p-adic properties of values of the modular j-function

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1. Introduction and Statement of Results

As usual, let $q := e^{2\pi iz}$ and let $j(z)$ be the classical modular function

$$j(z) = \sum_{n=-1}^{\infty} c(n)q^n = q^{-1} + 744 + 196884q + \cdots .$$

The values of $j(z)$ and its coefficients play many important roles in mathematics. For example, its values generate class fields and its coefficients appear as dimensions of a graded representation of the Monster via the Moonshine phenomenon. In a recent paper, Kaneko [K] produced an interesting connection between the values of $j(z)$ at Heegner points, the so-called singular moduli, and its coefficients. Using recent formulas of Zagier, he systematically expresses the coefficients $c(n)$ in terms of singular moduli. In this paper we want to illustrate some peculiar p-adic properties of certain values of the j-function in connection with class numbers and the integer 720.

To motivate our first result, we begin by recalling some classical formulas for values of the Riemann zeta-function at negative odd integers. To state these formulas, we begin by recalling some standard facts and notation. If $k \geq 4$ is even, then let $E_k(z)$ denote the usual weight k Eisenstein series for the full modular group $SL_2(\mathbb{Z})$

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n.$$

As usual, B_k is the k-th Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$, and H/Γ denotes the usual fundamental domain of the action of the modular group $\Gamma = SL_2(\mathbb{Z})$ on the

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upper half of the complex plane. If $k \geq 4$ is even, then

$$\frac{2}{\zeta(1-k)} = 60k - \sum_{\tau \in H/\Gamma} e_\tau \mathrm{ord}_\tau(E_k(z)) j(\tau). \tag{1}$$

If $\tau \in H/\Gamma$, then e_τ is defined by

$$e_\tau := \begin{cases} 1/2 & \text{if } \tau = i, \\ 1/3 & \text{if } \tau = e^{2\pi i/3}, \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

These formulas follow from Euler's formula for $\zeta(1-k)$ and the Fourier expansion for $E_k(z)$. For example, consider the case where $k = 10$. The divisor of $E_{10}(z)$ is supported by simple zeros at $\tau = i$ and $e^{2\pi i/3}$, and so the fact that $j(i) = 1728$ and $j(e^{2\pi i/3}) = 0$ confirms (1):

$$\frac{2}{\zeta(-9)} = -264 = 600 - \frac{j(e^{2\pi i/3})}{3} - \frac{j(i)}{2}.$$

Here we prove a p-adic analog of (1), where the values of the Kubota-Leopoldt zeta-function at $s = -1$ are given by a p-adic limit of certain precise expressions in the values of the j-function.

Theorem 1. *If $\tau \in H/\Gamma$ is a point for which $j(\tau) \in \mathbb{Q}$, and $p \leq 7$ is a prime for which*

$$j(\tau) \equiv \begin{cases} 0 \pmod{p} & \text{if } p \leq 5, \\ 6 \pmod{7} & \text{if } p = 7, \end{cases}$$

then as p-adic numbers we have

$$\frac{2}{\zeta_p^*(-1)} = -\frac{744}{p-1} - \lim_{n \to +\infty} \left(\sum_{a=0}^{n} \sum_{b=0}^{p^a - 1} j\left(\frac{p^{n-a}\tau + b}{p^a} \right) \right).$$

Note the similarity between (1) and the p-adic formulas appearing in Theorem 1. On the left hand side we have a zeta-value, and on the right hand side we have an expression involving values of the j-function. In (1) these expressions are traces of j-values over the divisor of an Eisenstein series, and in Theorem 1 these expressions are p-adic limits (as $n \to +\infty$) of the 'traces' of j-values over the images of τ under the upper triangular Möbius transformations with determinant p^n.

There are some important differences. First of all, the formulas in Theorem 1 only pertain to those zeta values at $s = 1 - k = -1$ (i.e. where $k = 2$). This is a byproduct of the fact that Theorem 1 is a statement about the arithmetic of p-adic modular forms of weight $k = 2$. Furthermore, Theorem 1 does not appear to be a statement related to the arithmetic of Eisenstein series. However, this is not true. The proof of Theorem 1 requires the arithmetic of the Eisenstein series (at least E_4 and E_6) in a vital way.

