ARITHMETIC OF CERTAIN HYPERGEOMETRIC MODULAR FORMS

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Abstract. In a recent paper, Kaneko and Zagier studied a sequence of modular forms $F_k(z)$ which are solutions of a certain second order differential equation. They studied the polynomials

$$\widetilde{F}_k(j) = \prod_{\tau \in H/\Gamma - \{i,\omega\}} (j - j(\tau))^{\mathrm{ord}_\tau(F_k)},$$

where $\omega = e^{2\pi i/3}$ and H/Γ is the usual fundamental domain of the action of $SL_2(\mathbb{Z})$ on the upper half of the complex plane. If $p \geq 5$ is prime, they proved that $\widetilde{F}_{p-1}(j) \pmod p$ is the nontrivial factor of the locus of supersingular j-invariants in characteristic p. Here we consider the irreducibility of these polynomials, and consider their Galois groups.

1. Introduction and Statement of Results.

If $k \geq 4$ is even, then let M_k denote the finite dimensional $\mathbb{C}$-vector space of weight k holomorphic modular forms on $SL_2(\mathbb{Z})$. As usual, we identify a modular form $f(z)$ with its Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n,$$

where $q := e^{2\pi i z}$. In a recent paper [K-Z], Kaneko and Zagier examined, for even integers $k \equiv 0, 4, 6, 10 \pmod{12}$, the unique normalized modular form $F_k(z) \in M_k$ that is a solution to the second order differential equation

$$\theta_{k+2}\theta_k F_k(z) - \frac{k(k+2)}{144} E_4(z) F_k(z) = 0. \tag{1.1}$$

Here θ_k is the differential operator defined on $f(z) \in M_k$ by

$$\theta_k f(z) := q\frac{d}{dq}(f(z)) - kE_2(z)f(z)/12.$$

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For each $4 \leq k \equiv 0, 4, 6, 10 \pmod{12}$, define integers $\delta_k, \epsilon_k \in \{0, 1\}$ by

$$4\delta_k + 6\epsilon_k = r, \tag{1.2}$$

where $k \equiv r \pmod{12}$. For such k, define the polynomial $\widetilde{F}_k(j) \in \mathbb{Q}[j]$ by the identity

$$F_k(z) = \Delta(z)^{[k/12]} E_4(z)^{\delta_k} E_6(z)^{\epsilon_k} \widetilde{F}_k(j(z)). \tag{1.3}$$

Here $j(z)$ denotes the usual j-function

$$j(z) = q^{-1} + 744 + 196884q + \cdots . \tag{1.4}$$

Kaneko and Zagier [K-Z] found beautiful congruences relating these "hypergeometric" modular forms to the loci of supersingular j-invariants. More precisely, if $p \geq 5$ is prime, then they proved that

$$j^{\delta_{p-1}}(j-1728)^{\epsilon_{p-1}} \widetilde{F}_{p-1}(j) \equiv \prod_{\substack{E/\overline{\mathbb{F}}_p \\ \text{supersingular}}} (j - j(E)) \pmod{p}.$$

The analogous property is enjoyed by the classical Eisenstein series $E_{p-1}(z)$.

These modular forms have further properties in common with the Eisenstein series. A classical theorem of F. K. C. Rankin and Swinnerton-Dyer [R-S] asserts that the zeros τ of $E_k(z)$ are simple. Moreover, if τ is such a zero, then $j(\tau)$ is real and in the interval $[0, 1728]$. In a recent paper [Th. 3, K], Kaneko observed, thanks to the theory of orthogonal polynomials, that the $F_k(z)$ also enjoy these properties.

Here we address analogs of some conjectured properties of the Eisenstein series. If H/Γ denotes the usual fundamental domain of $SL_2(\mathbb{Z})$, then it is widely believed that the polynomial

$$\mathcal{E}_k(j) = \prod_{\tau \in H/\Gamma - \{i, \omega\}} (j - j(\tau))^{\mathrm{ord}_\tau(E_k)} \in \mathbb{Q}[j]$$

is irreducible over $\mathbb{Q}$. In fact, it is believed that the Galois group of the splitting field of $\mathcal{E}_k(j)$ is the full symmetric group $S_{d(k)}$, where $d(k)$ is the degree of $\mathcal{E}_k(j)$. Extensive numerical evidence suggests the following analog.

Conjecture. *If $4 \leq k \equiv 0, 4, 6, 10 \pmod{12}$, then $\widetilde{F}_k(j)$ is irreducible over $\mathbb{Q}$. Furthermore, if $\widetilde{F}_k(j)$ has degree d_k, then the Galois group of its splitting field over $\mathbb{Q}$ is S_{d_k}.*

Little is presently known about such questions for Eisenstein series. In particular, it is not known that infinitely many of the $\mathcal{E}_k(j)$ are irreducible over $\mathbb{Q}$. In 1996, Kaneko and Niiho [K-N] studied these questions for $\widetilde{F}_{12n}(j)$ and provided an infinite subclass which are irreducible. Here we provide further results on these questions. We identify several infinite classes of $\widetilde{F}_k(j)$ that are irreducible. In addition, we make the observation that many of these $\widetilde{F}_k(j)$ have Galois groups which are not subgroups of A_{d_k}, the alternating of degree d_k.

Theorem 1.1. *Suppose that $p \geq 5$ is prime, $r \in \{0, 4, 6, 10\}$ and that s is a non-negative integer. Furthermore, suppose that n is a positive integer of the form*

$$n = \begin{cases} \frac{p-1}{6} \cdot p^s & \text{if } r = 0 \text{ and } p \equiv 1 \pmod 6, \\ \frac{p^2-1}{6} \cdot p^{2s} & \text{if } r = 0 \text{ and } p \equiv 5 \pmod 6, \end{cases}$$

$$n = \begin{cases} 20 \cdot 5^{2s} & \text{if } r = 4 \text{ and } p = 5, \\ \frac{p-5}{6} & \text{if } r = 4,\ p \equiv 5 \pmod 6 \text{ and } p \geq 11, \\ \frac{p^3-p}{6} \cdot p^{2s} & \text{if } r = 4,\ p \equiv 5 \pmod 6 \text{ and } p \geq 11, \end{cases}$$

