HECKE OPERATORS AND THE q-EXPANSION OF MODULAR FORMS

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1. Introduction and Statement of Results.

In a recent paper [Br-K-O], the author, Bruinier, and Kohnen investigated the values of a certain sequence of modular functions in connection with the arithmetic properties of meromorphic modular forms on $SL_2(\mathbb{Z})$. In this partially expository note we describe some of the combinatorial consequences of these results for the Fourier coefficients of modular forms. Suppose that

$$f(z) = \sum_{n=h}^{\infty} a_f(n) q^n \qquad (q := e^{2\pi i z} \ \ \text{throughout})$$

is a weight $k \in 2\mathbb{Z}$ meromorphic modular form on $SL_2(\mathbb{Z})$. It is well known that $f(z)$ is distinguished by its weight k and its "first few coefficients". The Riemann-Roch Theorem provides the number of coefficients which are sufficient for distinguishing such a form $f(z)$. Here we obtain a combinatorial extension of this classical fact; specifically, we give universal recursion relations which produce the entire Fourier expansion of any meromorphic modular form on $SL_2(\mathbb{Z})$ in terms of the weight k and its first few coefficients.

To state this result, we require a few preliminaries. If $k \geq 2$ is even, then let $E_k(z)$ denote the usual Eisenstein series

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{1.1}$$

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Here B_k denotes the usual kth Bernoulli number and $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$ (note: throughout this note all divisors of integers are taken to be positive). If $k > 2$, then $E_k(z)$ is a weight k modular form on $SL_2(\mathbb{Z})$. Although the Eisenstein series

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \tag{1.2}$$

is not a modular form, it will play an important role. As usual, let $j(z)$ denote the modular function on $SL_2(\mathbb{Z})$ which is holomorphic on $\mathcal{H}$, the upper half of the complex plane, with Fourier expansion

$$j(z) := q^{-1} + 744 + 196884q + 21493760q^2 + \cdots . \tag{1.3}$$

We require a specific sequence of modular functions $j_m(z)$. To define this sequence, let

$$j_0(z) := 1 \qquad \text{and} \qquad j_1(z) := j(z) - 744. \tag{1.4}$$

If $m \geq 2$, then define $j_m(z)$ by

$$j_m(z) := j_1(z) \mid T_0(m), \tag{1.5}$$

where $T_0(m)$ is the usual normalized mth weight zero Hecke operator defined by

$$g(z) \mid T_0(m) = \sum_{\substack{d|m \\ ad=m}} \sum_{b=0}^{d-1} g\left(\frac{az+b}{d}\right). \tag{1.6}$$

Each $j_m(z)$ is a monic polynomial in $j(z)$ of degree m. Here we list the first few:

$$\begin{aligned}
j_0(z) &= 1,\\
j_1(z) &= j(z) - 744 = q^{-1} + 196884q + \cdots ,\\
j_2(z) &= j(z)^2 - 1488j(z) + 159768 = q^{-2} + 42987520q + \cdots ,\\
j_3(z) &= j(z)^3 - 2232j(z)^2 + 1069956j(z) - 36866976 = q^{-3} + 2592899910q + \cdots .
\end{aligned}$$

Let $\mathfrak{F}$ denote the usual fundamental domain of the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$. By assumption, $\mathfrak{F}$ does not include the cusp at ∞. Throughout, let $i := \sqrt{-1}$ and let $\omega := (1+\sqrt{-3})/2$. If $\tau \in \mathfrak{F}$, then define e_τ by

$$e_\tau := \begin{cases} 1/2 & \text{if } \tau = i,\\ 1/3 & \text{if } \tau = \omega,\\ 1 & \text{otherwise.} \end{cases} \tag{1.7}$$

The next result contains the universal polynomial recursion formulae.

Theorem 1. *For every positive integer n define $F_n(x_1,\ldots,x_n)\in\mathbb{Q}[x_1,\ldots,x_n]$ by*

$$F_n(x_1,\ldots,x_n) := -\frac{2x_1\sigma_1(n)}{n}$$
$$+ \sum_{\substack{m_1,\ldots,m_{n-1}\geq 0,\\ m_1+2m_2+\cdots+(n-1)m_{n-1}=n}} (-1)^{m_1+\cdots+m_{n-1}}\cdot\frac{(m_1+\cdots+m_{n-1}-1)!}{m_1!\cdots m_{n-1}!}\cdot x_2^{m_1}\cdots x_n^{m_{n-1}}.$$

If $f(z)=q^h+\sum_{n=1}^{\infty}a_f(h+n)q^{h+n}$ is a weight k meromorphic modular form on $SL_2(\mathbb{Z})$, then for every positive integer n we have

$$a_f(h+n) = F_n(k, a_f(h+1),\ldots,a_f(h+n-1)) - \frac{1}{n}\sum_{\tau\in\mathfrak{F}} e_\tau \mathrm{ord}_\tau(f)\cdot j_n(\tau).$$