Using the fact that $\zeta_p^*(-1) = -(p-1)/12$, Theorem 1 may be interpreted as a curious collection of p-adic formulas for the integer 720:

$$0 = 720 + (p-1) \lim_{n \to +\infty} \left(\sum_{a=0}^{n} \sum_{b=0}^{p^a - 1} j\left(\frac{p^{n-a}\tau + b}{p^a} \right) \right). \tag{3}$$

Example. To illustrate Theorem 1, let $\tau = i$. Since $j(i) = 1728 \equiv 0 \pmod 2$, let $p = 2$ in Theorem 1. Formula (3), which is equivalent to Theorem 1, asserts that if H_n is defined by

$$H_n := 720 + \sum_{a=0}^{n} \sum_{b=0}^{2^a - 1} j\left(\frac{2^{n-a} i + b}{2^a} \right),$$

then H_n tends to zero 2-adically. If $n = 0, 1, \ldots, 5$, then

$$\begin{aligned} H_0 &= 720 + j(i) = 2448 = 2^4 \cdot 3^2 \cdot 17, \\ H_1 &= 2^5 \cdot 3^2 \cdot 5 \cdot 401, \\ H_2 &= 2^6 \cdot 3^2 \cdot 5 \cdot 167 \cdot 341927, \\ H_3 &= 2^7 \cdot 3^2 \cdot 131 \cdot 89604100069763687, \\ H_4 &= 2^8 \cdot 3^2 \cdot 11 \cdots, \\ H_5 &= 2^9 \cdot 3^2 \cdot 124909 \cdots, \end{aligned}$$

Combining similar arguments with recent work of the first author and Bruinier and Kohnen [Th. 9, B-K-O], one obtains the following p-adic formulas for class numbers $H(-D)$ of quadratic orders with discriminant $-D$. First we recall the notion of a Heegner point. A complex number τ of the form $\tau = \frac{-b+\sqrt{b^2-4ac}}{2a}$ with $a, b, c \in \mathbb{Z}$, $\gcd(a,b,c) = 1$ and $b^2 - 4ac < 0$ is known as a *Heegner point*, and its discriminant is the integer $d_\tau := b^2 - 4ac$.

Theorem 2. *Suppose that $-D < -4$ is a fundamental discriminant of an imaginary quadratic field, and let $\tau \in H/\Gamma$ be a Heegner point of discriminant $-D$. If $K = \mathbb{Q}(j(\tau))$, then the following are true:*

(1) *If $D \equiv 3 \pmod 8$, then as 2-adic numbers we have*

$$H(-D) = -\frac{1}{720} \lim_{n \to +\infty} \mathrm{Tr}_{K/\mathbb{Q}} \left(\sum_{a=0}^{n} \sum_{b=0}^{2^a-1} j\left(\frac{2^{n-a}\tau + b}{2^a} \right) \right).$$

(2) *If $D \equiv 1 \pmod 3$, then as 3-adic numbers we have*

$$H(-D) = -\frac{1}{360} \lim_{n \to +\infty} \mathrm{Tr}_{K/\mathbb{Q}} \left(\sum_{a=0}^{n} \sum_{b=0}^{3^a-1} j\left(\frac{3^{n-a}\tau + b}{3^a} \right) \right).$$

(3) *If* $D \equiv 2, 3 \pmod 5$*, then as* 5*-adic numbers we have*

$$H(-D) = -\frac{1}{180} \lim_{n \to +\infty} \mathrm{Tr}_{K/\mathbb{Q}} \left(\sum_{a=0}^{n} \sum_{b=0}^{5^a-1} j\left(\frac{5^{n-a}\tau + b}{5^a} \right) \right).$$

(4) *If* $D \equiv 1, 2, 4 \pmod 7$*, then as* 7*-adic numbers we have*

$$H(-D) = -\frac{1}{120} \lim_{n \to +\infty} \mathrm{Tr}_{K/\mathbb{Q}} \left(\sum_{a=0}^{n} \sum_{b=0}^{7^a-1} j\left(\frac{7^{n-a}\tau + b}{7^a} \right) \right).$$

Example. Here we illustrate Theorem 2 using the Heegner point $\tau = (3 + \sqrt{-7})/2$, using the classical evaluation $j(\tau) = -15^3$. Since $H(-7) = 1$, Theorem 2 (3) implies that

$$I_n := 180 + \sum_{a=0}^{n} \sum_{b=0}^{5^a-1} j\left(\frac{5^{n-a}\tau + b}{5^a} \right)$$

tends to zero 5-adically as $n \to +\infty$. For $n = 0, 1, 2$ and 3 we find that

$$\begin{aligned} I_0 &= -3195 = -3^2 \cdot 5 \cdot 71, \\ I_1 &= -3^2 \cdot 5^2 \cdot 11^2 \cdot 41118261707401, \\ I_2 &= -3^2 \cdot 5^3 \cdot 1123 \cdots, \\ I_3 &= -3^2 \cdot 5^4 \cdot 109 \cdots. \end{aligned}$$

