ and k is even,

$$\frac{1}{B_k} \sum_{a=0}^{n-1} R_r(a, n) B_k(a/n) = n^{r-k} \sum_{d|n} d^{k-r/2} \prod_{p|d} \left(1 - \frac{1}{p^k}\right) \eta(d), \text{ where}$$

$$\eta(d) = \begin{cases} 1 & \text{if } 2 \nmid d \\ 0 & \text{if } d \equiv 2 \pmod{4} \\ (-1)^{r/4} 2^{r/2} & \text{if } 4 \mid d. \end{cases}$$

(iv) If $r \equiv 0 \pmod{4}$ and k is odd,

$$\sum_{a=0}^{n-1} R_r(a, n) B_k(a/n) = 0.$$

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