HYPERGEOMETRIC GENERATING FUNCTIONS FOR VALUES OF DIRICHLET AND OTHER L-FUNCTIONS

Jeremy Lovejoy and Ken Ono

Abstract. Although there is a vast literature on the values of L-functions at non-positive integers, the recent appearance of some of these values as the coefficients of specializations of knot invariants comes as a surprise. Using work of Andrews, we revisit this old subject and provide uniform and general results giving such generating functions as specializations of basic hypergeometric functions. For example, we obtain such generating functions for all non-trivial Dirichlet L-functions.

1. Introduction and Statement of Results.

Suppose that χ_{12} is the Dirichlet character with modulus 12 defined by

$$\chi_{12}(n) := \begin{cases} 1 & \text{if } n \equiv 1, 11 \pmod{12}, \\ -1 & \text{if } n \equiv 5, 7 \pmod{12}, \\ 0 & \text{otherwise,} \end{cases}$$

and let $L(s, \chi_{12}) = \sum_{n=1}^{\infty} \frac{\chi_{12}(n)}{n^s}$ be its associated L-function. In a recent paper, Zagier [Z] obtained the following generating function for the values of $L(s, \chi_{12})$ at negative odd integers

$$\begin{aligned} -2e^{-t/24} \sum_{n=0}^{\infty} (1-e^{-t})(1-e^{-2t}) \cdots (1-e^{-nt}) &= \sum_{n=0}^{\infty} \frac{L(-2n-1, \chi_{12})}{n!} \cdot \left(\frac{-t}{24}\right)^n \\ &= -2 - \frac{23}{12}t - \frac{1681}{576}t^2 - \cdots . \qquad (1.1) \end{aligned}$$

The t-series expansion on the left hand side is obtained using the Taylor series expansion for e^{-t}. In other recent works (see [A-J-O], [C-O], [H1], [H2]), similar generating functions have been obtained for the values of certain L-functions at non-positive integers. These examples were derived from q-series identities associated to the "summation of the tails" of a modular form and the combinatorics of q-difference equations.

These L-function values are generalized Bernoulli numbers (for example, see [Ap]), and there is a vast literature on the subject. These numbers play many important roles in number theory. However, their recent appearance in knot theory, algebraic geometry, and

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mathematical physics comes as a surprise. In the recent works of Hikami [H1, H2] and Zagier [Z], they appear as the coefficients of specializations of Kashaev invariants in knot theory.

Motivated by these examples, it is natural to seek general results which uniformly provide such L-values as the coefficients of generating functions like (1.1) and the multidimensional series appearing in [H1] and [H2]. Here we provide such results.

If $0 < c \leq 1$ is a rational number, then let $\zeta(s,c)$, for $\Re(s) > 1$, denote the Hurwitz zeta function

$$\zeta(s,c) := \sum_{n=0}^{\infty} \frac{1}{(n+c)^s}. \tag{1.2}$$

These functions possess an analytic continuation to $\mathbb{C}$ with the exception of a simple pole at $s = 1$ with residue 1. These zeta functions, which generalize Riemann's zeta-function, are the building blocks of many important L-functions. All of the L-functions in [H1, H2, Z] are finite linear combinations of functions of the form

$$\begin{aligned} L_{a,b}(s) = (2a)^{-s}\left(\zeta\left(s,\frac{b}{2a}\right) - \zeta\left(s,\frac{a+b}{2a}\right)\right) &= \sum_{n=0}^{\infty}\left(\frac{1}{(2an+b)^s} - \frac{1}{(2an+a+b)^s}\right) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(ma+b)^s}, \end{aligned} \tag{1.3}$$

where $0 < b < a$ are integers. We obtain multidimensional basic hypergeometric generating functions for all of these L-functions at non-positive integers. (For a survey of basic hypergeometric series, see [A-A-R] or [G-R].)

