THE LIFE AND WORK OF R. A. RANKIN (1915–2001)

BRUCE C. BERNDT[1], WINFRIED KOHNEN, AND KEN ONO[2]

Robert Alexander Rankin, an eminent Scottish number theorist and, for several decades, one of the world's foremost experts in modular forms, died on January 27, 2001 in Glasgow at the age of 85. He was one of the founding editors of *The Ramanujan Journal*. For this and the next issue of the *The Ramanujan Journal*, several well-known mathematicians have prepared articles in Rankin's memory. In this opening paper, we provide a short biography of Rankin and discuss some of his major contributions to mathematics. At the conclusion of this article, we provide a complete bibliography of all of Rankin's writings divided into five categories: 1) Research and Expository Papers; 2) Books; 3) Books Edited; 4) Obituaries; 5) Other Writings.

1. His Life

Robert Rankin was born on October 27, 1915 at Garlieston, Wigtownshire, Scotland. His father, Reverend Oliver Shaw Rankin, was a parish minister in Sorbie, Wigtownshire, between 1912 and 1937. His mother was Olivia Teresa Shaw Rankin born in 1884 in Würzburg, Germany. Oliver and Olivia were first cousins, their mothers being sisters. Robert was named after his paternal grandfather, who was a pastor in Lamington, Lanarkshire, when his grandson was born. He attended Garlieston School and already at a young age developed an affinity for the Gaelic culture and language. Late in his teenage years, he spent four to six weeks working in a general store on the island of Barra in the Outer Hebrides in order to learn spoken Gaelic. From Garlieston, Robert went to Fettes College, an independent school in Edinburgh. He then obtained a scholarship to Clare College, Cambridge which he entered in 1934. The lectures of J. E. Littlewood and A. E. Ingham particularly interested him. In 1936 Rankin was a Wrangler in Part II of the Mathematical Tripos, and in 1937 he took Part III of the Tripos graduating in the same year. Also in 1937, his father became Professor of Old Testament Language, Literature, and Theology at the University of Edinburgh; among his scholarly achievements were his books, *The Origins of the Festival of Hanukkah* (1930) and *Israel's Wisdom Literature: its Bearing on Theology and the History of Religion* (1936). He died in 1954.

[1]Research partially supported by grant MDA904-00-1-0015 from the National Security Agency.

[2]Research partially sponsored by the an Alfred P. Sloan Foundation Research Fellowship, the National Science Foundation, a David and Lucile Packard Foundation Research Fellowship, and an H. I. Romnes Fellowship.

[3]2000 AMS Classification Numbers: Primary, 01A70; Secondary, 11-03, 11F03.

Upon receiving his B.A. degree, Robert began doctoral studies under Ingham with his work on the differences between consecutive primes, for which he was awarded the Rayleigh Prize in 1939. He was eventually to publish five papers [?], [?], [?], [?], [?] on this subject. Some years later, Ingham remarked [?], "Robert was the most serious of all my gifted pupils." However, Ingham left Cambridge for the University of London, and so in 1939 Rankin became a research student of G. H. Hardy. In that same year, Rankin also was elected a Fellow of Clare College, Cambridge, a position he held until 1951. Not surprisingly, in view of Hardy's association with Ramanujan, Rankin turned to Ramanujan's mathematics for his doctoral dissertation, making seminal contributions on Ramanujan's tau-function. His three papers on this topic [?], [?], [?], published in the *Proceedings of the Cambridge Philosophical Society* in 1939 and 1940, were perhaps his most famous and influential papers, about which more will be written later in this essay. His interest in Ramanujan became a lasting one, and he returned to Ramanujan's work on both mathematical and historical levels several times during his life, especially in his last twenty years.

When World War II came, Rankin's initial inclination was to join the British army as a soldier, but it was decided that his war efforts were better utilized at the Ministry of Supply at Fort Halstead in Kent, where he began work in 1940 on the development of rockets. He developed a theory which allowed the trajectory of a rocket to be calculated from its initial conditions during its burning phase. Although his theory was corroborated by experimental evidence, the British government paid little attention to his work at the time, and he was transferred to a place near Aberystwyth in North Wales, where he continued to work on rockets until the end of the War. His work was classified until the end of the War, and in 1949 Rankin was allowed to publish his work [?], *The Mathematical Theory of the Motion of Rotated and Unrotated Rockets*, in the *Philosophical Transactions of the Royal Society*, at that time, the longest paper ever to be published in the *Transactions*. In 1966, when Rankin visited Moscow to attend the International Congress of Mathematicans, he was curious to learn if the Moscow University Library contained any of his books and papers. He discovered that the only item of his listed in the Library catalogue was a Russian translation of his paper on rockets [?], published in 1951.

In North Wales he met his future wife, Mary Llewellyn, who was working as a secretary in the Ministry of Supply. She was a cousin of the famous contralto, Kathleen Ferrier. Mary and Robert married in 1942 in Maestag, Wales, and their first daughter, Susan, was born in Maesteg in June, 1943. Their son, Charles, was also born in Maesteg, in March, 1947. Their second and third daughters, Fenella and Olivia, were born in Cambridge in 1950 and in Birmingham in 1954, respectively.

In 1945, Rankin returned to Clare College, Cambridge where he was appointed Faculty Assistant Lecturer. In 1947, he assumed the position of Assistant Tutor, and in the following year, he was promoted to the rank of Lecturer. In 1949, he became a Praelector at Cambridge. During this time, Rankin also served as Secretary and Editor of the *Proceedings of the Cambridge Philosophical Society*.

While at Cambridge, he published two unusual papers. The first [?], inspired by Dorothy L. Sayers's crime novel about bell ringing, *The Nine Taylors*, is on campanology, the study of the change-ringing of church bells. In this traditional English method of ringing church bells the challenge is to ring a set of bells (a peal) in all possible orders (the changes) with no repetition allowed apart from the start and finish, which are normally in "rounds," with the bells being rung in order of descending pitch. The sequence of changes is partitioned into blocks, known as "leads," of a standard form, and one considers the sequence consisting of the last changes of the leads, called the lead ends. Each lead end is obtained from the previous one by a permutation depending on the type of lead, "plain," "bob," or "single," and one attempts to choose a sequence of leads so that all possible changes occur. A change is equivalent to a permutation of the integers, $1, 2, \ldots, q$, where q is the number of bells taking part in the changes. Using the theory of permutation groups, Rankin determined conditions on the possibility of ringing certain peals. For example, in the method of Grandsire Triples, there are seven bells with a total number of 5040 changes, and Rankin's theorem shows that a full extent of 5040 changes is impossible using plains and bobs only. A sequel [?] was published in 1966. In 1999, R. G. Swan [?] published a simpler proof of Rankin's main theorem.

The second unusual paper [?] appeared in the *Proceedings of the Royal Irish Academy* in 1948 and was written in Gaelic under the pen, Rob Alasdair Mac Fhraing, the Gaelic equivalent of Robert Alexander Rankin. The paper's translated title is: *The numbering of Fionn's and Dubhan's men and the story of Josephus and the 40 Jews*, and it concerns a Scottish–Gaelic version of a well-known ancient story about fifteen Christians and fifteen Turks. The mathematical problem is as follows. The numbers $1, 2, \ldots, n$ are arranged in a circle, and every mth number is deleted. The main task is to find the last number to be deleted, and Rankin constructed an algorithm for doing so.

Finding a reviewer for a paper written in Gaelic is a daunting task. For each language other than English, the editors of *Mathematical Reviews* have lists of mathematicians who can read the language. The list of those able to read Gaelic contained but one name – Robert Rankin. So when R. P. Boas, the editor of *Mathematical Reviews*, sought a reviewer for [?], he had but one choice and sent [?] to Rankin for review, not realizing that the requested reviewer was indeed the author!

In 1951, Rankin left Cambridge to become Mason Professor of Pure Mathematics at Birmingham University, in succession to G. N. Watson. However, Rankin held this post for only three years and returned to Scotland in 1954 to assume the post of Professor of Mathematics at the University of Glasgow, succeeding T. M. MacRobert, who had retired from the Chair of Mathematics. In an obituary of Rankin, D. Martin [?] writes,

> Thus began a period of 28 years during which Robert's powerful intellect, exceptionally accurate memory and tremendous energy, along with his absolute integrity and unstinted devotion to duty, enabled him to render signal service to the university.

This corroborates a conversation the first author had with Rankin in early 1967 on a train to Glasgow from Edinburgh, after attending a meeting of the Edinburgh Mathematical Society. The first author had expressed some remorse that Rankin's heavy administrative duties as Dean of the Faculty of Science had unfortunately taken him away from mathematical research. He surprised his travel companion by replying that he in no way regretted his administrative duties, that he enjoyed them, that it was an honor to be of such service, and that he was performing tasks that were truly worthwhile.