$$n = \begin{cases} 7^s & \text{if } r = 6,\ p = 7 \text{ and } s \geq 1, \\ \frac{p-7}{6} & \text{if } r = 6,\ p \equiv 1 \pmod 6 \text{ and } p \geq 13, \\ \frac{p-1}{6} \cdot p^s & \text{if } r = 6,\ p \equiv 1 \pmod 6,\ p \geq 13, \text{ and } s \geq 1, \\ \frac{p^2-7}{6} & \text{if } r = 6 \text{ and } p \equiv 5 \pmod 6, \\ \frac{p^2-1}{6} \cdot p^{2s} & \text{if } r = 6,\ p \equiv 5 \pmod 6 \text{ and } s \geq 1, \end{cases}$$

$$n = \begin{cases} 19 & \text{if } r = 10 \text{ and } p = 5, \\ 20 \cdot 5^{2s} & \text{if } r = 10,\ p = 5, \text{ and } s \geq 1, \\ 220 \cdot 11^{2s} & \text{if } r = 10 \text{ and } p = 11, \\ \frac{p-11}{6} & \text{if } r = 10,\ p \equiv 5 \pmod 6 \text{ and } p \geq 17, \\ \frac{p^3-p}{6} \cdot p^{2s} & \text{if } r = 10,\ p \equiv 5 \pmod 6 \text{ and } p \geq 17. \end{cases}$$

If $k = 12n + r$, then the polynomial $\widetilde{F}_k(j)$ is irreducible in $\mathbb{Q}[j]$.

Theorem 1.2. *Suppose that $p \geq 5$ is prime, $r \in \{0, 4, 6, 10\}$ and that s is a non-negative integer. Furthermore, suppose that n is a positive integer of the form*

$$n = \begin{cases} \frac{p-1}{6} \cdot p^s & \text{if } r = 0 \text{ and } p \equiv 1 \pmod{12}, \\ \frac{p^2-1}{6} \cdot p^{2s} & \text{if } r = 0 \text{ and } p \equiv 5 \pmod 6, \end{cases}$$

$$n = \begin{cases} 20 \cdot 5^{2s} & \text{if } r = 4 \text{ and } p = 5, \\ \frac{p-5}{6} & \text{if } r = 4,\ p \equiv 5 \pmod{12} \text{ and } p \geq 17, \\ \frac{p^3-p}{6} \cdot p^{2s} & \text{if } r = 4,\ p \equiv 5 \pmod{12} \text{ and } p \geq 17, \end{cases}$$

$$n = \begin{cases} 7^s & \text{if } r = 6,\ p = 7 \text{ and } s \geq 1, \\ \frac{p-7}{6} & \text{if } r = 6,\ p \equiv 7 \pmod{12} \text{ and } p \geq 19, \\ \frac{p-1}{6} \cdot p^s & \text{if } r = 6,\ p \equiv 7 \pmod{12},\ p \geq 19, \text{ and } s \geq 1, \end{cases}$$

$$n = \begin{cases} 220 \cdot 11^{2s} & \text{if } r = 10 \text{ and } p = 11, \\ \frac{p-11}{6} & \text{if } r = 10,\ p \equiv 11 \pmod{12} \text{ and } p \geq 23, \\ \frac{p^3-p}{6} \cdot p^{2s} & \text{if } r = 10,\ p \equiv 11 \pmod{12} \text{ and } p \geq 23. \end{cases}$$

If $k = 12n + r$, then $\widetilde{F}_k(j) \in \mathbb{Q}[j]$ is irreducible and its Galois group is not a subgroup of A_{d_k}.

2. Preliminaries

We begin by recalling some important notation. If n is a positive integer, then the *Pochhammer symbol* $(a)_n$ is defined

$$(a)_n := a(a+1)(a+2)\cdots(a+n-1). \tag{2.1}$$

If $n=0$, then let $(a)_n := 1$. Gauss' ${}_2F_1$ hypergeometric functions are defined by

$${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \mid x\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} \cdot x^n. \tag{2.2}$$

If $4 \leq k \equiv r \pmod{12}$, where $r \in \{0,4,6,10\}$, then Kaneko and Zagier provided the following description of the $\widetilde{F}_k(j)$:

$$\widetilde{F}_k(j) = 1728^{\frac{(k-r)}{12}} \binom{\frac{(k+4(\delta_k-1)-6\epsilon_k)}{12}}{\frac{(k-r)}{12}} {}_2F_1\left(\begin{matrix} -\frac{(k-r)}{12} & \frac{(k+r+2)}{12} \\ & \frac{(2\delta_k+2)}{3} \end{matrix} \mid \frac{j}{1728}\right). \tag{2.3}$$

By (2.2), it follows that $\widetilde{F}_k(j)$ is a polynomial of degree $(k-r)/12$ in j.

For our purposes, it suffices to consider the polynomials, for $r \in \{0,4,6,10\}$ and positive integers n, given by

$${}_2F_1\left(\begin{matrix} -n & n+\beta_r \\ & \gamma_r \end{matrix} \mid x\right) \in \mathbb{Q}[x], \tag{2.4}$$

where β_r and γ_r are defined by

$$\beta_r := (r+1)/6, \tag{2.5}$$

and

$$\gamma_r := \begin{cases} 2/3 & \text{if } r = 0, 6, \\ 4/3 & \text{if } r = 4, 10. \end{cases} \tag{2.6}$$

Each ${}_2F_1\left(\begin{matrix} -n & n+\beta_r \\ & \gamma_r \end{matrix} \mid x\right)$ is, up to scalar multiple and change of variable, the polynomial $\widetilde{F}_k(j)$ where $k = 12n + r$.

For convenience, we shall study the polynomials $B_r(n;x)$ which are defined by

$$B_r(n;x) = \sum_{m=0}^{n} c_r(n,m) x^m := x^n \cdot {}_2F_1\left(\begin{matrix} -n & n+\beta_r \\ & \gamma_r \end{matrix} \mid \frac{2}{x}\right). \tag{2.7}$$

We choose to work with these polynomials since they are monic. Clearly, $B_r(n;x)$ is irreducible in $\mathbb{Q}[x]$ if and only if $\widetilde{F}_{12n+r}(j)$ is irreducible in $\mathbb{Q}[j]$. Moreover, they have the same Galois groups. We begin with an elementary description of these polynomials.