We shall employ the standard notation

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}). \tag{1.4}$$

and throughout we assume that $|q| < 1$ and that the other parameters are restricted to domains that do not contain any singularities of the series or products under consideration. For integers $k \geq 2$, define the series $F_k(z;q)$ by

$$F_k(z;q) = \sum_{n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{(q;q)_{n_{k-1}}(z;q)_{n_{k-1}} z^{n_{k-1}+2n_{k-2}+\cdots+2n_1} q^{n_1^2+n_2^2+\cdots+n_{k-2}^2}}{(q;q)_{n_{k-1}-n_{k-2}}(q;q)_{n_{k-2}-n_{k-3}}\cdots(q;q)_{n_2-n_1}(q;q)_{n_1}(-z;q)_{n_1+1}}. \tag{1.5}$$

Theorem 1. *If $0 < b < a$ are integers and $k \geq 2$, then*

$$e^{-(k-1)b^2t}F_k(e^{-abt};e^{-a^2t}) = \sum_{n=0}^{\infty} L_{a,b}(-2n)\cdot\frac{((1-k)t)^n}{n!},$$

$$\frac{d}{dz}\left(z^{(2k-2)b}q^{(k-1)b^2}F_k(z^a q^{ab};q^{a^2})\right)|_{z=1,q=e^{-t}} = (2k-2)\sum_{n=0}^{\infty} L_{a,b}(-2n-1)\cdot\frac{((1-k)t)^n}{n!}.$$

The $\frac{d}{dz}$ series in Theorem 1 is obtained by differentiating summand by summand in z, and then setting $z = 1$ and $q = e^{-t}$. Although we omit its closed form expression for brevity, we note that it is easily obtained from (1.5) using the standard rules for differentiation.

Suppose that $\chi \mod f$ is a non-trivial Dirichlet character (i.e. a homomorphism from $(\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times$) with Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1.6}$$

Such a χ is even (resp. odd) if $\chi(-1) = 1$ (resp. $\chi(-1) = -1$). We give basic hypergeometric generating functions for the values of $L(s, \chi)$ at non-positive integers. Define the series $G(z; q)$ by

$$G(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q;q)_n (1/z;q)_n (zq;q)_{n+1} q^{n(n+1)/2}}{(q;q)_{2n+1}}. \tag{1.7}$$

If $\chi \mod f$ is a Dirichlet character, then define $G_\chi(z;q)$ by

$$G_\chi(z;q) := \sum_{r=1}^{[f/2]} \chi(r) z^{-r} q^{r^2} G(z^f q^{-2rf}; q^{f^2}). \tag{1.8}$$

Theorem 2. *Suppose that $\chi \mod f$ is a non-trivial Dirichlet character.*
1) If χ is odd, then

$$G_\chi(1; e^{-t}) = \sum_{n=0}^{\infty} L(-2n, \chi) \cdot \frac{(-t)^n}{n!}.$$

2) If χ is even, then

$$\frac{d}{dz}\left(G_\chi(z;q)\right)|_{z=1, q=e^{-t}} = -\sum_{n=0}^{\infty} L(-2n-1, \chi) \cdot \frac{(-t)^n}{n!}.$$

Suppose that χ is a nontrivial even (resp. odd) Dirichlet character. It is a classical fact that if $n \geq 0$ then $L(-2n, \chi) = 0$ (resp. $L(-2n-1, \chi) = 0$). Therefore, Theorem 2 provides the L-values at non-positive integers for all Dirichlet L-functions.

In Section 2, we give formulas for $F_k(z;q)$ and $G(z;q)$. To prove these identities, we employ important results of Andrews. In Section 3 we recall classical facts about Mellin integral representations of L-functions, and then combine them with the identities of Section 2 to prove Theorems 1 and 2. In Section 4 we give several examples of Theorems 1 and 2.

2. q-SERIES IDENTITIES.

For convenience, we make use of the following abbreviation

$$(a_1, a_2, ..., a_k)_n := (a_1;q)_n (a_2;q)_n \cdots (a_k;q)_n. \tag{2.1}$$

Theorem 2.1. *If $k \geq 2$, then the following identity is true*

$$F_k(z;q) = \sum_{n=0}^{\infty} (-1)^n z^{(2k-2)n} q^{(k-1)n^2}.$$

Theorem 2.2. *The following identity is true*

$$G(z;q) = \sum_{n=0}^{\infty} z^{-n} q^{n^2} - \sum_{n=1}^{\infty} z^n q^{n^2}.$$

Remarks. The series $F_2(z;q)$ is a specialization of the classical Rogers-Fine identity [F]

$$F_2(z;q) = \sum_{n=0}^{\infty} \frac{(z)_n z^n}{(-z)_{n+1}} = \sum_{n=0}^{\infty} (-1)^n z^{2n} q^{n^2},$$

while the series $G(z;q)$ is a specialization of an identity of Andrews [A2] related to "false" theta functions in Ramanujan's lost notebook.