After serving as Dean of the Faculty of Science, Rankin served as Clerk of the Senate from 1971 to 1978 and Dean of the Faculties from 1986 to 1988. From his arrival in Glasgow in 1954 until his retirement in 1982, Rankin served as Head of the Mathematics Department at the University of Glasgow. About his long stint as Head, D. Munn [?] opines,

> Robert ... ran it with exemplary efficiency .... He expected his staff to share his commitment to excellence; and, in return, he was rewarded by loyalty. It was a privilege to have been part of this team. During his long tenure, many changes were introduced. In particular, class syllabuses were revised and modernised, staff numbers were increased, and research activities encouraged.

Similar sentiments are expressed by Martin [?] in his assessment,

> In running his department, Robert thought out everything with meticulous care and consulted others before making important decisions. He gave much thought to staff matters, and always tried to be fair. He appreciated both good teaching and good research.

Rankin and his family spent the 1963–1964 academic year at Indiana University, where he prepared the first draft of his book [?], *Modular Forms and Functions*, eventually published by the Cambridge University Press in 1977. The book was the first comprehensive treatment of modular forms in English and has served as a standard reference since its publication. Earlier, he had written a monograph [?] on modular groups.

A turning point in Rankin's lifelong interest in Ramanujan came in 1965 after Watson died. Rankin visited Watson's widow, and she brought him to her husband's cluttered attic office. Among the papers Rankin found were the notes that Watson and B. M. Wilson had accumulated in their attempt to edit Ramanujan's notebooks [?] in the late 1920s and 1930s, and a sheaf of 138 pages of original unpublished mathematics of Ramanujan. Since Watson had been a Fellow at Trinity College, Cambridge, Rankin and Mrs. Watson agreed that he should sift through all of Watson's papers and send those that were worth saving to Trinity College Library for preservation. Over a period of three years, Rankin sorted out material and sent it in batches, including the aforementioned items, to Trinity College. In his obituary of Watson [?], Rankin did not mention the Ramanujan manuscript. It was rediscovered by G. E. Andrews in the spring of 1976, and he christened his finding, "the lost notebook." Rankin was not fond of this appellation, since the manuscript had never been really lost and was not really a notebook, but in view of the famous notebooks of Ramanujan [?], Andrews's designation was natural. For a history of the lost notebook [?], see papers by either Andrews [?] or Berndt [?].

Rankin wrote two very informative papers [?], [?] on the history and genesis of Ramanujan's papers, including his notebooks, "lost notebook," other unpublished fragments, and his letters to Hardy. Ramanujan's illness was a focus of Rankin's paper [?], for which he prepared by visiting nursing homes where Ramanujan was a patient and by unearthing medical records and bills. This study was later resumed by the British physician, D. A. B. Young [?], who made a firm diagnosis of the cause of Ramanujan's death. Young's paper and the three aforementioned papers by Rankin are reproduced in Rankin's book [?] with the first author. In an earlier book [?], Berndt and Rankin had collected as many letters to, from, and about Ramanujan as they could find and wrote extensive commentary on them. In particular, the authors trace the history of each formula communicated in Ramanujan's first two famous letters to Hardy in 1913.

In an important contribution to understanding Ramanujan's mathematics, Rankin [?] elucidated some of Ramanujan's congruences for both the partition and tau functions from an unpublished, handwritten manuscript of Ramanujan. This manuscript was published for the first time in handwritten form along with the lost notebook in 1988 [?]. A typed version, together with commentary, an historical account of related work since the writing of the manuscript, and added details and proofs of some of Ramanujan's claims, was published by two of the present authors in 1999 [?].

Rankin's interest in the history of mathematics extended far beyond Ramanujan. Rankin was twice elected President of the *Edinburgh Mathematical Society*, and in his Presidential address he recounted the first fifty years of the Society. Later at the Society's centenary in 1983, Rankin [?] compiled the Society's complete history.

Robert's wife, Mary, died suddenly in her sleep in June, 1996. After this devastating loss, his health steadily declined. In his last five years, his four children and his devotion to mathematics and its history, especially that connected with Ramanujan, served as sustenance.

Rankin remained active up until perhaps a week before his death, when he and Berndt completed their book [?]. In December, 2000, against the advice of his doctors, Rankin travelled to London, where he presented a paper [?] before the *British Society for the History of Mathematics* on Hugh Blackburn, one of Rankin's predecessors in the Chair at Glasgow and a "little-known friend of Lord Kelvin." Sharing the program with him was his daughter, Fenny Smith, an historian of mathematics, who opened the meeting with a paper explaining the most practical reasons for the adoption of the Hindu–Arabic numerals over the Roman ones.

Rankin was elected to the London Mathematical Society in 1946. He served as its Vice-President from 1966 to 1968 and was a Council member in 1947–1951, 1962–1963, and 1965–1968. He was elected a Fellow of the Royal Society of Edinburgh in 1955 and received the Society's Keith Prize for papers written in 1961–1963. In 1987, he received the Senior Whitehead Prize of the *London Mathematical Society*, and in 1998 he received the Society's most prestigious honor, the De Morgan Medal. (A short history of the Medal may be found after this article.)

Rankin was first and foremost a Scot. He had a long abiding interest in Gaelic language and culture. This interest was fostered at Fettes by his contemporary, George Campbell Hay, who later became a well-known Gaelic poet. Rankin's interest in Gaelic was more than casual; he was a Gaelic scholar. He wrote a definitive paper [?], published in the *Transactions of the Gaelic Society of Glasgow* in 1948, on the Gaelic poem, *Oran na comhachag* (Song of the owl), dating from the seventeenth century. In his paper, Rankin examined various versions of the poem and different accounts of its history. He later wrote a sequel [?] on the place-names mentioned in the poem. Rankin's life-long friendship with Hay motivated Rankin to write his personal reflections on Hay [?]. In this article one can find Hay's poem, entitled "Maths," perhaps his first poem, but "a piece of doggerel," in Rankin's words [?, p. 3].

Maths

Five terms I've graced the bottom set
Without a hope of a punt[4] as yet,
For when I should be doing Maths
My mind will run on other paths.
But after all why should I care?
Equations only make me swear,
And riders make me tear my hair,
And graphs just drive me to despair:
I loathe the sound of stock and share.
X, Y, and Z I cannot bear.
Whether they're plain or cube or square.
I loathe them one and all – so there!
Revision papers give me pain,
Pythagoras benumbs my brain,
All algebra's a beastly blain!
Geometry's a useless strain!
All Maths are but a vile excrescence;
They pain me by their very presence.

In 1957, Rankin was elected Honorary President of the Glasgow Gaelic Society. He served as an external examiner at University College, Galway in Ireland by examining mathematical papers written in Irish Gaelic.

Rankin enjoyed walking in the Scottish hills, especially on Sunday afternoons. In particular, he enjoyed walking in the Kilpatrick Hills, not far from his Glasgow home. Lying in these hills are a dozen well-tooled stones, whose purpose and history had been lost. Rankin not only uncovered their purpose, but a good deal of local history as well. His findings are published in his booklet [?].

Rankin had a strong interest in music. He was an able organist and harpsichordist and, in fact, assembled his own clavichord (from a kit). The Bach chorale preludes were perhaps his favorite organ pieces. He especially enjoyed playing the harpsichord prior to going out for the evening with Mary. Invariably, Robert was ready to depart before Mary, and so he would utilize his waiting time by playing Bach on the harpsichord. However, when Mary was ready, Robert did not cease playing but insisted on completing the piece in its entirety. Robert also enjoyed the music of, especially,

[4]"Punt" is the poet's slang for a transfer to a higher class.

Chopin and Mozart, but he did not entirely share Mary's love of opera, which sounded to him like "a pain in the singer's stomach."

Not only did Rankin build a clavichord, but he also constructed a large canoe in his living room. It was then placed on end in the double story foyer of the Rankins' large home at 10 University Square. Whenever the first author visited their home during the 1966–67 academic year, he wondered how the boat could ever be removed from their home. In fact, the family enjoyed many trips to Loch Lomond with the canoe, which could be maneuvered out of the front door and fixed on top of the family car. Before building the canoe, Robert had constructed a small dinghy, also in his living room. The family still remembers the multitude of boiling kettles and pans required to produce steam to bend the wood into the right shape. This boat, named *a' Comhachag* for the owl of the poem which he loved, was powered with an outboard motor, and often, on a trailer, accompanied the family on trips to Loch Lomond (sometimes with the canoe along too). *The Owl* made possible many enjoyable (and safe, since no one was allowed to board any small craft without a life jacket) explorations of some of the small islands in the loch.

It is possible that these trips enabled Robert to relive in some way one of the most memorable holidays of his youth, one of which took place the summer before he went to Cambridge: a sailing trip around the lochs and islands of the west coast of Scotland, with his brother Kenneth, his friend George Hay, and another family friend, a time which he remembered with pleasure all his life.

Rankin attempted to instill his love of mathematics into his children on automobile trips by asking them if license plate numbers were prime, were divisible by certain integers, etc. The outcome of his efforts is inconclusive.

His interest in his children's education led him to serve as governor of two schools. First, he was Chairman of the Governors at Laurel Bank, the school which all his daughters attended, and where Mary subsequently taught. After he retired from this post at Laurel Bank, he was for many years a governor at Strathallan School in Perthshire, and was still, at his death, a valued Honorary Governor.