The first few polynomials F_n are

$$F_1(x_1) := -2x_1,$$
$$F_2(x_1,x_2) := -3x_1+\frac{x_2^2}{2},$$
$$F_3(x_1,x_2,x_3) := -\frac{8x_1}{3}-\frac{x_2^3}{3}+x_2x_3,$$
$$F_4(x_1,x_2,x_3,x_4) := -\frac{7x_1}{2}-x_2^2x_3+x_2x_4+\frac{x_2^4}{4}+\frac{x_3^2}{2}.$$

The $n=1$ case of Theorem 1 implies that

$$a_f(h+1) = 60k-744h-\sum_{\tau\in\mathfrak{F}} e_\tau \mathrm{ord}_\tau(f)\cdot j(\tau).$$

Example 1. Since

$$\Delta(z)=\sum_{n=1}^{\infty}\tau(n)q^n = q-24q^2+252q^3-\cdots,$$

the unique normalized weight 12 cusp form on $SL_2(\mathbb{Z})$, is nonvanishing in $\mathcal{H}$, Theorem 1 implies that

$$\tau(n+1)=F_n(12,\tau(2),\ldots,\tau(n)).$$

Here we also examine the infinite product expansion of modular forms. In [B1, B2], Borcherds investigated the infinite product expansion of those modular forms possessing a Heegner divisor (note: Zagier [Z] has recently obtained a more elementary proof). For such forms $f(z)$, he provided a striking description for the infinite product expansion. For example, if the integers $c(n)$ are the exponents in the infinite product expansion of the classical Eisenstein series $E_4(z)$

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = (1-q)^{-240}(1-q^2)^{26760}\cdots = \prod_{n=1}^{\infty}(1-q^n)^{c(n)},$$

his theorem implies that there is a weight 1/2 meromorphic modular form

$$G(z) = \sum_{n\geq -3} b(n)q^n = q^{-3} + 4 - 240q + 26760q^4 + \cdots - 4096240q^9 + \cdots$$

on $\Gamma_0(4)$ with the property that $c(n) = b(n^2)$ for every positive integer n.

In more recent work [Br-O], Bruinier and the author have obtained a different type of result for a wider class of forms using p-adic modular forms. However, these results do not provide formulae for the exponents in the infinite product expansion of a generic modular form. The next result provides such formulae.

Theorem 2. *Suppose that $f(z) = \sum_{n=h}^{\infty} a_f(n)q^n$ is a weight k meromorphic modular form on $SL_2(\mathbb{Z})$ for which $a_f(h) = 1$, and let $c(n)$ denote the complex numbers for which*

$$f(z) = q^h \prod_{n=1}^{\infty}(1-q^n)^{c(n)}.$$

If n is a positive integer, then

$$c(n) = 2k + \frac{1}{n}\sum_{\tau\in\mathfrak{F}} e_\tau \mathrm{ord}_\tau(f) \sum_{d\mid n}\mu(n/d)j_d(\tau).$$

Example 2. Let $f(z) = \Delta(z) = q - 24q^2 + 252q^3 - \cdots$ be the unique normalized cusp form of weight 12 on $SL_2(\mathbb{Z})$. Since it is nonvanishing on $\mathcal{H}$, Theorem 2 easily gives the classical infinite product formula

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

Example 3. Here we revisit the infinite product for $E_4(z)$. Since the divisor of $E_4(z)$ is supported at a simple zero at $\tau = \omega$, Theorem 2 implies that

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = (1-q)^{-240}(1-q^2)^{26760}\cdots = \prod_{n=1}^{\infty}(1-q^n)^{c(n)},$$

where

$$c(n) = 8 + \frac{1}{3n}\sum_{d|n} \mu(n/d) j_d(\omega).$$

By (2.4), it turns out that

$$\sum_{n=0}^{\infty} j_n(\omega)q^n = \frac{E_6(z)}{E_4(z)} = 1 - 744q + 159768q^2 - 36866976q^3 + \cdots .$$

Using these terms, Theorem 2 gives

$$\begin{aligned} c(1) &= 8 + j_1(\omega)/3 = 8 - 744/3 = -240, \\ c(2) &= 8 + (j_2(\omega) - j_1(\omega))/6 = 26760, \\ c(3) &= 8 + (j_3(\omega) - j_1(\omega))/9 = -4096240. \end{aligned}$$

In §2 we sketch the proof of a fundamental formula describing the action of Ramanujan's Theta-operator (see also [Br-K-O]), and in §3 we use this result to prove Theorems 1 and 2.