Proofs of Theorem 2.1 and 2.2. Two sequences (α_n, β_n) are said to form a Bailey pair with respect to a if for every $n \geq 0$,

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_n}{(q)_{n-k}(aq)_{n+k}}.$$

Given such a pair, Andrews [A3] has shown that for any natural number k and complex numbers b_i, c_i we have (subject to convergence conditions)

$$\begin{aligned} &\frac{(\frac{aq}{b_k}, \frac{aq}{c_k})_\infty}{(aq, \frac{aq}{b_kc_k})_\infty} \sum_{r\geq 0} \frac{(b_1, c_1, ..., b_k, c_k)_r}{(\frac{aq}{b_1}, \frac{aq}{c_1}, ... \frac{aq}{b_k}, \frac{aq}{c_k})_r} \left(\frac{a^kq^k}{b_1c_1...b_kc_k}\right)^r \alpha_r \\ &= \sum_{n_k\geq n_{k-1}\geq ...\geq n_1\geq 0} \frac{(b_k, c_k)_{n_k}...(b_1, c_1)_{n_1}}{(\frac{aq}{b_{k-1}}, \frac{aq}{c_{k-1}})_{n_k}...(\frac{aq}{b_1}, \frac{aq}{c_1})_{n_2}} \\ &\times \frac{(\frac{aq}{b_{k-1}c_{k-1}})_{n_k-n_{k-1}}...(\frac{aq}{b_1c_1})_{n_2-n_1}}{(q)_{n_k-n_{k-1}}...(q)_{n_2-n_1}} \left(\frac{aq}{b_kc_k}\right)^{n_k} ... \left(\frac{aq}{b_1c_1}\right)^{n_1} \beta_{n_1}. \end{aligned} \tag{2.2}$$

The theorems follow by inserting the right Bailey pairs into the above equation and making appropriate specializations of the parameters. Substituting the Bailey pair with respect to a [A2],

$$\alpha_j = \frac{(a)_j(1-aq^{2j})(-1)^jq^{j(j-1)/2}}{(q)_j(1-a)} \quad \text{and} \quad \beta_j = \begin{cases} 1, & j=0 \\ 0, & \text{otherwise,} \end{cases}$$

yields a limiting case of Andrews' multidimensional generalization of Watson's transformation [A1],

$$\begin{aligned} &\frac{(\frac{aq}{b_k}, \frac{aq}{c_k})_\infty}{(aq, \frac{aq}{b_kc_k})_\infty} \sum_{r\geq 0} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, ..., b_k, c_k)_r}{(q, \sqrt{a}, -\sqrt{a}, \frac{aq}{b_1}, \frac{aq}{c_1}, ... \frac{aq}{b_k}, \frac{aq}{c_k})_r} \left(\frac{a^kq^k}{b_1c_1...b_kc_k}\right)^r (-1)^r q^{r(r-1)/2} \\ &= \sum_{n_{k-1}\geq ...\geq n_1\geq 0} \frac{(b_k, c_k)_{n_{k-1}}...(b_2, c_2)_{n_1}}{(\frac{aq}{b_{k-1}}, \frac{aq}{c_{k-1}})_{n_{k-1}}...(\frac{aq}{b_1}, \frac{aq}{c_1})_{n_1}} \\ &\times \frac{(\frac{aq}{b_{k-1}c_{k-1}})_{n_{k-1}-n_{k-2}}...(\frac{aq}{b_1c_1})_{n_1}}{(q)_{n_{k-1}-n_{k-2}}...(q)_{n_1}} \left(\frac{aq}{b_kc_k}\right)^{n_{k-1}} ... \left(\frac{aq}{b_2c_2}\right)^{n_1}. \end{aligned} \tag{2.3}$$

Keeping in mind that $(x)_r/x^r \to (-1)^rq^{r(r-1)/2}$ and $(y/x)_r \to 1$ as $x \to \infty$, let $a = z^2, b_1 = -z, c_k = z, b_k = q$, and all remaining $b_i, c_i \to \infty$ in (2.3). After a bit of simplification, this is Theorem 2.1.