Rankin was also deeply committed to environmental issues. He chaired the Clyde Estuary Amenity Council from 1969 until 1982 when there was much concern about developments along the Clyde River estuary.

For Rankin, his family, Gaelic history and culture, the church, the Scottish hills, the environment, music, mathematical history, Ramanujan, teaching, administration, and research were the focal points of his life. In the remainder of this paper, we focus on his research.

2. His Work

For Rankin, mathematical research was generally done alone, perhaps because colleagues at the universities he served generally did not share his research interests. Only four of his more than 100 papers have coauthors, and only one, D. G. Kendall, wrote more than one paper with Robert, although the first author of this paper coauthored one paper and two books with him. Here we provide a brief introduction, divided by subject matter, to the many contributions Rankin made to mathematics.

2.1. **Gaps Between Consecutive Primes.** Let $p_1 = 2, p_2 = 3, \ldots$ denote the prime numbers in increasing order. Studying the distance between consecutive primes is one of the most famous subjects in analytic number theory. In his first published paper [?], Rankin improved upon P. Erdös's work [?] on the problem of bounding from below the largest difference between consecutive primes. He introduced what is now called "Rankin's method," which remains today an important technique in comparative prime number theory. If, as customary,

$$\Psi(x,y) := \#\{n \leq x : P^+(n) \leq y\},$$

where $P^+(n)$ denotes the largest prime factor of n, then Rankin devised an ingenious method for deriving an upper bound for $\Psi(x,y)$. Let $\log_k x$ denote the kth iterated logarithm of $\log x$ (i.e., $\log_k x = \log\log\cdots\log x$), where k logarithms are indicated. Improving on his result in [?], Rankin proved in [?] that for infinitely many values of n,

$$d_n := p_{n+1} - p_n > (e^\gamma - \epsilon)\log p_n \frac{\log_2 p_n \, \log_4 p_n}{(\log_3 p_n)^2}, \tag{2.1}$$

where γ denotes Euler's constant and ϵ is any small positive number. Two improvements on the constant e^γ have been made since Rankin's paper [?]. The first was by H. Maier and C. Pomerance [?], and the second was by J. Pintz [?]. Erdös offered \$10,000 to the person who can prove (**??**) with e^γ replaced by ∞ [?, p. 194]. The problem of finding a lower bound for d_n is known in the literature as the Erdös–Rankin problem. H. Cramér's famous conjecture is that $d_n = O\left((\log p_n)^2\right)$, but this is currently far out of reach.

In [?], [?], and [?], Rankin considered the problem of finding an upper bound for

$$\ell := \lim_{n\to\infty}\inf \frac{p_{n+1} - p_n}{\log p_n}.$$

Using the prime number theorem, one can easily show that $\ell \leq 1$, and in 1940, Erdös [?] proved that $\ell < 1$. In [?], Rankin proved that $\ell < \frac{42}{43} = 0.976\ldots$. Currently, the best result is $\ell < 0.248\ldots$, due to Maier [?] in 1985.

2.2. **Sums of Powers of Linear Forms.** Rankin wrote three papers on sums of powers of linear forms [?], [?], [?]. The principal problem was to find upper bounds for the minimal nontrivial values of such sums. Such questions are motivated by problems arising in the study of the "geometry of numbers." As is well known, one of the most powerful tools in the subject is Minkowski's Convex Body Theorem.

In its generality, this result is best possible. However, in special circumstances one may expect improvements in the relevant constants which have important number theoretic consequences. Rankin's two papers [?], [?], published in *Annals of Mathematics* in 1949, are written with such improvements in mind. Specifically, suppose that

$$L_j(x_1, \ldots, x_n) = \sum_{k=1}^{n} a_{jk}x_k,$$

for $1 \leq j \leq n$, are n linear forms with determinant $D \neq 0$. Furthermore, suppose that $n = r + 2s$, where r and s are integers for which $L_1, L_2, \ldots, L_r$ are real and where $L_{r+\nu}$

and $L_{r+s+\nu}, 1 \le \nu \le s$, are complex conjugates. If $\beta \ge 1$ is a real number, then for each nonzero integral point $P = (x_1, \dots, x_n)$, define $g(P)$ by

$$g(P) = \left\{ \sum_{j=1}^{n} |L_j(P)|^\beta \right\}^{1/\beta},$$

and let $M(g)$ be the minimum of $g(P)$ over all nonzero integral points P.

Rankin's papers are concerned with the problem of bounding $M(g)$ from above and determining

$$M_\beta := \sup\{M(g)\},$$

where the supremum is taken over all systems of n linear forms with fixed D, r, and s. Minkowski's Convex Body Theorem implies that

$$M(g) \le M_\beta \le 2J_\beta^{-1/n}, \tag{2.2}$$

where J_β is the volume of the body determined by those integral points for which $g(P) \le 1$.

In papers [**?**] and [**?**], Rankin improved on earlier works of van der Corput, Schaake, Hlawka, and Hua which lowered the upper bound in (**??**) for various β. For convenience, let $\alpha = 1/\beta$. In [**?**], Rankin proved that if $2 < \beta \le 2n$, then

$$M(g) \le M_\beta \le 2^{-\alpha}(1 + \alpha n)^{1/n}(1 + R_\beta)^{-1/n} \cdot 2J_\beta^{-1/n}, \tag{2.3}$$

where

$$R_\beta = \frac{2^{-\alpha n}}{1 + 2\alpha n} - (\sqrt{2} - 1)^{1+2\alpha n}\left(1 + \frac{\sqrt{2}}{1 + 2\alpha n}\right).$$

In [**?**], Rankin proved that if $1 \le \beta \le 2$, then

$$M(g) \le M_\beta \le (1 + Q_\beta)^{-1/n} \cdot 2^{\alpha - 1}(1 + (1 - \alpha)n)^{1/n} \cdot 2J_\beta^{-1/n}, \tag{2.4}$$

where

$$Q_\beta = (1 - \alpha)n(\sqrt{2} - 1)^{2n(1-\alpha)+2}.$$

For many choices of n and β, (**??**) and (**??**) represent substantial improvements on Minkowski's bound (**??**).

2.3. **Modular groups.** In several papers Rankin investigated properties of subgroups Γ of the full modular group $\Gamma(1) = SL_2(\mathbf{Z})$. Recall that Γ by definition is a congruence subgroup if it contains the principal congruence subgroup

$$\Gamma(N) = \{M \in \Gamma(1) \mid M \equiv I \pmod N\},$$

(where I is the identity matrix) for some natural number N. For instance, the Hecke subgroups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \ : \ c \equiv 0 \pmod N \right\}$$

and their conjugates

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \ : \ b \equiv 0 \pmod N \right\}$$

are congruence subgroups.

In [?], Rankin studied the question of whether a single linear congruence

$$Aa + Bb + Cc + Dd \equiv 0 \pmod{N}$$

determines a subgroup of $\Gamma(1)$. Here A, B, C, and D are fixed integers, N is a positive integer, and M denotes the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. As an example of his results, he proved that if $(N, 6) = 1$, then every subgroup arising in this way is conjugate to $\Gamma_0(N)$. If $(N, 6) > 1$, other possibilities occur. For instance, if $N = 4$, then

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \ : \ 2b + c \equiv 0 \pmod{4} \right\}$$

is a subgroup of $\Gamma(1)$ which is not conjugate to $\Gamma_0(4)$.

In another paper [?], Rankin studied, for positive integers n, the normal closure $\Delta(n)$ in $\Gamma(1)$ of the parabolic element T^n, where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(1)$. More specifically, $\Delta(n)$ is the subgroup of $\Gamma(1)$ generated by all the conjugate parabolic matrices $M^{-1}T^nM$, where the matrices M vary through $\Gamma(1)$. He provided a new and simple proof (using elementary properties of indefinite binary quadratic forms) of the classical result that $\Delta(n) = \Gamma(n)$ for $n \leq 5$, and that $\Delta(n)$ has infinite index in $\Gamma(n)$ for $n \geq 6$. These facts were first pointed out by M. Knopp [?], and they may also be deduced, with some effort, from classical works of Fricke and Klein [?, pp. 354–360].

In his paper [?] from 1954, Rankin derived several results on the fundamental domains of the so-called horocyclic subgroups $\Gamma \subset SL_2(\mathbf{R})$ (in German, "Grenzkreisgruppen"). These groups had been defined and studied extensively by H. Petersson (for example, see [?]). Among other results, Rankin's paper is best known for the theorem that a horocyclic group is a Fuchsian group of the first kind, a notion first introduced by H. Poincaré [?]. Consequently, this paper played a significant historical role by providing a bridge between the "Hamburger Schule" of Petersson and those researchers studying automorphic forms whose knowledge was based largely on Ford's book [?] (the only comprehensive book on automorphic forms available in English at the time).