Example. For another example, we consider the Heegner point $\tau = (1 + \sqrt{-15})/2$, using the evaluation (see [§II.6, Si])

$$j(\tau) = \frac{-191025 - 85995\sqrt{5}}{2}.$$

Now $H(-15) = 2$, and so Theorem 2 (4) implies that

$$J_n := 240 + \mathrm{Tr}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}} \left(\sum_{a=0}^{n} \sum_{b=0}^{7^a-1} j\left(\frac{7^{n-a}\tau + b}{7^a} \right) \right)$$

tends to zero 7-adically as $n \to +\infty$. For $n = 0, 1$ and 2 we find that

$$\begin{aligned} J_0 &= -190785 = -3 \cdot 5 \cdot 7 \cdot 23 \cdot 79, \\ J_1 &= -3 \cdot 5 \cdot 7^2 \cdot 3678217 \cdots, \\ J_2 &= -2^2 \cdot 3 \cdot 5 \cdot 7^3 \cdots. \end{aligned}$$

In the next section we prove Theorem 1, and we sketch the proof of Theorem 2.

2. Proof of Theorems 1 and 2

We begin this section with two essential facts. First of all, we construct holomorphic modular forms with prescribed divisor whose Fourier expansions are trivial modulo primes $p \leq 7$. For our purposes, it suffices to observe that

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{4}$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n. \tag{5}$$

Proposition 2.1. *Suppose that $\tau \in H/\Gamma$ is a point for which $j(\tau) \in \mathbb{Q}$. If $p \leq 7$ is a prime for which*

$$j(\tau) \equiv \begin{cases} 0 \pmod{p} & \text{if } p \leq 5, \\ 6 \pmod{7} & \text{if } p = 7, \end{cases}$$

then there is a holomorphic integer weight modular form with p-integral coefficients, say $\mathcal{E}_{\tau,p}(z)$, for which $\mathcal{E}_{\tau,p}(\tau) = 0$ and

$$\mathcal{E}_{\tau,p}(z) \equiv 1 \pmod{p}.$$

Proof. Define $\mathcal{E}_{\tau,p}(z)$ by

$$\mathcal{E}_{\tau,p}(z) := \Delta(z)(j(z) - j(\tau)). \tag{6}$$

Here $\Delta(z)$ is the classical cusp form

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \frac{E_4(z)^3}{j(z)} = \frac{E_6(z)^2}{j(z) - 1728}. \tag{7}$$

It follows that $\mathcal{E}_{\tau,p}(z)$ is a holomorphic modular form of weight 12 for which $\mathcal{E}_{\tau,p}(\tau) = 0$.

If $p \leq 5$ is prime, then the desired congruence follows from (4), (6) and (7). If $p = 7$, then (5), (6), (7) and the fact that $1728 \equiv 6 \pmod{7}$ implies the desired congruence.

□

Now we recall some important facts about Ramanujan's Theta-operator in connection with the values of the j-function. Ramanujan's Theta-operator is the differential operator defined by

$$\Theta\left(\sum_{n=h}^{\infty} a(n) q^n\right) := \sum_{n=h}^{\infty} n a(n) q^n. \tag{8}$$

It is a classical fact that if $f(z) = \sum_{n=h}^{\infty} a(n)q^n$ is a weight k meromorphic modular form on $SL_2(\mathbb{Z})$, then

$$\Theta(f) = (\tilde{f} + kfE_2)/12, \tag{9}$$

where $\tilde{f}$ is a meromorphic modular form of weight $k+2$ on $SL_2(\mathbb{Z})$. Here E_2 denotes the weight 2 quasi-modular form

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n.$$

Although Θ is simple to define, its arithmetic nature is deeper and is dictated by the $\tilde{f}$ appearing in (9). There is an explicit formula for $\Theta(f)$ in terms of a natural sequence of modular functions $j_m(z)$. Let $j_0(z) := 1$, and for every positive integer m let $j_m(z)$ be the unique modular function whose Fourier expansion is of the form

$$j_m(z) = q^{-m} + \sum_{n=1}^{\infty} c_m(n)q^n,$$

which is also holomorphic on the upper half of the complex plane. Notice that if m is a positive integer, then a convenient description of these functions is given by using the normalized weight 0 Hecke operators $T(m)$:

$$j_m(z) := (j(z) - 744 \mid T(m)) = \sum_{\substack{ad=m, d>0,\\ 0\le b\le d-1}} \left(j\left(\frac{az+b}{d}\right) - 744\right). \tag{10}$$

The first few j_m are:

$$\begin{aligned}
&j_0(z) = 1,\\
&j_1(z) = j(z) - 744 = q^{-1} + 196884q + \cdots,\\
&j_2(z) = j(z)^2 - 1488j(z) + 159768 = q^{-2} + 42987520q + \cdots,\\
&j_3(z) = j(z)^3 - 2232j(z)^2 + 1069956j(z) - 36866976 = q^{-3} + 2592899910q + \cdots
\end{aligned}$$

Each j_m is a monic degree m polynomial in j with integer coefficients. The conclusions of the next theorem may be found in [A-K-N, B-K-O]:

Theorem 2.2. *For every point $\tau \in H/\Gamma$, define $H_\tau(z)$ by*

$$H_\tau(z) := \sum_{n=0}^{\infty} j_n(\tau)q^n.$$

(1) *For every point $\tau \in H/\Gamma$, we have*

$$H_\tau(z) = \frac{E_4^2(z)E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z)-j(\tau)}.$$

In particular, H_τ is a weight 2 meromorphic modular form.