Next insert in (2.2) the Bailey pair with respect to q [A4],

$$\alpha_j = (-z)^{-j}q^{j(j+1)/2}(1-z^{2j+1}) \quad \text{and} \quad \beta_j = \frac{(z)_{j+1}(q/z)_j}{(q^2)_{2j}}.$$

The $k=1$ case is Andrews' generalized false theta identity [A2],

$$\sum_{n=0}^{\infty} \frac{(z)_{n+1}(q/z, b, c)_n \left(\frac{q^2}{bc}\right)^n}{(q)_{2n+1}} = \frac{(q^2/b, q^2/c)_\infty}{(q, q^2/bc)_\infty} \sum_{n=0}^{\infty} \frac{(b,c)_n \left(\frac{q^2}{bc}\right)^n}{(q^2/b, q^2/c)_n} q^{n(n+1)/2}((-z)^{n+1} + (-z)^{-n}).$$

Letting $b = q, z = zq$, and $c \to \infty$ gives Theorem 2.2.

□

3. Proofs of Theorems 1 and 2.

To prove Theorems 1 and 2, we require the following classical results regarding the Mellin integral representations of L-functions.

Proposition 3.1. *Suppose that Ψ is a periodic function with modulus f with mean value zero, and let*

$$L(s)=\sum_{n=1}^{\infty}\frac{\Psi(n)}{n^s}=f^{-s}\sum_{r=1}^{f}\Psi(r)\zeta\left(s,\frac{r}{f}\right).$$

As $t\searrow 0$ we have

$$\sum_{n=1}^{\infty}\Psi(n)e^{-n^2t}\sim\sum_{n=0}^{\infty}L(-2n)\cdot\frac{(-t)^n}{n!},$$
$$\sum_{n=1}^{\infty}\Psi(n)ne^{-n^2t}\sim\sum_{n=0}^{\infty}L(-2n-1)\cdot\frac{(-t)^n}{n!}.$$

Proof. By the hypothesis on Ψ, $L(s)$ has an analytic continuation to $\mathbb{C}$. Suppose that $H(t)$ is the asymptotic expansion as $t\searrow 0$ given by

$$H(t)=\sum_{n=0}^{\infty}b(n)t^n\sim\sum_{n=1}^{\infty}\Psi(n)e^{-n^2t}. \tag{3.1}$$

Using the Mellin integral representation for $L(2s)$ (for $\Re(s)>1$), and by letting $T=n^2t$, it turns out that

$$\begin{aligned}\int_0^{\infty}\left(\sum_{n=1}^{\infty}\Psi(n)e^{-n^2t}\right)t^{s-1}dt&=\sum_{n=1}^{\infty}\Psi(n)\int_0^{\infty}e^{-n^2t}t^{s-1}dt\\&=\sum_{n=1}^{\infty}\frac{\Psi(n)}{n^{2s}}\int_0^{\infty}e^{-T}T^{s-1}dT\\&=\Gamma(s)L(2s).\end{aligned}\tag{3.2}$$

For any $N>0$, this combined with (3.1) implies that

$$\begin{aligned}\Gamma(s)L(2s)&=\int_0^{\infty}\left(\sum_{n=0}^{N-1}b(n)t^n+O(t^N)\right)t^{s-1}dt\\&=\sum_{n=0}^{N-1}\frac{b(n)}{s+n}+F(s),\end{aligned}$$

where $F(s)$ is analytic for $\Re(s)>-N$. Therefore, $b(n)$ is the residue at $s=-n$ of $\Gamma(s)L(2s)$, and so

$$b(n)=\frac{(-1)^n}{n!}\cdot L(-2n).$$

The same argument applies to the asymptotic expansion of $\sum_{n=1}^{\infty}\Psi(n)ne^{-n^2t}$.

□

We now prove Theorems 1 and 2 using Theorems 2.1 and 2.2, and Proposition 3.1.