Proposition 2.1. *Let n be a non-negative integer.*
1) If $r=0$ or 6, then

$$B_r(n;x)=\sum_{m=0}^{n}\frac{(-1)^{n-m}\binom{n}{m}\cdot(6n+r+1)(6n+r+7)\cdots(6n+6(n-m)+r-5)x^m}{2\cdot 5\cdot 8\cdots(3(n-m)-1)}.$$

2) If $r=4$ or 10, then

$$B_r(n;x)=\sum_{m=0}^{n}\frac{(-1)^{n-m}\binom{n}{m}\cdot(6n+r+1)(6n+r+7)\cdots(6n+6(n-m)+r-5)x^m}{4\cdot 7\cdot 10\cdots(3(n-m)+1)}.$$

(Note: Empty products are taken to be 1).

Proof. These follow from a simple calculation using the fact that $\frac{(-n)_m}{m!}=(-1)^m\binom{n}{m}$.

□

We also require the functions $f_r(n), g_r(n)$ and $h_r(n)$ defined by

$$f_r(n):=\frac{(12n+r+1)(36n^2+6rn+6n+3\gamma_r r-15\gamma_r)}{(3n+3\gamma_r)(6n+r+1)(12n+r-5),} \tag{2.8}$$

$$g_r(n):=-\frac{(12n+r-5)(12n+r+1)(12n+r+7)}{(3n+3\gamma_r)(6n+r+1)(12n+r-5)}, \tag{2.9}$$

$$h_r(n):=-\frac{9n\left(2n+(-1)^{\frac{r}{2}+1}\right)(12n+r+7)}{(3n+3\gamma_r)(6n+r+1)(12n+r-5)}. \tag{2.10}$$

Proposition 2.2. *If $r\in\{0,4,6,10\}$, then the $B_r(n;x)$ satisfy the three term recurrence relation*

$$B_r(n+1;x)=(f_r(n)x+g_r(n))\cdot B_r(n;x)+h_r(n)x^2B_r(n-1;x),$$

where $B_r(-1;x)=0$ and $B_r(0;x)=1$.

Proof. These recurrence relations follow from the definition of the $B_r(n;x)$ and the classical contiguous relation [p. 100, AAR]

$$\begin{aligned}&2b(c-a)(b-a-1)\cdot\ {}_2F_1\left(\begin{matrix}a-1 & b+1\\ & c\end{matrix}\mid x\right)\\ &-((1-2x)(b-a-1)_3+(b-a)(b+a-1)(2c-b-a-1))\cdot\ {}_2F_1\left(\begin{matrix}a & b\\ & c\end{matrix}\mid x\right)\\ &-2a(b-c)(b-a+1)\cdot\ {}_2F_1\left(\begin{matrix}a+1 & b-1\\ & c\end{matrix}\mid x\right)=0.\end{aligned}$$

One lets $a=-n, b=n+\beta_r$ and $c=\gamma_r$.

□

3. The discriminants of the $B_r(n;x)$

Here we compute the discriminants of many of the $B_r(n;x)$. For convenience, we let

$$D_r(n) := \text{ Discriminant of } B_r(n;x). \tag{3.1}$$

We express these discriminants in terms of n, r, the constant terms of $B_r(s;x)$ for $1 \leq s \leq n$, the value $B_r(n;2)$, and the recurrence functions $h_r(1), h_r(2), \ldots h_r(n-1)$.

Theorem 3.1. *If $r \in \{0,4,6,10\}$ and $n \geq 1$ is an integer for which $B_r(n;1) \neq 0$, then*

$$D_r(n) = (-1)^{\frac{n(n-1)}{2}} \left(\frac{n(n-\gamma_r+\beta_r)}{(2n+\beta_r-1)} \right)^n \cdot \frac{c_r(n,0)}{B_r(n;2)} \prod_{j=1}^{n-1} h_r(j)^j c_r(j,0)^2.$$

To prove Theorem 3.1, we begin with a lemma on the derivatives of ${}_2F_1$ hypergeometric functions at their zeros.

Lemma 3.2. *Suppose that $\alpha \notin \{0,1\}$ is a zero of ${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \mid x\right)$. If $a+1 \neq b$, $c \neq 0$ and $b \neq c$, then*

$${}_2F_1{}'\left(\begin{matrix} a & b \\ & c \end{matrix} \mid \alpha\right) = -\frac{a(c-b)}{\alpha(\alpha-1)(a-b+1)} \cdot {}_2F_1\left(\begin{matrix} a+1 & b-1 \\ & c \end{matrix} \mid \alpha\right).$$

Proof. To prove the lemma, we require the following two facts [p. 95-96, AAR]:

$$ {}_2F_1\left(\begin{matrix} A & B \\ & C \end{matrix} \mid x\right) = (1-x)\, {}_2F_1\left(\begin{matrix} A+1 & B \\ & C \end{matrix} \mid x\right) + \frac{(C-B)x}{C} \cdot {}_2F_1\left(\begin{matrix} A+1 & B \\ & C+1 \end{matrix} \mid x\right), \tag{3.2}$$

and

$$x \cdot {}_2F_1{}'\left(\begin{matrix} A & B \\ & C \end{matrix} \mid x\right) = A \cdot {}_2F_1\left(\begin{matrix} A+1 & B \\ & C \end{matrix} \mid x\right) - A \cdot {}_2F_1\left(\begin{matrix} A & B \\ & C \end{matrix} \mid x\right). \tag{3.3}$$

By (3.3), where $A = a, B = b$ and $C = c$, we find that

$${}_2F_1{}'\left(\begin{matrix} a & b \\ & c \end{matrix} \mid \alpha\right) = \frac{a}{\alpha} \cdot {}_2F_1\left(\begin{matrix} a+1 & b \\ & c \end{matrix} \mid \alpha\right). \tag{3.4}$$