Rankin also wrote an important paper on noncongruence subgroups of $\Gamma(1)$. The theory of modular functions on such groups remains a mystery, and is replete with open questions and problems (for example, see [?]). In [?], Rankin developed general facts about such groups, and rediscovered (see Section 6 of [?]) large families of noncongruence subgroups of finite index. These families were also given by Lehner [?] in his important monograph on automorphic functions and discontinuous groups.

2.4. **Eisenstein Series and Poincaré Series.** Rankin is best known for his substantial contributions to the theory of modular forms. His book [?] is one of the classic texts on the subject. A *modular form* on $\Gamma(1) = SL_2(\mathbf{Z})$ of weight k is any holomorphic function, say $f(z)$, on the upper half of the complex plane, $\mathcal{H} = \{z : \text{Im } z > 0\}$, for which

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),$$

for every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. (Here, for purposes of simplicity, we have assumed that the multiplier system is identically equal to 1 and that k is an integer.) Eisenstein series and Poincaré series are natural examples of modular forms. For each even integer $k \geq 4$, the Eisenstein series

$$E_k(z) := \frac{1}{2} \sum (cz+d)^{-k},$$

where the summation is over all coprime pairs of integers c and d, is a modular form of weight k on $\Gamma(1)$. These series are fundamental modular forms; in fact, the two Eisenstein series $E_4(z)$ and $E_6(z)$ generate the algebra of all modular forms on $\Gamma(1)$.

Poincaré series are generalizations of Eisenstein series. If $k \geq 4$ is even and m is an integer, then the Poincaré series $G_k(z, m)$ is defined by

$$G_k(z,m) = \frac{1}{2} \sum q^m|_k L,$$

where $q = \exp(2\pi i z)$, the sum is over all $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ with distinct bottom rows, and where the operator $|_k$ is defined by

$$f|_k L = (cz+d)^{-k} f(Lz).$$

Observe that these series contain the classical Eisenstein series arising from the case $m = 0$. These series are holomorphic on $\mathcal{H}$. Moreover, if $m \leq 0$, then $G_k(z, m)$ is non-zero and has a Fourier expansion of the form

$$G_k(z,m) = q^m + \sum_{n=1}^{\infty} a_{k,m}(n) q^n.$$

If $m > 0$ and $k \in \{4, 6, 8, 10, 14\}$, then it turns out that $G_k(z, m)$ is identically zero. On the other hand, if $k \geq 12$ is even and d_k denotes the dimension of the space of cusp forms of weight k on $\Gamma(1)$, then $G_k(z, m)$ is not identically zero for each $1 \leq m \leq d_k$. It is natural to seek an extension for larger ranges of m. In [?], Rankin proved that there are positive integers k_0 and B with the property that $G_k(z, m)$ is not identically zero whenever $k \geq k_0$ is an even integer and when $m \leq k^2 \exp(-B \log k / \log \log k)$. These results have been generalized by C. J. Mozzochi [?] and J. Lehner [?] to modular forms on other groups.

Rankin was also concerned about the zeros of Eisenstein series [?]. It can be shown by direct elementary calculations that $E_4(e^{2\pi i/3}) = 0$ and that $E_6(i) = 0$. K. Wohlfahrt [?] asked if all the zeros of $E_k(z)$ in the standard fundamental region, $\{z : |z| \geq 1, -\frac{1}{2} \leq \text{Re } z < \frac{1}{2}\}$, are on the unit circle, and he showed this was the case for $4 \leq k \leq 26$. Rankin [?] proved this conjecture for $k = 28, 30, 32, 34, 38$ and showed that, in any case, all the zeros of $E_k(z)$ are "near" $|z| = 1$. Shortly thereafter, F. K. C. Rankin (Robert's daughter, Fenny) and H. P. F. Swinnerton-Dyer [?] proved that indeed all the zeros of $E_k(z)$ are on $|z| = 1$. The zeros of Eisenstein series are important in establishing Ramanujan's formulas for the power series coefficients of quotients of Eisenstein series. In particular, see papers by Hardy and Ramanujan [?], [?, pp. 310–321] and Berndt, P. Bialek, and A. J. Yee [?]. In 1982, Rankin greatly extended the work of his daughter and Swinnerton-Dyer. He proved [?] that the Poincaré series on $\Gamma(1)$ with order $m \leq 0$ have their zeros on the arc mentioned above. Furthermore, T. Asai, M. Kaneko and H. Ninomiya [?] showed that all the zeros of the function $j - 744$ and its images under the usual Hecke operators lie on the unit circle, where j is the classical modular invariant. At present, this phenomenon does not seem to be fully understood.

We conclude this section by briefly mentioning a series of four papers Rankin wrote late in his career on the diagonalization of Eisenstein series. These works pertain to modular forms on congruence subgroups of $\Gamma(1)$. In 1970, A. O. L. Atkin and J. Lehner published their important paper "Hecke operators on $\Gamma_0(m)$" [?]. This paper, which was subsequently generalized by T. Miyake [?] and W.–C. W. Li [?], shows that spaces of integer weight cusp forms decompose naturally as direct sums of spaces of "oldforms" together with a space of "newforms." Each space of newforms admits a basis of cusp forms which are normalized Hecke eigenforms (also known as newforms). Moreover, these newforms satisfy the "multiplicity one" phenomenon, the assertion that to each pair of newforms there are infinitely many primes p for which the eigenvalues of the Hecke operator T_p differ.

In papers [?], [?], [?], and [?], Rankin considered the problem of developing an analogous theory for the complementary spaces, the spaces of Eisenstein series. He obtained necessary and sufficient conditions dictating whether the analogous theorems continue to hold for Eisenstein series. For example, in [?] Rankin proved that the Eisenstein series on congruence subgroups of square-free level N are diagonalizable for all the Hecke operators.

2.5. **Rankin-Cohen Brackets and Differential Operators on Modular forms.**

Rankin conducted fundamental work on the theory of differential operators acting on spaces of modular forms. We begin with a basic fact: If $f(z)$ is a modular form of weight k, then in general the first derivative $f'(z)$ is not a modular form (and the higher derivatives are neither). Indeed, $f'(z)$ transforms as

$$f'\left(\frac{az+b}{cz+d}\right) = kc(cz+d)^{k+1}f(z) + (cz+d)^{k+2}f'(z),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the relevant group. However, if we can find another function which transforms with the additional summand $c(cz+d)$ in a similar way as above, then perhaps we can "combine" both and produce a true modular form. For example, the "nearly modular" Eisenstein series

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}$$

of weight 2, where $\sigma_1(n) = \sum_{d|n} d$, transforms as

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 E_2(z) + \frac{12c}{2\pi i}(cz+d)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. Consequently, we obtain the classical and well-known fact that the differential operator

$$12 \cdot \frac{1}{2\pi i} \frac{d}{dz} - kE_2$$

maps modular forms of weight k to those of weight $k+2$.

In [?] and [?], Rankin substantially generalized these observations. In [?], he initiated the study and classification of those polynomials $P(X_0, X_1, \ldots, X_n)$ which have the property that, for each modular form f of weight $k > 0$ on the (horocyclic) group Γ, the function $P(f, f', \ldots, f^{(n)})$ is a modular form on Γ (of some weight k' depending on k). In fact, using certain Wronskian determinants, he constructed a basis for all modular forms which are polynomials (or rational functions) in a given f and its derivatives. As an application, he proved that every meromorphic modular form on $\Gamma(1)$ is a rational function in Δ and its first three derivatives, where Δ is the usual discriminant function of weight 12. In the same paper, he deduced the elegant differential equation

$$13\Delta_1^4 + 10\Delta^2\Delta_1\Delta_2 - 24\Delta\Delta_1^2\Delta_2 - 2\Delta^3\Delta_4 + 3\Delta^2\Delta_2^2 = 0$$

(where we have written Δ_r instead of $\Delta^{(r)}$).

In [?], Rankin obtained generalizations of these results for collections of modular forms. One important special case of these results was independently discovered by H. Cohen [?] many years later. There are many combinatorial consequences of these constructions. Suppose that f_1 and f_2 are entire modular forms on Γ of weights k_1 and k_2, respectively. If ν is a non-negative integer, then

$$F_\nu(f_1, f_2) = \sum_{\mu=0}^{\nu} (-1)^{\nu-\mu} \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)\Gamma(k_2+\nu)}{\Gamma(k_1+\mu)\Gamma(k_2+\nu-\mu)} f_1^{(\mu)} f_2^{(\nu-\mu)}$$

is a modular form of weight $k_1 + k_2 + 2\nu$ on Γ. Moreover, it turns out that this is a cusp form if ν is positive. These bilinear operators F_ν are usually called the "Rankin-Cohen Brackets," and they play an important role in the theory of modular forms, and he [?] used them in his work on class number relations. Cohen also used them in his proof of the famous Doi-Naganuma "lifting theorem" [?]. In fact, Cohen's result is a more general lifting theorem. These brackets play an important role in recent work of P. Cohen, Y. Manin and D. Zagier [?] on liftings of automorphic forms to certain pseudodifferential operators.