Proof of Theorem 1. By Theorem 2.1, we have

$$z^{(2k-2)b}q^{(k-1)b^2}F_k\left(z^aq^{ab};q^{a^2}\right)=\sum_{n=0}^{\infty}(-1)^nz^{(2k-2)(an+b)}q^{(k-1)(an+b)^2}. \tag{3.3}$$

By letting $z=1$ and $q=e^{-t}$ in (3.3), we obtain a power series in t. To see that the t-series is well defined, use the fact that the constant term in the Taylor expansion of $1-e^{-mt}$ is zero for every positive integer m. Each summand in (1.5) therefore has the property that the numerator contains $2n_{k-1}$ such factors while the denominator has n_{k-1} many (note: the $(-z;q)_{n_1+1}$ does not contribute any). Therefore, Theorem 1 for $L_{a,b}(-2n)$ follows from Proposition 3.1.

To obtain Theorem 1 for $L_{a,b}(-2n-1)$, we apply the argument above to the series which is obtained by differentiating (3.3) in z summand by summand before letting $z=1$ and $q=e^{-t}$. As above, the form of (1.5) implies that the resulting t-series is well defined.

□

Proof of Theorem 2. By Theorem 2.2, if $0<r<f$, then

$$z^{-r}q^{r^2}G(z^fq^{-2rf};q^{f^2})=\sum_{n=0}^{\infty}z^{-(fn+r)}q^{(fn+r)^2}-\sum_{n=1}^{\infty}z^{(fn-r)}q^{(fn-r)^2}. \tag{3.4}$$

Recall the definition of $G_\chi(z;q)$ (see (1.8))

$$G_\chi(z;q):=\sum_{r=1}^{[f/2]}\chi(r)z^{-r}q^{r^2}G(z^fq^{-2rf};q^{f^2}). \tag{3.5}$$

By (3.4), this implies that

$$G_\chi(z;q)=\sum_{r=1}^{[f/2]}\sum_{n=0}^{\infty}\chi(r)\left(z^{-(fn+r)}q^{(fn+r)^2}-z^{(fn+f-r)}q^{(fn+f-r)^2}\right).$$

Therefore, we have

$$G_\chi(1;e^{-t})=\sum_{r=1}^{[f/2]}\sum_{n=0}^{\infty}\chi(r)\left(e^{-(fn+r)^2t}-e^{-(fn+f-r)^2t}\right),$$

$$\frac{d}{dz}\left(G_\chi(z;q)\right)|_{z=1,q=e^{-t}}=-\sum_{r=1}^{[f/2]}\sum_{n=0}^{\infty}\chi(r)\left((fn+r)e^{-(fn+r)^2t}+(fn+f-r)e^{-(fn+f-r)^2t}\right).$$

An inspection of (1.7) shows that both t-series are well defined. Since $\chi(f/2)=0$ for even f, Proposition 3.1 implies that if χ is odd, then

$$G_\chi(1;e^{-t})=\sum_{n=0}^{\infty}L(-2n,\chi)\cdot\frac{(-t)^n}{n!}.$$

Similarly if χ is even, then Proposition 3.1 implies that

$$\frac{d}{dz}\left(G_\chi(z;q)\right)_{z=1,q=e^{-t}}=-\sum_{n=0}^{\infty}L(-2n-1,\chi)\cdot\frac{(-t)^n}{n!}.$$

This completes the proof of Theorem 2.

□

4. Examples

Here we give examples of Theorems 1 and 2.

Example 1. Here we give the $k = 2$ example of Theorem 1 for the function

$$L_{2,1}(s) = \sum_{n=1}^{\infty} \frac{\chi_{-1}(n)}{n^s},$$

where χ_{-1} is the unique non-trivial Dirichlet character modulo 4. Theorem 1 implies that

$$\begin{aligned} e^{-t}F_2(e^{-2t}; e^{-4t}) &= e^{-t}\sum_{n=0}^{\infty} \frac{(1-e^{-2t})(1-e^{-6t})\cdots(1-e^{-2t-4(n-1)t})e^{-2nt}}{(1+e^{-2t})(1+e^{-6t})\cdots(1+e^{-2t-4nt})} \\ &= \frac{1}{2} + \frac{1}{2}t + \frac{5}{4}t^2 + \frac{61}{12}t^3 + \frac{1385}{48}t^4 + \cdots \\ &= \sum_{n=0}^{\infty} L(-2n, \chi_{-1}) \cdot \frac{(-t)^n}{n!}. \end{aligned}$$