(2) *If $f = \sum_{n=h}^{\infty} a_f(n)q^n$ is a nonzero weight k meromorphic modular form on $SL_2(\mathbb{Z})$ for which $a_f(h) = 1$, then*

$$\Theta(f) = \frac{kE_2 f}{12} - f f_\Theta,$$

where f_Θ is defined by

$$f_\Theta := \sum_{\tau \in H/\Gamma} e_\tau \mathrm{ord}_\tau(f) H_\tau(z).$$

Corollary 2.3. *If $\tau \in H/\Gamma$, then*

$$\frac{\Theta(j(z)-j(\tau))}{j(z)-j(\tau)} = -H_\tau(z) = -1 - \sum_{n=1}^{\infty} j_n(\tau)q^n.$$

Therefore, the series appearing in Corollary 2.3 is a meromorphic modular form of weight 2. The following theorem of Serre [Th. 7, Se] is vital for all of the results in this paper.

Theorem 2.4. *If $p \leq 7$ is prime and*

$$f = \sum_{n=0}^{\infty} a(n)q^n$$

is a p-adic modular form of weight $k \neq 0$, then

$$a(0) = \frac{\zeta_p^*(1-k)}{2} \cdot \lim_{n \to +\infty} a(p^n).$$

Proof of Theorem 1. If s is a non-negative integer, then consider the modular form

$$P_{\tau,s}(z) := \mathcal{E}_{\tau,p}(z)^{p^s} \cdot \frac{\Theta(j(z)-j(\tau))}{j(z)-j(\tau)}. \tag{11}$$

By Proposition 2.1 and Corollary 2.3, $P_{\tau,s}(z)$ is a holomorphic modular form for which

$$P_{\tau,s}(z) \equiv -H_\tau(z) \pmod{p^{s+1}}. \tag{12}$$

Furthermore, $P_{\tau,s}(z)$ has weight $2 + 12 \cdot p^s$. By letting $s \to +\infty$, we find that $H_\tau(z)$ is a p-adic modular form of weight 2. The conclusion of Theorem 1 now follows from (10) and Theorem 2.4.

□

Sketch of the Proof of Theorem 2. Let $\tau_1, \tau_2, \ldots \tau_{H(-D)} \in H/\Gamma$ be the Heegner points of discriminant $-D$. Then let $F_D(z)$ be the modular function defined by

$$F_D(z) := \prod_{t=1}^{H(-D)} (j(z) - j(\tau_t)). \tag{13}$$

Gross and Zagier proved that if [Cor. 2.5, G-Z] τ is a Heegner point with discriminant d_τ, then

$$\begin{aligned}
|d_\tau| \equiv 3 \pmod 8 \quad &\Longrightarrow \quad j(\tau) \equiv 0 \pmod{2^{15}},\\
|d_\tau| \equiv 1 \pmod 3 \quad &\Longrightarrow \quad j(\tau) \equiv 1728 \pmod{3^6},\\
|d_\tau| \equiv 2, 3 \pmod 5 \quad &\Longrightarrow \quad j(\tau) \equiv 0 \pmod{5^3},\\
|d_\tau| \equiv 1, 2, 4 \pmod 7 \quad &\Longrightarrow \quad j(\tau) \equiv 1728 \pmod{7^2}.
\end{aligned}$$

Arguing as in the proof of Theorem 1, for every integer s the holomorphic modular form $Q_{D,s}(z)$ defined by

$$Q_{D,s}(z) := \frac{\Theta(F_D(z))}{F_D(z)} \cdot \prod_{t=1}^{H(-D)} \mathcal{E}_{\tau_t,p}(z)^{p^s}$$

satisfies the congruence

$$Q_{D,s}(z) \equiv \frac{\Theta(F_D(z))}{F_D(z)} = -\sum_{t=1}^{H(-D)} H_{\tau_t}(z) \pmod{p^{s+1}}.$$

By letting $s \to +\infty$, we find that $\frac{\Theta(F_D(z))}{F_D(z)}$ is a weight 2 p-adic modular form. Since the numbers $j(\tau_t)$ are algebraic integers which form a complete set of Galois conjugates over $\mathbb{Q}$, the result follows from Theorem 2.4.

□

Remark. It would be interesting to see a direct and elementary proof of the p-adic convergence of the limits appearing in Theorems 1 and 2.

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