These brackets are also essential in the theory of periods of modular forms, which in turn are vital to questions in arithmetic geometry such as the Birch and Swinnerton-Dyer Conjecture. In a famous paper, Zagier [?] gave a different proof of the well-known result of Eichler, Shimura, and Manin on the algebraicity of quotients of periods of cuspidal Hecke eigenforms f [?, Chap. 5]. To obtain this result, he computed the Petersson scalar product of f against the Rankin-Cohen bracket F_ν of two Eisenstein series, a quantity which may be interpreted as a product of the two periods of f (the case $\nu = 0$ was already due to Rankin [?]).

The Rankin-Cohen brackets have been generalized to the context of Siegel modular forms and Jacobi forms; this provides further evidence of their importance. More precisely, Ibukiyama [?] has generalized these operators to certain differential operators defined on r-tuples of Siegel modular forms of degree n with values in spaces of vector-valued Siegel modular forms, and he has characterized them using invariant pluri-harmonic polynomials. The case $r > 2$ was studied in order to obtain vector-valued Siegel modular forms. If $r = 2$ and n is arbitrary, the existence and uniqueness of these operators and recursive formulas were given by Eholzer and Ibukiyama [?]. In the case where $n = r = 2$, Choie and Eholzer [?] have provided the expected explicit formulas.

These differential operators also play an important role in the theory of Jacobi forms; for example, see the famous monograph by Eichler and Zagier on the subject [?]. Rankin-Cohen operators, in this context, are also considered in the recent work by Choie and Eholzer [?] and Böcherer [?] . In Böcherer's work the bracket operators on Jacobi forms are reduced to Maass operators (this works also for Siegel modular forms).

2.6. **The Rankin-Selberg Method.** Rankin is perhaps best known for his role in originating what has become to be known as the "Rankin-Selberg method." This method has its origins in Rankin's paper in 1939 on Ramanujan's tau-function [?], and in Selberg's paper [?], [?] in 1940, and was first developed for congruence subgroups. In 1965, Selberg [?], [?], in his paper on the estimation of coefficients of modular forms, extended the method to subgroups of finite index. Roughly, the Rankin-Selberg method is a very powerful tool which allows one to obtain, under rather weak hypotheses, the meromorphic continuation (and a functional equation) for the Mellin transform of the constant term of a Γ-invariant function, where Γ is an appropriate subgroup.

Let us describe this method in some detail in the special case of the full modular group $\Gamma(1)$. First recall the definition of the non-holomorphic Eisenstein series

$$E(z,s) = \frac{1}{2}\sum \frac{y^s}{|mz+n|^{2s}} \qquad (z \in \mathcal{H},\ \mathrm{Re}(s) > 1),$$

where the summation extends over all nonzero pairs of integers (m,n), and where $z = x+iy$. Obviously $E(z,s)$ is $\Gamma(1)$-invariant. As is well-known, if we put

$$E^*(z,s) = \pi^{-s}\Gamma(s)E(z,s) \qquad (\mathrm{Re}(s) > 1),$$

then $E^*(z,s)$ has a meromorphic continuation in s to the entire complex plane. Moreover, it is holomorphic up to simple poles at $s = 0, 1$, with residue $\frac{1}{2}$ (independent of z) at $s = 1$. In addition, we have the functional equation

$$E^*(z, 1-s) = E^*(z,s).$$

Indeed, using the Poisson summation formula, we can show that

$$E^*(z,s) = y^s\zeta^*(2s) + y^{1-s}\zeta^*(2s-1) + 2\sqrt{y}\sum_{N\neq 0}|N|^{s-1/2}\sigma_{1-2s}(N)K_{s-1/2}(2\pi|N|y)e^{2\pi i N x},$$

where we have put

$$\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \ \ \sigma_s(N) = \sum_{d|N, d>0} d^s,$$

and where $K_{s-1/2}$ denotes the modified Bessel function of order $s - 1/2$. The claims made above easily follow from the fact that K_s is an even function of s.

Suppose now that $u : \mathcal{H} \to \mathbf{C}$ is a smooth $\Gamma(1)$-invariant function such that $u(z) = \mathcal{O}(y^{-\alpha})$ as $y \to \infty$, for some $\alpha > 0$. Let

$$u(z) = \sum_{n \in \mathbf{Z}} a_n(y) e^{2\pi i n x}$$

be the Fourier expansion of u, and let

$$A_0(s) = \int_0^\infty a_0(y) y^{s-1} dy \qquad (0 < \mathrm{Re}(s) < \alpha)$$

be the Mellin transform of the constant term $a_0(y)$. Then we have the identity

$$\int_{\mathcal{F}} u(z) E(z,s) d\omega = \zeta(2s) A_0(s-1) \qquad (1 < \mathrm{Re}(s) < \alpha + 1), \tag{2.5}$$

where $\mathcal{F}$ is a fundamental domain for $\Gamma(1)$ and $d\omega = \frac{dxdy}{y^2}$ is the $SL_2(\mathbf{R})$-invariant measure. In particular, using the properties of the Eisenstein series $E(z,s)$, we can deduce that $\zeta(2s)A_0(s-1)$ has a meromorphic continuation with at most a (simple) pole at $s = 1$ of residue equal to

$$\frac{\pi}{2} \int_{\mathcal{F}} u(z)\, d\omega.$$

By inserting the appropriate Γ-factor, we obtain a functional equation for

$$\zeta(2s) A_0(s-1).$$

The proof of (**??**) is elegant and simple, and this proof is now a standard tool in analytic number theory. We first observe that $E(z,s)$, up to the factor $\frac{1}{2}\zeta(2s)$, can be rewritten as the sum of $\mathrm{Im}(M \circ z)^s$, with M ranging over $\Gamma(1)_\infty \backslash \Gamma(1)$, where $\Gamma(1)_\infty$ is the subgroup of upper triangular matrices in $\Gamma(1)$. Then we unfold the integral, observing that a fundamental domain for $\Gamma(1)_\infty$ is the half-strip $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $y > 0$.

In [**?**], Rankin applied this method in the case $u(z) = y^k |f(z)|^2$, where $f(z) = \sum_{n=1}^\infty a(n) e^{2\pi i n z}$ is a nonzero cusp form of weight k on $\Gamma(1)$. As is easily seen

$$A_0(s) = \frac{\Gamma(s+k)}{(4\pi)^{s+k}} L_2(f, s+k) \qquad (\mathrm{Re}(s) >> 0),$$

in this case, where

$$L_2(f,s) = \sum_{n \geq 1} \frac{|a(n)|^2}{n^s} \qquad (\mathrm{Re}(s) >> 0)$$

is the so-called Rankin-Dirichlet series. Observe that the coefficients of $L_2(f,s)$ are nonnegative (and so $L_2(f,s)$ must have a singularity at its abscissa of convergence if the latter is not $-\infty$). Since $\zeta(s) \neq 0$ on $\mathrm{Re}(s) = 1$, it follows immediately that $L_2(f,s)$ extends to a holomorphic function except for a simple pole at $s = k$. The residue at $s = k$ is an absolute constant times the Petersson scalar product $\langle f, f \rangle$, a concept due earlier to Petersson but discovered independently by Rankin. Furthermore, $L_2(f,s)$, when completed with the appropriate Γ-factors, satisfies a functional equation under $s \mapsto 2k - 1 - s$. Using standard and classical methods from analytic number theory, Rankin deduced that

$$\sum_{n\le x} |a(n)|^2 = \alpha\langle f, f\rangle x^k + \mathcal{O}(x^{k-2/5}),$$

where $\alpha > 0$ is an absolute constant. In particular, it follows that

$$a(n) = \mathcal{O}(n^{k/2-1/5}),$$

for all cusp forms of positive weight on subgroups of finite index in $\Gamma(1)$, and for any multiplier system. This estimate is a considerable improvement upon Hecke's [**?**, Satz 8], [**?**, p. 484] well-known estimate $a(n) = \mathcal{O}(n^{k/2})$ for the Fourier coefficients of f. Note that Weil's bound for Kloosterman sums [**?**] implies that $a(n) = \mathcal{O}_\epsilon(n^{k/2-1/4+\epsilon})$ for any $\epsilon > 0$ (see L. A. Parson's paper [**?**] for conditions on f ensuring this bound), while the celebrated theorem of Deligne [**?**], proving the Ramanujan-Petersson conjecture, asserts that $a(n) = \mathcal{O}_\epsilon(n^{k/2-1/2+\epsilon})$ ($\epsilon > 0$) on subgroups $\Gamma_0(N)$, where k is any positive integer, and the multiplier system is any Dirichlet character modulo N. The latter estimate, by the way, is best possible as was demonstrated in another paper by Rankin [**?**]. Rankin [**?**], [**?**] wrote two excellent surveys on the coefficients of cusp forms, with the latter paper concentrating on the Ramanujan tau-function.

The Rankin-Selberg method has many beautiful applications to both holomorphic and nonholomorphic modular forms. For example, using the explicit Fourier expansion of $E(z, s)$, we can obtain a rather simple and independent proof of the fact that $\zeta(s)$ has no zeros on the line $\mathrm{Re}(s) = 1$ (which is equivalent to the Prime Number Theorem). The method of Rankin-Selberg also generalizes to higher groups, such as GL_n. It is a powerful and vital tool in the theory of automorphic forms; for example, see [**?**].