Set $A = b-1, B = a+1$ and $C = c$, then the symmetry of A and B in (3.2) implies that

$$\begin{aligned} {}_2F_1{}'\left(\begin{matrix} a & b \\ & c \end{matrix} \mid \alpha\right) = \frac{a}{\alpha(1-\alpha)} \cdot {}_2F_1\left(\begin{matrix} a+1 & b-1 \\ & c \end{matrix} \mid \alpha\right) \\ - \frac{a(c-a-1)}{c(1-\alpha)} \cdot {}_2F_1\left(\begin{matrix} a+1 & b \\ & c+1 \end{matrix} \mid \alpha\right). \end{aligned}$$

Replace the last summand by applying (3.2) once more, with $A = a, B = b$ and $C = c$. The fact that α is a zero of ${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \mid x\right)$ implies that

$$ {}_2F_1{}'\left(\begin{matrix} a & b \\ & c \end{matrix} \mid \alpha\right) = \frac{a}{\alpha(1-\alpha)} \cdot {}_2F_1\left(\begin{matrix} a+1 & b-1 \\ & c \end{matrix} \mid \alpha\right) + \frac{a(c-a-1)}{\alpha(c-b)} \cdot {}_2F_1\left(\begin{matrix} a+1 & b \\ & c \end{matrix} \mid \alpha\right). $$

Using (3.4), this last expression is equivalent to the claimed formula for ${}_2F_1{}'\left(\begin{matrix} a & b \\ & c \end{matrix} \mid \alpha\right)$.

□

We obtain the following convenient fact using this lemma.

Proposition 3.3. *If $n \geq 1$ and $\alpha \neq 1$ is a zero of $B_r(n;x)$, then*

$$ B_r'(n;\alpha) = \frac{\alpha n(n - \gamma_r + \beta_r)}{(2n + \beta_r - 1)(\alpha - 2)} \cdot B_r(n-1;\alpha). $$

Proof. By definition, we have that $B_r(n;x) = x^n \cdot {}_2F_1\left(\begin{matrix} -n & n+\beta_r \\ & \gamma_r \end{matrix} \mid \frac{2}{x}\right)$. Therefore, it follows that

$$ B_r'(n;x) = \frac{n}{x} B_r(n;x) - 2x^{n-2} \cdot {}_2F_1{}'\left(\begin{matrix} -n & n+\beta_r \\ & \gamma_r \end{matrix} \mid \frac{2}{x}\right). $$

Since $\alpha \neq 0$ (i.e. $B_r(n;0) \neq 0$) is a root of $B_r(n;x)$, the claim follows from Lemma 3.2 and the definition of $B_r(n-1;x)$.

□

We require one final proposition for the proof of Theorem 3.1. Suppose that $f(x)$ (resp. $g(x)$) is a degree n (resp. m) polynomial with roots $x_1, x_2, \ldots, x_n$ (resp. $y_1, y_2, \ldots, y_m$). Furthermore, suppose that

$$ f(x) = \sum_{j=0}^{n} a(i)x^j \quad \text{and} \quad g(x) = \sum_{j=0}^{m} b(j)x^j. $$

The resultant $R(f,g)$ of these polynomials satisfies

$$ R(f,g) = a(n)^m b(m)^n \prod_{i=1}^{n}\prod_{j=1}^{m}(x_i - y_i) = a(n)^m \prod_{i=1}^{n} g(x_i) = (-1)^{mn} R(g,f). \tag{3.5} $$

In particular, notice that if $D(f)$ is the discriminant of $f(x)$, then

$$ R(f,f') = (-1)^{\frac{n(n-1)}{2}} D(f). \tag{3.6} $$

Proposition 3.4. *If $n \geq 0$, then*

$$R(B_r(n;x), B_r(n+1;x)) = R(B_r(n+1;x), B_r(n;x)) = \prod_{j=1}^{n} h_r(j)^j c_r(j,0)^2.$$

Proof. Since $n(n+1)$ is even, (3.5) implies that the first equality holds for all n. If α is a root of $B_r(n;x)$, then Proposition 2.2 implies that

$$B_r(n+1;\alpha) = \alpha^2 h_r(n) B_r(n-1;\alpha). \tag{3.7}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $B_r(n;x)$ repeated with multiplicity. Since each $B_r(j;x)$ is monic, (3.5) and (3.7) implies that

$$\begin{aligned} R(B_r(n;x), B_r(n+1;x)) &= \prod_{j=1}^{n} B_r(n+1;\alpha_j) \\ &= h_r(n)^n c_r(n,0)^2 \prod_{j=1}^{n} B_r(n-1;\alpha_j) \\ &= h_r(n)^n c_r(n,0)^2 R(B_r(n-1;x), B_r(n;x)). \end{aligned} \tag{3.8}$$

Arguing inductively with (3.8), we find that

$$\begin{aligned} R(B_r(n;x), B_r(n+1;x)) &= h_r(n)^n c_r(n,0)^2 R(B_r(n-1;x), B_r(n;x)) \\ &= h_r(n)^n h_r(n-1)^{n-1} c_r(n,0)^2 c_r(n-1,0)^2 R(B_r(n-2;x), B_r(n-1;x)) \\ &\ \vdots \\ &= \prod_{j=1}^{n} h_r(j)^j c_r(j,0)^2. \end{aligned}$$

□

Proof of Theorem 3.1. Begin by noticing that $x = 0$ is not a zero of $B_r(n;x)$. Now suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $B_r(n;x)$ repeated with multiplicity. Since $B_r(n;x)$ is monic, Proposition 3.3 and (3.6) imply that

$$\begin{aligned} (-1)^{n(n-1)/2} D_r(n) &= R(B_r(n;x), B_r'(n;x)) \\ &= \prod_{j=1}^{n} B_r'(n;\alpha_j) \\ &= \prod_{j=1}^{n} \frac{\alpha_j (n - \gamma_r + \beta_r) n}{(\alpha_j - 2)(2n + \beta_r - 1)} \cdot B_r(n-1;\alpha_j) \\ &= \frac{c_r(n,0) n^n (n - \gamma_r + \beta_r)^n}{B_r(n;2)(2n + \beta_r - 1)^n} \cdot \prod_{j=1}^{n} B_r(n-1;\alpha_j). \end{aligned}$$

By (3.5), this implies that

$$D_r(n) = (-1)^{\frac{n(n-1)}{2}} \left(\frac{n(n-\gamma_r+\beta_r)}{(2n+\beta_r-1)}\right)^n \cdot \frac{c_r(n;0)}{B_r(n;2)} \cdot R(B_r(n;x), B_r(n-1;x)).$$

The formula now follows immediately from Proposition 3.4.