Theorem 1 provides a generating function for the values $L(-2n-1, \chi_{-1})$, where $n \geq 0$. Since χ_{-1} is odd, these values are zero. Theorem 1 implies that

$$\begin{aligned} 0 = 2\sum_{n=0}^{\infty} L(-2n-1, \chi_{-1}) \cdot \frac{(-t)^n}{n!} &= \frac{d}{dz}\left(z^2 q F_2(z^2q^2; q^4)\right)|_{z=1, q=e^{-t}} \\ &= \sum_{n=0}^{\infty} \frac{d}{dz}\left(\frac{(z^2q^2; q^4)_n z^{2n+2} q^{2n+1}}{(-z^2q^2; q^4)_{n+1}}\right)|_{z=1, q=e^{-t}} \\ &= \left(\frac{1}{2} + \frac{1}{2}t - \frac{1}{4}t^2 - \cdots\right) \\ &\quad + \left(-\frac{1}{2} + \frac{3}{2}t + \frac{17}{4}t^2 - \cdots\right) \\ &\quad + \left(-2t + \frac{15}{2}t^2 + \cdots\right) + \cdots\cdots \end{aligned}$$

Example 2. Here we compute the values of $L(s, \chi_{-1})$ at even non-positive integers again using Theorem 2. By Theorem 2, we find that

$$\begin{aligned} G_{\chi_{-1}}(1; e^{-t}) &= e^{-t}G(e^{8t}; e^{-16t}) \\ &= e^{-t}\sum_{n=0}^{\infty} \frac{(-1)^n(e^{-16t}; e^{-16t})_n(e^{-8t}; e^{-16t})_n(e^{-8t}; e^{-16t})_{n+1}e^{-8n(n+1)t}}{(e^{-16t}; e^{-16t})_{2n+1}} \\ &= \sum_{n=0}^{\infty} L(-2n, \chi_{-1}) \cdot \frac{(-t)^n}{n!} \\ &= \frac{1}{2} + \frac{1}{2}t + \frac{5}{4}t^2 + \frac{61}{12}t^3 + \frac{1385}{48}t^4 + \cdots. \end{aligned}$$

Example 3. Suppose that χ_5 is the Dirichlet character modulo 5 given by the Legendre symbol modulo 5. By definition, we have that

$$G_{\chi_5}(z; q) = z^{-1}qG(z^5q^{-10}; q^{25}) - z^{-2}q^4G(z^5q^{-20}; q^{25}),$$

and so we have

$$G_{\chi_5}(z;q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q^{25};q^{25})_n (z^{-5}q^{10};q^{25})_n (z^5 q^{15};q^{25})_{n+1} z^{-1} q^{\frac{25n(n+1)}{2}+1}}{(q^{25};q^{25})_{2n+1}}$$
$$- \sum_{n=0}^{\infty} \frac{(-1)^n (q^{25};q^{25})_n (z^{-5}q^{20};q^{25})_n (z^5 q^{5};q^{25})_{n+1} z^{-2} q^{\frac{25n(n+1)}{2}+4}}{(q^{25};q^{25})_{2n+1}}.$$

By differentiating in z summand by summand, then letting $z=1$ and $q=e^{-t}$, Theorem 2 gives

$$\frac{d}{dz}\left(G_{\chi_5}(z;q)\right)|_{z=1,q=e^{-t}} = \frac{2}{5} + 2t + \frac{67}{5}t^2 + \frac{361}{3}t^3 + \cdots +$$
$$= -\sum_{n=0}^{\infty} L(-2n-1,\chi_5)\cdot \frac{(-t)^n}{n!}.$$

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CNRS, LaBRI, 351 cours de la Libération, 33405 Talence Cedex, FRANCE
E-mail address: lovejoy@labri.fr

Dept. Math., University of Wisconsin, Madison, Wisconsin, 53706, USA.
E-mail address: ono@math.wisc.edu