The Rankin-Dirichlet series $L_2(f, s)$ is, up to a factor of certain Riemann zeta functions, the second symmetric power L-function attached to the automorphic representation π_f associated with f, when f is a Hecke eigenform. The symmetric power L-functions are natural objects which are presently the focus of intense study, and so the construction of the second symmetric power L-function is an extremely nice feature of the Rankin-Selberg theory. Many open questions remain. For example, it is conjectured that the higher symmetric power L-functions attached to such π_f have holomorphic continuations to $\mathbf{C}$. Although there are celebrated cases in which the conjecture has been settled (for example, in the case of the symmetric square, the conjecture was proved by Shimura [**?**], and was later proved in a different way by Zagier [**?**]), little else is known. For an excellent survey, see [**?**].

2.7. **Sums of Squares, Quadratic Forms, and Sphere Packings.** Rankin wrote extensively on sums of squares, quadratic forms, and sphere packings. We begin by discussing his work on sums of squares. If s is a positive integer and n is a nonnegative integer, then let

$$r_s(n) := \#\{(x_1, x_2, \ldots, x_s) \in \mathbf{Z}^s \ : \ x_1^2 + x_2^2 + \cdots + x_s^2 = n\}.$$

By the works of Jacobi, Legendre, and others, for certain small even s and positive n, there are elegant formulas expressing $r_s(n)$ in terms of "divisor functions." For example, if $n \geq 1$, then

$$r_2(n) = 4 \sum_{d|n} \chi_{-1}(d),$$

where $\chi_{-1}(n) = (-1)^{(n-1)/2}$ for odd n, and is zero for even n. Like his predecessors, Rankin was interested in computing such exact formulas. In one his last papers [**?**], he provided a beautiful account of some unpublished research from 1944 on elementary methods for deducing such divisor function formulas.

These elegant formulas arise naturally in the theory of modular forms. If $\Theta(z) = 1 + 2q + 2q^4 + 2q^9 + \cdots$, then it is easy to see that

$$\Theta(z)^s = \sum_{n=0}^{\infty} r_s(n) q^n.$$

Since $\Theta(z)$ is an entire modular form of weight $1/2$, $\Theta(z)^s$ is an entire modular form of weight $s/2$. By the theory of modular forms, for each positive integer s,

$$\Theta(z)^s = \mathcal{E}_s(z) + f_s(z),$$

where $\mathcal{E}_s(z)$ is an explicit and easily computable Eisenstein series, and $f_s(z)$ is a cusp form. As a consequence, if s is even, then

$$r_s(n) = \rho_s(n) + c_s(n),$$

where $\rho_s(n), n \geq 0$, the coefficients of $\mathcal{E}_s(z)$, can be written as explicit divisor functions, and $c_s(n), n \geq 1$, are the coefficients of the cusp form $f_s(z)$. Whenever $f_s(z) \neq 0$, the coefficients $c_s(n)$ are appreciably smaller than $\rho_s(n)$ for large n.

It is easy to see that elegant divisor function formulas occur for those s for which $f_s(z) = 0$. Recent theorems of the third author [**?**], S. Milne [**?**], and D. Zagier [**?**] provide infinite families of formulas for $r_s(n)$ in terms of more complicated expressions of divisor functions. With a delightfully simple argument [**?**], Rankin answered the natural question: How often is $f_s(z) = 0$? He proved that $f_s(z) \neq 0$ for all $s > 8$. When $f_s(z)$ is nonzero, there are nonetheless examples which have nice descriptions. Extending earlier work of J. W. L. Glaisher [**?**], Rankin obtained such an elegent formula when $s = 20$ [**?**].

Rankin wrote two papers on the minimal points of perfect quadratic forms [**?**], [**?**]. A positive definite quadratic form $f(\mathbf{x})$ is said to be "perfect" if it is completely determined by its set of minimal points. Suppose that $f(\mathbf{x})$ is a perfect positive definite quadratic form in n-variables, and that $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ are its minimal points. It is well known that $n(n+1)/2 \leq p \leq 2^n - 1$. Now form all the possible $n \times n$ determinants Δ_μ having the x_ν's as row vectors. It is a classical result that

$$0 \leq \Delta_\mu^2 \leq \gamma_n^n, \tag{2.6}$$

where γ_n is Hermite's constant. Recall that Hermite's constant is the greatest minimum of any quadratic form in n variables of determinant 1. In 1954, H. Davenport and G. L. Watson [**?**] proved the existence of perfect forms for which the upper bound above is fairly sharp. Rankin considered the natural question of finding a smaller upper bound for $\Delta^2 := \min\{\Delta_\mu^2\}$. Improving on his earlier work in [?], he proved that $\Delta^2 = 1$ if $n \leq 6$, and that

$$\Delta^2 \leq \frac{n(n!)}{(n+1)(2n)!} \cdot \{(n+1)\gamma_n\}^n,$$

for $n > 6$ [?]. For large n, this is a considerable improvement over the upper bound in (??).

Rankin wrote several papers on the "sphere packing" problem. There is now a vast literature on the subject [?]. Here we discuss only Rankin's paper of 1947 in *Annals of Mathematics* [?], extending earlier work of H. F. Blichfeldt [?]. Let C_n be an n-dimensional hypercube of edge L, and let $N(L)$ denote the maximum number of unit hyperspheres which can be "packed" in C_n. The packing constant ρ_n is defined by

$$\rho_n := \lim_{L\to\infty} K_n N(L)/L^n,$$

where $K_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$. Since a packing does not need to be regular (i.e., the centers of the hyperspheres need not form a lattice), we may also define the regular packing constant ρ_n' by

$$\rho_n' := \lim_{L\to\infty} K_n N'(L)/L^n,$$

where $N'(L)$ denotes the number of unit hyperspheres which can be packed in C_n so that their centers form a lattice. In [?] Rankin proved that both ρ_n and ρ_n' exist and are intimately connected to Hermite's constant γ_n.

In the same paper Rankin considered Kepler's Conjecture, perhaps the most famous of the sphere packing problems. The conjecture asserts that the density of a packing of congruent spheres in three dimensions is never greater than $\pi/\sqrt{18} \sim 0.74048\ldots$. This conjecture was proved in 1998 by T. C. Hales [?]. Rankin obtained $0.827\ldots$ as an upper bound for these densities, a record for Kepler's Conjecture which stood until 1958 when C. A. Rogers lowered the upper bound to $0.779\ldots$ [?].

In summary, Rankin made seminal contributions in a wide variety of areas. We have expounded on some of his most notable accomplishments, but, undoubtedly, we have overlooked some important results.

3. Acknowledgements

In writing this article the authors have profited immensely from conversations and correspondence with Rankin's daughter, Fenny Smith, and obituaries written by T. Dalyell [?], D. Martin [?], D. Munn [?], and I. Tweddle [?]. The authors also thank Antal Balog, Heini Halberstam, and Marvin Knopp for their helpful comments and suggestions.

References

[1] G. E. Andrews, "An introduction to Ramanujan's lost notebook," *Amer. Math. Monthly* **86** (1979), 89–108; reprinted in [?, pp. 165–184].

[2] T. Asai, M. Kaneko and H. Ninomiya, "Zeros of certain modular functions and an application," *Comm. Math. Univ. Sancti Pauli* **46** (1997), 93–101.

[3] A. O. L. Atkin and J. Lehner, "Hecke operators on $\Gamma_0(m)$," *Math. Ann.* **185** (1970), 134–160.

[4] B. C. Berndt, "The remaining 40% of Ramanujan's lost notebook," in *Number Theory and its Applications*, Surikaisekikenkyuusho Kokyuuroku, No. 1060, RIMS Kyoto University, Kyoto, 1998, pp. 111–118.

[5] B. C. Berndt, P. B. Bialek, and A. J. Yee, "Formulas of Ramanujan for the power series coefficients of certain quotients of Eisenstein series," International Mathematics Research Notices **2002**, No. 21, 1077–1109.

[6] B. C. Berndt and K. Ono, "Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary," Sém. Lotharingien de Combinatoire 42 (1999), 63 pp.; in *The Andrews Festschrift*, D. Foata and G.–N. Han, eds., Springer–Verlag, Berlin, 2001, pp. 39–110.

[7] B. C. Berndt and R. A. Rankin, *Ramanujan: Letters and Commentary*, American Mathematical Society, Providence, RI, 1995; London Mathematical Society, London, 1995.

[8] B. C. Berndt and R. A. Rankin, *Ramanujan: Essays and Surveys*, American Mathematical Society, Providence, RI, 2001; London Mathematical Society, London, 2001.

[9] H. F. Blichfeldt, "The minimum value of quadratic forms and the closest packing of spheres," *Math. Ann.* **101** (1929), 605–608.

[10] S. Böcherer, "Bilinear holomorphic differential operators for the Jacobi group," *Comm. Math. Univ. Sancti Pauli* **47** (1998), 135–154.