□

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by showing that the $B_r(n;x)$ are p-Eisenstein for those n given in Theorem 1.1. This clearly implies that $\widetilde{F}_k(j) \in \mathbb{Q}[j]$ is irreducible.

Proof of Theorem 1.1. We re-index $B_r(n;x)$ in this proof to simplify certain calculations, otherwise, we assume the notation from earlier sections. Setting $c'_r(n,m) = c_r(n, n-m)$, we have

(4.1)

$$B_r(n;x) = \sum_{m=0}^{n} c'_r(n,m)x^{n-m} = \sum_{m=0}^{n} \frac{(-1)^m \binom{n}{m}(6n+r+1)\cdots(6n+6m+r-5)x^{n-m}}{(3\gamma_r+3)\cdots(3m+(3\gamma_r-3))}.$$

It suffices to show that p divides all coefficients, apart from the leading term, and that p^2 does not divide the constant term. We present the proof for the case where $r=0$ and explain the additional observations needed for the case $r=4$ along the way. The final two cases, $r=6$ and $r=10$ are closely related to these first two. Our task is simplified by the fact that both 3 and 6 are coprime to p, so the arithmetic sequences in the numerator and denominator run through the full set of residue classes modulo p.

If $r=0$, let $p=6d+1$ be a prime congruent to 1 modulo 6. If $n=(p-1)/6=d$, then

$$\begin{aligned} c'_0(d,m) &= \frac{(-1)^m \binom{d}{m}(p)(p+6)\cdots(6(n+m)-5)}{2\cdot 5\cdots(3m-1)} \\ &= \frac{(-1)^m \binom{d}{m}(p)\cdots(2p-6(d-m)-5)}{2\cdot 5\cdots((p-1)/2-3(d-m)-1)}. \end{aligned} \tag{4.2}$$

The p in the numerator appears in every non-leading coefficient, and no other multiples of p occur in the numerator or denominator. The constant term $c'_0(d,d)$ is divisible by this single power of p, so the series is p-Eisenstein. The argument is entirely the same if $r=4$, $p \equiv 5 \pmod 6$ and $n=(p-5)/6$.

Now suppose that $n=(p-1)p^s/6 = dp^s$, where $s \geq 1$. Since $p^t \equiv 1 \pmod 6$ for any t, we can easily describe the multiples of p in the coefficients,

$$\begin{aligned} c'_0(dp^s,m) &= \frac{(-1)^m \binom{dp^s}{m}(6dp^s+1)(6dp^s+7)\cdots(6(dp^s+m)-5)}{2\cdot 5\cdots(3m-1)} \\ &= \frac{(-1)^m \binom{dp^s}{m}(6dp^s+1)\cdots(6dp^s+p^t)\cdots p^{s+1}\cdots(6(dp^s+m)-5)}{2\cdot 5\cdots 2p^t\cdots 2p^s\cdots(3m-1)}. \end{aligned}$$

The first multiple of p^t occurs in the numerator when $m = (p^t - 1)/6$, and appears in the denominator when $m = (2p^t - 2)/3 = 4(p^t - 1)/6$. Every multiple of p in the denominator is cancelled by an equal power in the numerator, and thus $c_0'(n, m)$ is p-integral for all m. But p^{s+1} appears as a factor in the numerator when $m \geq (p^s - 1)/6$, and since the denominator factors are bounded above by $p^{s+1}/2$, there is an extra power of p on the numerator for such m. The binomial coefficient $\binom{dp^s}{m}$ is divisible by p for $1 \leq m \leq p^s - 1$, so $c_r(n, m)$ is divisible by p for all m. Finally, the numerator of $c_0'(n, n)$ clearly contains exactly one more power of p than the denominator, so the constant term is not divisible by p^2.

Next, consider the case that $r = 0$ and $p = 6d + 5$, and let $c = (p^2 - 1)/6 = dp + 5d - 4$. If $n = c$, then

$$c_0'(c, m) = \frac{(-1)^m \binom{c}{m}(p^2)(p^2+6)\cdots(p^2+6p)\cdots(2p^2 - 6(c-m) - 4)}{2 \cdot 5 \cdots p \cdots 4p \cdots ((p^2-1)/2 - 3(c-m) - 1)}. \tag{4.3}$$

The first multiple of p in the denominator occurs when $m = 2d+1$, and thus every coefficient is divisible by p. Furthermore, there is one extra power of p in the numerator.

If $n = (p^2 - 1)p^{2s}/6 = cp^{2s}$, then

$$\begin{aligned} c_0'(n, m) &= \frac{(-1)^m \binom{n}{m}(6n+1)(6n+7)\cdots(6(n+m)-5)}{2 \cdot 5 \cdots (3m-1)} \\ &= \frac{(-1)^m \binom{n}{m}(6n+1)\cdots(6n+p^{2t})\cdots(6n+5p^{2t+1})\cdots p^{2s+2}\cdots(6(n+m)-5)}{2 \cdots 2p^{2t} \cdots p^{2t+1} \cdots p^{2s+1} \cdots (3m-1)}. \end{aligned} \tag{4.4}$$

Since $p^2 \equiv 1 \pmod 6$, the even powers of p behave exactly like the previous case: the first multiple of p^{2t} appears as a factor in the numerator when $m = (p^{2t} - 1)/6$, and in the denominator when $m = 4(p^{2t} - 1)/6$.