2. An explicit formula for Ramanujan' s Θ-operator

Recall that Ramanujan's Theta-operator is defined by

$$\Theta\left(\sum_{n=h}^{\infty} a(n)q^n\right) := \sum_{n=h}^{\infty} na(n)q^n. \tag{2.1}$$

We refer to Θ as Ramanujan's operator since he observed that [R]

$$\Theta(E_4) = (E_4E_2 - E_6)/3 \quad \text{and} \quad \Theta(E_6) = (E_6E_2 - E_8)/2. \tag{2.2}$$

If $f(z) = \sum_{n=h}^{\infty} a_f(n)q^n$ is a weight k meromorphic modular form on $SL_2(\mathbb{Z})$, then

$$\Theta(f) = (\tilde{f} + kfE_2)/12, \tag{2.3}$$

where $\tilde{f}$ is a meromorphic modular form of weight $k+2$ on $SL_2(\mathbb{Z})$. Here we give an explicit description of the form $\tilde{f}$ appearing above in terms of the modular functions $j_m(z)$.

To state these results, for every point $\tau \in \mathcal{H}$ let $H_\tau(z) := \sum_{n=0}^{\infty} j_n(\tau)q^n$. Asai, Kaneko, and Ninomiya [Th. 3, A-K-N] proved that

$$H_\tau(z) = \sum_{n=0}^{\infty} j_n(\tau)q^n = \frac{E_4^2(z)E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z) - j(\tau)}. \tag{2.4}$$

In particular, for every τ it turns out that $H_\tau(z)$ is a weight 2 meromorphic modular form. The utility of (2.4) was already known; for example, it can be used to prove the famous denominator formula for the monster Lie algebra.

In [Br-K-O], the following result is proved.

Theorem 2.1. *If $f = \sum_{n=h}^{\infty} a_f(n)q^n$ is a nonzero weight k meromorphic modular form on $SL_2(\mathbb{Z})$ for which $a_f(h) = 1$, then*

$$\frac{\Theta(f)}{f} = \frac{kE_2}{12} - f_\Theta,$$

where f_Θ is defined by

$$f_\Theta := \sum_{\tau \in \mathfrak{F}} e_\tau \mathrm{ord}_\tau(f) H_\tau(z).$$

For convenience, we recall the following straightforward fact (see [Prop. 2.1, Br-K-O]) which explains the connection between Theorem 2.1 and the combinatorial properties of the infinite product exponents of a given modular form.

Proposition 2.2. *Let $f = \sum_{n=h}^{\infty} a_f(n)q^n$ be a meromorphic function in a neighborhood of $q = 0$, and suppose that $a_f(h) = 1$. Then there are uniquely determined complex numbers $c(n)$ such that*

$$f = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)},$$

where the product converges in a small neighborhood of $q = 0$. Moreover, the following identity is true

$$\frac{\Theta(f)}{f} = h - \sum_{n=1}^{\infty} \sum_{d|n} c(d) d q^n.$$

Sketch of the Proof of Theorem 2.1. As before, let $\mathfrak{F}$ denote the standard fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$. We cut off $\mathcal{F}$ by a horizontal line $\mathcal{L} := \{iC - t \ : -\frac{1}{2} \leq t \leq \frac{1}{2}\}$ where $C > 0$ is chosen so large that all poles and zeros of f, apart from those at the cusp at infinity, are contained in $\{z \in \mathcal{H} \ : \ \mathrm{Im}\,(z) < C\}$.

For simplicity, suppose that $f(z)$ has no zero or pole on the boundary $\partial\mathcal{F}$ except possibly i or ω, and let γ be the closed path with positive orientation consisting of $\mathcal{L}$ and γ_1 where γ_1 is the part of $\partial\mathfrak{F}$ below $\mathcal{L}$ modified in the usual way (i.e. as in the standard proof of the "$k/12$-valence formula").

We integrate $\frac{1}{2\pi i}\frac{f'(z)}{f(z)} j_n(z)$ along γ. By the residue theorem, taking into account that $j_n(z)$ is holomorphic on $\mathcal{H}$, this integral is equal to

$$\sum_{\tau \in \mathfrak{F} - \{\omega, i\}} \mathrm{ord}_\tau(f) j_n(\tau).$$

On the other hand, the integral can be evaluated separately along the different pieces of γ. If we let r tend to zero, we then find that

$$\sum_{\tau\in\mathfrak{F}-\{\omega,i\}} \operatorname{ord}_\tau(f) j_n(\tau) = \\ -\frac{1}{3}\operatorname{ord}_\omega(f) j_n(\omega) - \frac{1}{2}\operatorname{ord}_i(f) j_n(i) + \frac{1}{2\pi i}\int_\rho \frac{F'(q)}{F(q)} J_n(q)dq - \frac{k}{2\pi i}\int_\sigma \frac{j_n(z)}{z}dz. \tag{2.5}$$

Here $F(q) = f(z)$ as before and $J_n(q) := j_n(z)$. Furthermore, ρ is a small circle around $q = 0$ with negative orientation and not containing any pole or zero of $F(q)$ except possibly 0, and σ is the part of the unit circle in the upper half-plane that connects ω and i, with positive orientation.