[11] D. Bump, "The Rankin-Selberg method: A survey," in *Number Theory, Trace Formulas and Discrete Groups, Symposium in Honor of Atle Selberg, Oslo, Norway*, Academic Press, Boston, 1989, pp. 49–109.

[12] Y. Choie and W. Eholzer, "Rankin-Cohen operators for Jacobi and Siegel modular forms, *J. Number Theory* **68** (1998), 160–177.

[13] H. Cohen, "Sums involving the values at negative integers of L-functions of quadratic characters," *Math. Ann.* **217** (1975), 271–285.

[14] H. Cohen, "A lifting of modular forms in one variable to Hilbert modular forms in two variables," in *Modular Functions of One Variable, VI*, J.–P. Serre and D. B. Zagier, eds., Lecture Notes in Mathematics, Vol. 627, Springer–Verlag, Berlin (1977), pp. 175–196.

[15] P. Cohen, Y. Manin and D. Zagier, "Automorphic pseudodifferential operators," in *Algebraic Aspects of Integrable Systems: in Memory of Irene Dorfman*, A. S. Fokas, et al., eds., Progr. Nonlinear Diff. Eqn. Appl. **26**, Birkhäuser, Boston, 1997, pp. 17–47.

[16] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Springer–Verlag, New York, 1999.

[17] T. Dalyell, "Professor Robert Rankin," *The Independent*, 10 February 2001, p. 7.

[18] H. Davenport and G. L. Watson, "The minimal points of a positive definite quadratic form," *Mathematika* **1** (1954), 14–17.

[19] P. Deligne, "La conjecture de Weil. I.," Inst. Hautes Études Sci. Publ. Math. (1974), No. 43, 273–307.

[20] W. Eholzer and T. Ibukiyama, "Examples of invariant pluri-harmonic polynomials and Rankin-Cohen type differential operators," *Int. J. Math.* **9** (1998), 443–463.

[21] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progr. in Math. **55**, Birkhäuser, Boston, 1985.

[22] P. Erdös, "On the difference of consecutive primes," *Quart. J. Math. (Oxford)* **6** (1935), 124–128.

[23] P. Erdös, "The difference of consecutive primes," *Duke Math. J.* **6** (1940), 438–441.

[24] L. R. Ford, *Automorphic Functions*, McGraw-Hill, New York, 1929.

[25] J. W. L. Glaisher, "On the number of representations of a number as a sum of $2r$ squares, where $2r$ does not exceed 18," *Proc. London Math. Soc.* (2) **5** (1907), 479–490.

[26] J. Hadamard, *Non-Euclidean Geometry in the Theory of Automorphic Functions*, J. J. Gray and A. Shenitzer, eds., American Mathematical Society, Providence, RI, 1999; London Mathematical Society, London, 1999.

[27] T. C. Hales, "An overview of the Kepler conjecture," preprint.

[28] G. H. Hardy and S. Ramanujan, "On the coefficients in the expansions of certain modular functions," *Proc. Roy. Soc. London, Sect. A* **95** (1918), 144–155.
[29] E. Hecke, "Theorie der Eisensteinschen Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik," *Abh. Math. Sem. Univ. Hamburg* **5** (1927), 199–224.
[30] E. Hecke, *Mathematische Werke*, Vandenhoeck & Ruprecht, Göttingen, 1970.
[31] T. Ibukiyama, "On differential operators on automorphic modular forms and invariant pluriharmonic polynomials," *Comm. Math. Univ. Sancti Pauli* **48** (1999), 103–117.
[32] F. Klein und R. Fricke, *Vorlesungen ueber die Theorie der Elliptischen Modulfunktionen*, Bd. 1, B. G. Teubner, Leipzig, 1890.
[33] M. Knopp, "A note on subgroups of the modular group," *Proc. Amer. Math. Soc.* 14 (1963), 95–97.
[34] S. Lang, *Introduction to Modular Forms*, Grundlehren Math. Wiss., No. 222, Springer–Verlag, Berlin, 1995.
[35] J. Lehner, *Discontinuous Groups and Automorphic Functions*, Math. Surveys **VIII**, American Mathematical Society, Providence, RI, 1964.
[36] J. Lehner, "On the nonvanishing of Poincaré series," *Proc. Edinburgh Math. Soc.* (2) **23** (1980), 225–228.
[37] W.-C. W. Li, "Newforms and functional equations," *Math. Ann.* **212** (1975), 285–315.
[38] R. A. Mac Fhraing, "Áireamh muinntir Fhinn is Dhubhain, Agus sgeul Iosephuis is an dá fhichead Iudhaich," *Proc. Roy. Irish Acad. Sect. A* **52** (1948), 87–93.
[39] H. Maier, "Primes in short intervals," *Michigan Math. J.* **32** (1985), 221–225.
[40] H. Maier and C. Pomerance, "Unusually large gaps between consecutive primes," *Trans. Amer. Math. Soc.* **322** (1990), 201–237.
[41] D. Martin, "Prof. Robert A. Rankin," *The Scotsman*, 5 February 2001, p. 14.
[42] S. Milne, "Infinite families of exact sums of squares formulas, Jacobi elliptic functions, continued fractions, and Schur functions," *Ramanujan J.* **6** (2002), 7–149.
[43] T. Miyake, "On automorphic forms on GL_2 and Hecke operators," *Ann. of Math.* (2) **94** (1971), 174–189.
[44] C. J. Mozzochi, "On the nonvanishing of Poincaré series," *Proc. Edinburgh Math. Soc.* (2) **32** (1989), 131–137.
[45] D. Munn, "Robert Rankin," *The Herald*, February 3, 2001, p. 16.
[46] K. Ono, "Representation of integers as sums of squares," *J. Number Thy.* **95** (2002), 253–258.
[47] L. A. Parson, "Generalized Kloosterman sums and the Fourier coefficients of cusp forms," *Trans. Amer. Math. Soc.* **217** (1976), 329–350.
[48] H. Petersson, "Zur analytischen theorie der Grenzkreisgruppen I," *Math. Ann.* **115** (1938), 23–67.
[49] J. Pintz, "Very large gaps between consecutive primes," *J. Number Theory* **63** (1997), 286–301.
[50] S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.
[51] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
[52] F. K. C. Rankin and H. P. F. Swinnerton-Dyer, "On the zeros of Eisenstein series," *Bull. London Math. Soc.* **2** (1970), 169–170.
[53] R. A. Rankin, "The difference between consecutive prime numbers," *J. London Math. Soc.* **13** (1938), 242–247.
[54] R. A. Rankin, "Contributions to the theory of Ramanujan's function $\tau(n)$ and similar functions. I. The zeros of the function $\sum_{n=1}^{\infty} \tau(n)/n^s$ on the line $\mathcal{R}s = 13/2$," Proc. Cambridge Philos. Soc. **35** (1939), 351–356.
[55] R. A. Rankin, "Contributions to the theory of Ramanujan's function $\tau(n)$ and similar functions. II. The order of the Fourier coefficients of integral modular forms," Proc. Cambridge Philos. Soc. **35** (1939), 357–373.
[56] R. A. Rankin, "Contributions to the theory of Ramanujan's function $\tau(n)$ and similar functions. III. A note on the sum of the Fourier coefficients of integral modular forms," Proc. Cambridge Philos. Soc. **36** (1940), 150–151.