However, the situation is more complicated for odd powers of p, and the contribution from the binomial coefficients is essential. We use the fact that if p^k is the highest power of p that divides m, then $\binom{cp^{2s}}{m}$ is divisible by p^{2s-k}. The first power of p^{2t+1} appears in the numerator when

$$m = (5p^{2t+1} - 1)/6 = 5(d+1)p(p+1)(p^{2t-2} + \cdots + p^2 + 1) + 5d + 4, \tag{4.5}$$

but it appears in the denominator when

$$m = (p^{2t+1} - 1)/3 = 2(d+1)p(p+1)(p^{2t-2} + \cdots + p^2 + 1) + 2d + 1. \tag{4.6}$$

If $t \leq s - 1$ and m is as in (4.5), then the arithmetic sequence in the denominator of $c_0'(n, m)$ contains $2t + 1$ more powers of p than the numerator. But $\binom{cp^{2s}}{m}$ is divisible by p^{2s}, so the coefficient is divisible by p. If m increases to the next multiple of p^k for some k, the factor of p^k that is lost in the binomial coefficient appears in the arithmetic sequence of the numerator instead. If $t = s$, and m is as in (4.6), then a factor of p^{2s+1} appears in the denominator. However, p^{2s+2} appears in the numerator when $m \geq (p^{2s} - 1)/6$, and the corresponding term on the denominator is p^{2s}, contributing an overall factor of p^2. The numerator of $c_0'(n,n)$ contains exactly one extra power of p, since every term on the denominator is strictly less than p^{2s+2}, so the series is p-Eisenstein.

The argument is similar if $r=4$, $p=6d+5$ and $n=cp^{2s+1}$, replacing (4.4) by

$$\begin{aligned}(4.7)\quad c_4'(n,m) &= \frac{(-1)^m\binom{n}{m}(6n+5)(6n+11)\cdots(6(n+m)-1)}{4\cdot 7\cdots(3m+1)}\\ &= \frac{(-1)^m\binom{n}{m}(6n+5)\cdots(6n+5p^{2t})\cdots(6n+p^{2t+1})\cdots p^{2s+3}\cdots(6(n+m)-1)}{2\cdots p^{2t}\cdots 2p^{2t+1}\cdots p^{2s+2}\cdots(3m+1)}.\end{aligned}$$

The key difference is that all of the exponents have been shifted by one because the factors in the numerator of $c_4'(n,m)$ are all congruent to 5 modulo 6 instead of 1 modulo 6, and the factors in the denominator are congruent to 1 modulo 3 instead of 2 modulo 3.

Equation (4.1) shows that if $r=6$, then the coefficients are nearly the same as in the case $r=0$ through the relation

$$c_6'(n,m) = \frac{c_0'(n,m)(6(n+m)+1)}{(6n+1)}. \tag{4.8}$$

Thus, if $B_0(n;x)$ is p-Eisenstein for some n and p does not divide $6n+1$ or $12n+1$, then $B_6(n;x)$ is p-Eisenstein for the same n. The only exceptions occur when $6n+1=p$ if $p\equiv 1 \pmod 6$ and when $6n+1=p^2$ if $p\equiv 5 \pmod 6$. In these cases, (4.2) and (4.3) show that the first factor in the arithmetic sequence of the numerator of $c_0'(n,m)$ is p or p^2. The analogous case for $r=6$ occurs if $6n+7=p$ or $6n+7=\tilde{p}^2$. When $p=7$, this is impossible (since $n\geq 1$), so the family of allowable n values starts at a higher power; $n=(7-1)7^s/6=7^s$ for $s\geq 1$.

Similarly, if $r=10$, then there is an analog to (4.8) that shows that $B_{10}(n;x)$ is p-Eisenstein for nearly all of the same n as $B_4(n;x)$; the differences are described in the Theorem.

□

Remark. Arguing as above, it can be shown that the $B_r(n;x)$ which are p-Eisenstein for a prime $p\geq 5$ are precisely those polynomials identified by Theorem 1.1.

5. Proof of Theorem 1.2

The polynomials $\widetilde{F}_k(j)$ in Theorem 1.2 are irreducible over $\mathbb{Q}$ by Theorem 1.1. To prove Theorem 1.2, it suffices to show that the discriminant $D_r(n)$ of $B_r(n;x)$, for the associated n, is not a square of a rational number. Through a careful analysis of the discriminant formula found in section 3, we prove that the power of p dividing $D_r(n)$ is odd. Before beginning the proof, we present a pair of technical lemmas.

Lemma 5.1. *Given an integer k and $-k<l<k$, let p be an odd prime such that $p\equiv 1 \pmod k$. Define $\lambda\equiv l \pmod k$ such that $-k\leq\lambda\leq -1$. If $n=dp^s$, where $dk<p$ and $(kdp^{s-1}+\lambda)p\leq(dp^s-1)k+l$, then the product*

$$P=\prod_{j=1}^{n-1}(kj+l)^j$$

satisfies

$$\operatorname{ord}_p(P) = \frac{dp^s}{2}\left(s + d\left(\frac{p^s-1}{p-1}\right)\right) - \frac{ld}{k}\left(\frac{p^s-1}{p-1}\right) + \frac{\lambda s d p^s}{k}. \tag{5.1}$$

Proof. The formula is found by directly counting the powers of p in P. Since $p^t \equiv 1 \pmod{k}$ for any t, the multiples of p^t that appear as factors in P have the form $(mp^t)^{(mp^t-l)/k}$, where $m \equiv l \pmod{k}$ and $0 \le m \le (kdp^{s-t}+\lambda)p^t$. Define ζ such that $0 \le \zeta \le k-1$ and $\zeta \equiv l \pmod{k}$ (note that if $l \ge 0$, then $\zeta = l$ and $\lambda = l-k$; if $l < 0$, then $\zeta = l+k$ and $\lambda = l$). Then the product can be written

$$P = (k+l)^1 \cdots (\zeta p)^{\frac{\zeta p - l}{k}} \cdots (\zeta p^t)^{\frac{\zeta p^t - l}{k}} \cdots ((kdp^{s-t}+\lambda)p^t)^{\frac{(kdp^{s-t}+\lambda)p^t)-l}{k}} \cdots ((dp^s-1)k+l)^{dp^s-1}$$

To find the total exponent of p in P, add the exponents appearing on each multiple of p, then add the exponents on multiples of p^2, and continue up to multiples of p^s. This is given by the sum

$$\begin{aligned}
&\frac{\zeta p - l}{k} + \frac{(k+\zeta)p - l}{k} + \cdots + \frac{\zeta p^2 - l}{k} + \ldots \frac{\zeta p^s - l}{k} + \ldots \frac{(kdp^{s-1}+\lambda)p - l}{k}\\
&\qquad + \frac{\zeta p^2 - l}{k} + \ldots \frac{\zeta p^s - l}{k} + \ldots \frac{(kdp^{s-2}+\lambda)p^2 - l}{k}\\
&\qquad\qquad \vdots\\
&\qquad\qquad + \frac{\zeta p^s - l}{k} + \ldots \frac{(kd+\lambda)p^s - l}{k}.
\end{aligned}$$

The final condition in the statement of the theorem ensures that all of these multiples of p actually appear in P.