By Proposition 2.2, we have that

$$\frac{qF'(q)}{F(q)} = \frac{\Theta(f)}{f} = h - \sum_{n=1}^{\infty}\sum_{d|n} c(d)dq^n,$$

where h is the order of F at $q = 0$. Therefore we find that

$$\frac{1}{2\pi i}\int_\rho \frac{F'(q)}{F(q)} J_n(q)dq = \sum_{d|n} c(d)d. \tag{2.6}$$

Since (2.5) holds for $f = \Delta$, and since Δ has no zeros on $\mathcal{H}$, we obtain from (2.5) that

$$\frac{1}{2\pi i}\int_\sigma \frac{j_n(z)}{z}dz = 2\sigma_1(n). \tag{2.7}$$

To prove that

$$\frac{\Theta(f)}{f} = \frac{kE_2}{12} - f_\Theta,$$

where f_Θ has the claimed form, one now simply argues coefficient by coefficient with (1.2), (2.5), (2.6) and (2.7).

□

3. Proof of Theorems 1 and 2

Proof of Theorem 1. By Theorem 2.1, if n is a positive integer, then

$$\sum_{d|n} c(d)d = 2k\sigma_1(n) + \sum_{\tau\in\mathfrak{F}} e_\tau \operatorname{ord}_\tau(f) j_n(\tau). \tag{3.1}$$

Therefore, to prove Theorem 1 it suffices to obtain a closed formula for

$$b(n)=\sum_{d|n}c(d)d \tag{3.2}$$

in terms of the coefficients of $f(z)$. In particular, a straightforward calculation reveals that it suffices to show that if $n\geq 1$, then

$$\begin{aligned} b(n)=n\sum_{\substack{m_1,\dots,m_n\geq 0,\\ m_1+2m_2+\dots+nm_n=n}} &(-1)^{m_1+\cdots+m_n} \\ &\times\frac{(m_1+\cdots+m_n-1)!}{m_1!\cdots m_n!}\cdot a_f(h+1)^{m_1}\cdots a_f(h+n)^{m_n}. \end{aligned} \tag{3.3}$$

To prove (3.3), observe that

$$0=b(n)+b(n-1)a_f(h+1)+b(n-2)a_f(h+2)+\cdots+b(1)a_f(h+n-1)+na_f(h+n),$$

and then use the fact that

$$0=s_n-s_{n-1}\sigma_1+s_{n-2}\sigma_2-\cdots+(-1)^{n-1}s_1\sigma_{n-1}+(-1)^n n\sigma_n.$$

Here the σ_i are the elementary symmetric functions in $X_1,\dots,X_n$ and the s_i are the power functions in these variables (i.e. $s_i:=X_1^i+\cdots+X_n^i$). By evaluating these identities at $(X_1,...,X_n)=(\lambda(1,n),...,\lambda(n,n))$ where the $\lambda(j,n)$ are the roots of the polynomial

$$X^n+a_f(h+1)X^{n-1}+a_f(h+2)X^{n-2}+\cdots+a_f(h+n),$$

one obtains (3.3). This final calculation requires the fact that

$$s_i=i\sum_{\substack{m_1,\dots,m_n\geq 0,\\ m_1+2m_2+\cdots+nm_n=i}}(-1)^{m_2+m_4+\dots}\frac{(m_1+m_2+\cdots+m_n-1)!}{m_1!m_2!\cdots m_n!}\cdot\sigma_1^{m_1}\cdots\sigma_n^{m_n}.$$

□

Proof of Theorem 2. By (3.1), it follows that

$$F(n):=\sum_{d|n}c(d)d=2k\sigma_1(n)+\sum_{\tau\in\mathfrak{F}}e_\tau\mathrm{ord}_\tau(f)j_n(\tau).$$

By Möbius Inversion, it follows that

$$c(n) = \frac{1}{n}\sum_{d|n}\mu(n/d)F(d)$$
$$= \frac{1}{n}\sum_{d|n}\mu(n/d)\left(2k\sigma_1(d) + \sum_{\tau\in\mathfrak{F}} e_\tau \mathrm{ord}_\tau(f) j_d(\tau)\right).$$

The formula for $c(n)$ follows from the elementary identity

$$n = \sum_{d|n}\mu(n/d)\sigma_1(d).$$

□

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