[57] R. A. Rankin, “The difference between consecutive prime numbers. II,” *Proc. Cambridge Math. Soc.* **36** (1940), 255–266.
[58] R. A. Rankin, “On the closest packing of spheres in n dimensions,” *Ann. of Math.* (2) **48** (1947), 1062–1081.
[59] R. A. Rankin, “The difference between consecutive prime numbers. III,” *J. London Math. Soc.* **22** (1947), 226–230.
[60] R. A. Rankin, “A campanological problem in group theory,” *Proc. Cambridge Philos. Soc.* **44** (1948), 17–25.
[61] R. A. Rankin, “On sums of powers of linear forms. III,” *Nederl. Akad. Wetensch., Proc.* **51** (1948), 846–853.
[62] R. A. Rankin, “On sums of powers of linear forms. I,” *Ann. of Math.* (2) **50** (1949), 691–698.
[63] R. A. Rankin, “On sums of powers of linear forms. II,” *Ann. of Math.* (2) **50** (1949), 699–704.
[64] R. A. Rankin, “The mathematical theory of the motion of rotated and unrotated rockets,” *Philos. Trans. Roy. London. Ser. A* **241** (1949), 457–585.
[65] R. A. Rankin, “The difference between consecutive prime numbers. IV,” *Proc. Amer. Math. Soc.* **1** (1950), 143–150.
[66] R. A. Rankin, “The scalar product of modular forms,” *Proc. London Math. Soc.* (3) **2** (1952), 198–217.
[67] R. A. Rankin, “A minimum problem for the Epstein zeta-function,” *Proc. Glasgow Math. Assoc.* **1** (1953), 149–158.
[68] R. A. Rankin, “On horocyclic groups,” *Proc. London Math. Soc.* (3) **4** (1954), 219–234.
[69] R. A. Rankin, “The construction of automorphic forms from derivatives of a given form,” *J. Indian Math. Soc. (N.S.)* **20** (1956), 103–116.
[70] R. A. Rankin, “The construction of automorphic forms from the derivatives of given forms,” *Michigan Math. J.* **4** (1957), 181–186.
[71] R. A. Rankin, “On the minimal points of positive definite quadratic forms,” *Mathematika* **3** (1956), 15–24.
[72] R. A. Rankin, “Oran na comhachaig,” *Trans. Gaelic Soc. Glasgow* **5** (1958), 122–171.
[73] R. A. Rankin, “On the representation of a number as the sums of any number of squares, and in particular of twenty,” *Acta Arith.* **7** (1961/62), 399–407.
[74] R. A. Rankin, “The difference between consecutive prime numbers. V,” *Proc. Edinburgh Math. Soc.* (2) **13** (1962/63), 331–332.
[75] R. A. Rankin, “On the minimal points of perfect quadratic forms,” *Math. Z.* **84** (1964), 228–232.
[76] R. A. Rankin, “Sums of squares and cusp forms,” *Amer. J. Math.* **87** (1965), 857–860.
[77] R. A. Rankin, “A campanological problem in group theory. II,” *Proc. Cambridge Philos. Soc.* **62** (1966), 11–18.
[78] R. A. Rankin, “George Neville Watson,” *J. London Math. Soc.* **41** (1966), 551–565.
[79] R. A. Rankin, “Lattice subgroups of free congruence groups,” *Invent. Math.* **2** (1967), 215–221.
[80] R. A. Rankin, “The zeros of Eisenstein series,” *Publ. Ramanujan Inst.* No. 1 (1968/69), 137–144.
[81] R. A. Rankin, *The Modular Group and Its Subgroups*, The Ramanujan Institute, Madras, 1969.
[82] R. A. Rankin, “Subgroups of the modular group defined by a single linear congruence,” *Acta Arith.* **24** (1973), 313–323.
[83] R. A. Rankin, “Subgroups of the modular group generated by parabolic elements of constant amplitude,” *Acta Arith.* **18** (1971), 145–151.
[84] R. A. Rankin, “An Ω result for the coefficients of cusp forms,” *Math. Ann.* **203** (1973), 239–250.
[85] R. A. Rankin, “Ramanujan’s unpublished work on congruences,” in *Modular Functions of One Variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, J.-P. Serre and D. B. Zagier, eds., Lecture Notes in Math., Vol. 601, Springer–Verlag, Berlin, 1977, pp. 3–15.
[86] R. A. Rankin, *Modular Forms and Functions*, Cambridge University Press, Cambridge, 1977.
[87] R. A. Rankin, “The vanishing ot Poincaré series,” *Proc. Edinburgh Math. Soc.* (2) **23** (1980), 151–161.
[88] R. A. Rankin, “The zeros of certain Poincaré series,” *Compositio Math.* **46** (1982), 255–272.

[89] R. A. Rankin, "Ramanujan's manuscripts and notebooks," *Bull. London Math. Soc.* **14** (1982), 81–97; reprinted in [**?**, pp. 117–128].

[90] R. A. Rankin, "The first hundred years (1883–1983)," *Proc. Edinburgh Math. Soc.* **26** (1983), 135–150.

[91] R. A. Rankin, "Ramanujan as a patient," *Proc. Indian Acad. Sci.* (*Math. Sci.*) **93** (1984), 79–100; reprinted in [**?**, pp. 41–64].

[92] R. A. Rankin, "George Campbell Hay as I knew him," *Chapman* 40 **8** (1985), 1–12.

[93] R. A. Rankin, "Fourier coefficients of cusp forms," *Math. Proc. Cambridge Philos. Soc.* **100** (1986), 5–29.

[94] "Ramanujan's tau-function and its generalizations," in *Ramanujan Revisited*, G. E. Andrews, R. A. Askey, B. C. Berndt, K. G. Ramanathan, and R. A. Rankin, eds., Academic Press, Boston, 1988, pp. 245–268.

[95] R. A. Rankin, "Diagonalizing Eisenstein series. I," in *Analytic Number Theory (Allerton Park, IL,* 1989*)*, B. C. Berndt, H. G. Diamond, H. Halberstam, and A. Hildebrand, eds., Birkhäuser, Boston, 1990, pp. 429–450.

[96] R. A. Rankin, "Ramanujan's manuscripts and notebooks, II," *Bull. London Math. Soc.* **21** (1989), 351–365; reprinted in [**?**, pp. 129–142].

[97] R. A. Rankin, "Diagonalizing Eisenstein series, II," in *A Tribute to Emil Grosswald: Number theory and Related Analysis*, M. Knopp and M. Sheingorn, eds., Contemp. Math. No. 143, American Mathematical Society, Providence, RI, 1993, pp. 525–537.

[98] R. A. Rankin, "Diagonalizing Eisenstein series. III," in *Discrete Groups and Geometry (Birmingham,* 1991*)*, W. J. Harvey, ed., London Mathematical Society Lecture Note Ser. 173, Cambridge University Press, Cambridge, 1992, pp. 196–208.

[99] R. A. Rankin, "Diagonalizing Eisenstein series. IV," in *The Rademacher Legacy to Mathematics (University Park, PA,* 1992*)*, G. E. Andrews, D. M. Bressoud, and L. A. Parson, eds., Contemp. Math. No. 166, American Mathematical Society, Providence, RI, 1994, pp. 107–118.

[100] R. A. Rankin, *March Stones in the Kilpatrick Hills*, Clydebank District Libraries and Museums Department, Glasgow, 1993.

[101] R. A. Rankin, "Place-names in the *comhachag* and other similar poems," *Scottish Gaelic Studies* **18** (1998), 111–130.

[102] R. A. Rankin, "Sums of squares: an elementary method," in *Number Theory*, R. P. Bambah, V. C. Dumir, and R. Hans–Gill, eds., Hindustan Book Co., Delhi, 1999, pp. 371–399.

[103] R. A. Rankin, "Hugh Blackburn: a little-known mathematical friend of Lord Kelvin," *Brit. Soc. Hist. Math.* **43** (2001), 7–14.

[104] P. Ribenboim, *The Book of Prime Number Records*, Springer–Verlag, New York, 1988.

[105] C. A. Rogers, "The packing of equal spheres," *Proc. London Math. Soc.* (3) **8** (1958), 609–620.

[106] A. J. Scholl, "On the Hecke algebra of a non-congruence subgroup," *Bull. London Math. Soc.* **29** (1997), 395–399.

[107] A. Selberg, "Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist," *Archiv. Math. Natur. B* **43** (1940), 47–50.

[108] A. Selberg, "On the estimation of Fourier coefficients of modular forms," *Proc. Symp. Pure Math.*, **VIII**, American Mathematical Society, Providence, RI, 1965, pp. 1–15.

[109] A. Selberg, *Collected Papers*, Vol. 1, Springer–Verlag, Berlin, 1989.

[110] F. Shahidi, "Symmetric power L-functions for $GL(2)$," in *Elliptic Curves and Related Topics*, H. Kisilevsky and M. R. Murty, eds., CRM Proc. and Lect. Notes, **4** (1994), pp. 159–182.

[111] G. Shimura, "On the holomorphy of certain Dirichlet series," *Proc. London Math. Soc.* **31** (1975), 79–98.

[112] R. G. Swan, "A simple proof of Rankin's campanological theorem," *Amer. Math. Monthly* **106** (1999), 159–161.

[113] I. Tweddle, "Robert Alexander Rankin, 1915–2001," *Brit. Soc. Hist. Math.* **43** (2001), 26–31.

[114] A. Weil, “Sur les courbes algébriques et les variétés qui s’en deduisent,” *Actualités, Sci. Ind.*, No. 1041, *Publ. Inst. Math. Univ. Strasbourg*, Vol. 7 (1945), Hermann, Paris, 1948, 85 pp.
[115] K. Wohlfahrt, “Über die Nullstellen einiger Eisensteinreihen,” *Math. Nachr.* **26** (1964), 381–383.
[116] D. A. B. Young, “Ramanujan’s illness,” *Notes Rec. Royal Soc. London* **48** (1994), 107–119; reprinted in [**?**, pp. 65–75].
[117] D. Zagier, “Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields,” in *Modular Functions of One Variable VI*, J.-P. Serre and D. B. Zagier, eds., Lecture Notes in Math., Vol. 627, Springer–Verlag, Berlin, 1977, pp. 105–169.
[118] D. Zagier, “A proof of the Kac-Wakimoto affine denominator formula for the strange series,” *Math. Res. Letters* **7** (2000), 597–604.

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA
E-mail address: berndt@math.uiuc.edu

Mathematisches Institut, Universität Heidelberg, INF 288, D–69120 Heidelberg, Germany
E-mail address: winfried@mathi.uni-heidelberg.de

Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53726, USA
E-mail address: ono@math.wisc.edu