Each line is an arithmetic sequence that has been uniformly shifted; the t-th line contains dp^{s-t} terms. The sum is thus

$$\begin{aligned}
&\frac{(\zeta-\lambda)p}{k} + \frac{(k+\zeta-\lambda)p}{k} + \ldots \frac{\zeta p^s - \lambda p}{k} + \ldots \frac{(kdp^{s-1}+\lambda-\lambda)p}{k} - \frac{l}{k}(dp^{s-1}) + \frac{\lambda p}{k}(dp^{s-1})\\
&\qquad \vdots\\
&\qquad + \frac{(\zeta-\lambda)p^s}{k} + \ldots \frac{(kd+\lambda-\lambda)p^s}{k} - \frac{l}{k}(d) + \frac{\lambda p^s}{k}(d).
\end{aligned}$$

This is simplified by the fact that $\zeta - \lambda = k$, giving

$$\begin{aligned}
&p + 2p + \ldots dp^s - \frac{l}{k}(dp^{s-1}) + \frac{\lambda p}{k}(dp^{s-1})\\
&\qquad \vdots\\
&\qquad + p^s + \ldots dp^s - \frac{l}{k}(d) + \frac{\lambda p^s}{k}(d).
\end{aligned}$$

Summing the arithmetic and geometric progressions and collecting like terms gives the formulas.

□

Lemma 5.2. *Given k, and $-k < l < k$, let p be an odd prime such that $p = dk + r$ with $0 \le r \le k-1$ and $r^2 \equiv 1 \pmod{k}$. Let $\lambda \equiv l \pmod{k}$ be defined such that $-k \le \lambda \le -1$, and let $\kappa = \lambda + k$. Also, let $\bar{\lambda} \equiv r\kappa \pmod{k}$ such that $\bar{\lambda}$ lies in the same range as λ, and define $\bar{\kappa} = \bar{\lambda} + k$. If $n = cp^{2s}$, where $ck < p^2$ and $(kcp^{2s-1} + \bar{\lambda})p \le kc(p^2 s - 1) + l$, let $\bar{d}$ be the largest integer such that $(k\bar{d} + \bar{\kappa})p^{2s+1} < k(cp^{2s} - 1) + l$. Then the product*

$$P = \prod_{j=1}^{n-1} (kj + l)^j$$

satisfies

$$\operatorname{ord}_p(P) = \frac{cp^{2s}}{2}\left(2s + c\left(\frac{p^{2s}-1}{p-1}\right)\right) - \frac{lc}{k}\left(\frac{p^{2s}-1}{p-1}\right) + \frac{2scp^{2s}(\frac{\lambda+\bar{\lambda}}{2})}{k} - \frac{l\bar{d}}{k} + \frac{\bar{\lambda}\bar{d}p^{2s+1}}{k} + \frac{p^{2s+1}\bar{d}(\bar{d}+1)}{2}. \tag{5.2}$$

The proof of Lemma 5.2 follows as in the proof of Lemma 5.1, and it is omitted for brevity.

Proof of Theorem 1.2. Suppose that $p \ge 5$ is a prime, and that n and r satisfy the condtions of Theorem 1.2. Recall from Theorem 3.1 the formula for the discriminant:

$$D_r(n) = (-1)^{\frac{n(n-1)}{2}} \left(\frac{n(n-\gamma_r+\beta_r)}{(2n+\beta_r-1)}\right)^n \cdot \frac{c_r(n,0)}{B_r(n;2)} \prod_{j=1}^{n-1} h_r(j)^j c_r(j,0)^2. \tag{5.3}$$

We will show that p occurs as a factor of $D_r(n)$ with odd multiplicity by presenting careful arguments in the case where $r = 0$ and noting the slight differences in proving the other cases.

Suppose that $r = 0$, $p = 12d' + 1$ and $n = dp^s$, where $d = 2d'$. The first term in (5.3) is

$$\left(\frac{n(n-\gamma_0+\beta_0)}{(2n+\beta_0-1)}\right)^n = \left(\frac{n(n-\frac{2}{3}+\frac{1}{6})}{(2n+16-1)}\right)^n = \left(\frac{n(6n-3)}{12n-5}\right)^n.$$

Since $p \ge 13$, the only multiples of p appear in n itself, and the total product of factors of p is $(p^s)^{dp^s}$. If $r = 6$, $p = 12d' + 7$, $s \ge 1$, and $n = dp^s$, where $d = 2d' + 1$, then the analagous term in (5.3) is $(n(6n+3)/(12n+1))^n$, which also has the factors $(p^s)^{dp^s}$. When $s = 0$, there are no powers of p in the $r = 0$ case, corresponding to the case where $n = (p-7)/6$ if $r = 6$.

We showed in section 4 that the monic polynomial $B_r(n;x)$ is p-Eisenstein for any value of r, so that p divides every non-leading coefficient, and p^2 does not divide the constant term $c_r(n,0)$. Thus $c_r(n,0)$ is divisible by exactly one multiple of p, and $B_r(n;2)$ contains no power of p, for $B_r(n;2) \equiv 2^n \pmod{p}$.

Now we turn to the product factors in (5.3). Since we wish to show that it is not a perfect square, we can ignore the factors of the form $c_0(j,0)^2$. We could also remove all of the square powers from the remaining terms and just consider the exponents modulo 2, but our lemmas count the prime factors when all of the factors are present. Since p is prime to 9, we ignore the factor of 9 in $h_r(j)$ throughout the rest of the proof. We have

$$\prod_{j=1}^{n-1} h_0(j)^j = \prod_{j=1}^{n-1} \left(\frac{-9j(2j-1)(12j+7)}{(3j+2)(6j+1)(12j-5)} \right)^j. \tag{5.4}$$

Since $n = dp^s$,

$$\prod_{j=1}^{n-1} \left(\frac{12j+7}{12j-5} \right) = (12dp^s - 5)^{dp^s-1} \prod_{j=1}^{dp^s-1} \frac{1}{12j-5}, \tag{5.5}$$

and the factors of p multiply to $p^{-d(p^s-1)/(p-1)}$. In the case where $r = 6$, $p = 12d' + 7$ and $n = dp^s$, the analog of (5.4) is easily found, and the corresponding formula for (5.5) now depends on the parity of s.

Lemma 5.1 counts the factors of p in the remainder of (5.4), since $p = 12d' + 1 \equiv 1 \pmod{2, 3, 6, 12}$ For example, this lemma states that the product $\prod_{j=1}^{dp^s-1}(3j+2)^j$ contains

$$\frac{2d'p^s}{2}\left(s + 2d'\left(\frac{p^s-1}{p-1}\right)\right) - \frac{2(2d')}{3}\left(\frac{p^s-1}{p-1}\right) - \frac{s(2d')p^s}{3}$$

total factors of p. Thus the parity of the total power of p in $D_0(n)$ is found by adding $1 + sdp^s$ to the factors in (5.4), giving

$$\begin{aligned} 1 + sdp^s + d\left(\frac{p^s-1}{p-1}\right)&\left(0 + \frac{1}{2} + \frac{2}{3} + \frac{1}{6} - 1\right) + sdp^s\left(-1 - \frac{1}{2} + \frac{1}{3} + \frac{5}{6}\right) \\ &= \frac{d}{3}\left(\frac{p^s-1}{p-1}\right) - \frac{sdp^s}{3} + 1 + sdp^s = 1 + \frac{d}{3}\left(2sp^s + \frac{p^s-1}{p-1}\right). \end{aligned}$$

Since $d = 2d'$ is even, this power is clearly odd. The argument is similar if $r = 6$ and $p = 12d' + 7$, although it must be argued that Lemma 5.1 can be applied to $\prod_{j=1}^{n-1}(6j+7)^j$, since the final factor is not divisible by p and can be removed.

Now suppose that $r = 0$, $p = 6d + 5$, and $n = cp^{2s}$. The case $p = 5$ must be treated separately, since the first factors in (5.3) are

$$\left(\frac{n(6n-3)}{12n-5}\right)^n = \left(\frac{c \cdot 5^{2s}(6c \cdot 5^{2s} - 3)}{5(12c \cdot 5^{2s-1} - 1)}\right)^{c \cdot 5^{2s}}.$$

The product of all of the factors of 5 is $5^{(2s-1)c \cdot 5^{2s}}$, and there is one multiple of 5 in $c_0(n, 0)$. Otherwise, if $p \neq 5$, the factors give $p^{2scp^{2s}}$. If $r = 4$ or $r = 10$ and $n = cp^{2s+1}$ then the product is $p^{(2s+1)cp^{2s+1}}$

We now use Lemma 5.2 to count the multiples of p in (5.4), for $p^2 \equiv 1 \pmod{2, 3, 6}$. First, consider the case where d is even. There is a potential concern when $p = 5$, for the initial terms in (5.3) only contain $1+(2s-1)cp^{2s}$ powers of p. However, the formula that we find will still be valid, for one of the conditions of Lemma 5.2 is not met by the product $\prod_{j=1}^{n-1}(12j-5)^j$ when $p=5$. In fact, the final multiple of p that is counted by the lemma occurs when $j=n=cp^{2s}$, and hence it is not actually contained in the product. This term appears on the denominator of (5.4), so the factors of p cancel exactly. If we write $c=(p^2-1)/6=dp+5d+4$, it is clear that $\bar{d}=d+1$ exactly when $k(5d+4)\geq\bar{\kappa}p$, and otherwise $\bar{d}=d$. After simplification and eliminating terms which are clearly even, the total power of p is thus

$$2+s(d+1)(p-1)p^{2s}+2(d+1)\frac{(p^{2s}-1)}{3}+\frac{d(p^{2s+1}+1)}{3}+\frac{p^{2s+1}+1}{6}.$$

Since d is even, every term except the final one is even. The final term is odd, since

$$\frac{p^{2s+1}+1}{6}=(d+1)(p^{2s}-p^{2s-1}+\cdots-p+1)\equiv 1-1+\cdots+1-1+1\equiv 1 \pmod 2.$$

If d is odd, then Lemma 5.2 gives the same formulas, and the congruence properties of p again show that the total exponent is odd.

Finally, if $r=4$ or $r=10$ and $n=cp^{2s+1}$, the argument is similar, although the formulas become more complicated. Lemma 5.2 nearly applies, replacing $2s$ by $2s+1$ throughout, but the odd exponent changes the relative weighting of $\bar{\kappa}$ and $\bar{\lambda}$ in (5.2). There are $s+1$ different odd powers of p so $\bar{\lambda}$ appears in the proportion $(s+1)/(2s+1)$, but there are only s even powers, so λ makes up the remaining $s/(2s+1)$ fraction. Also, since the highest power is p^{2s+2}, the $\bar{\lambda}$ in the second line of (5.2) is replaced by λ. For example, if $r=4$, $p=6d+5$ with d even, and $n=cp^{2s+1}$, then $\prod_{j=1}^{n-1}(6j+5)^j$ contains

$$\begin{aligned}\frac{cp^{2s+1}}{2}\left(2s+1+c\left(\frac{p^{2s+1}-1}{p-1}\right)\right)-\frac{5c}{6}\left(\frac{p^{2s+1}-1}{p-1}\right)-\frac{5(s+1)cp^{2s+1}}{6}-\frac{scp^{2s+1}}{3}\\ -\frac{5d}{6}-\frac{dp^{2s+2}}{6}+\frac{p^{2s+2}d(d+1)}{2}\end{aligned}$$

powers of p.

In all of the cases given in Theorem 1.2, the total exponent of p is odd, and so $B_r(n;x)$ is an irreducible polynomial whose Galois group is not a subgroup of A_{d_k}.

□

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