HECKE OPERATORS AND THE $q$-EXPANSION OF MODULAR FORMS

Ken Ono


1. Introduction and Statement of Results.

In a recent paper [Br-K-O], the author, Bruinier, and Kohnen investigated the values of a certain sequence of modular functions in connection with the arithmetic properties of meromorphic modular forms on $SL_2(\mathbb{Z})$. In this partially expository note we describe some of the combinatorial consequences of these results for the Fourier coefficients of modular forms. Suppose that

$$f(z) = \sum_{n=h}^{\infty} a_f(n) q^n \quad (q := e^{2\pi i z} \text{ throughout})$$

is a weight $k \in 2\mathbb{Z}$ meromorphic modular form on $SL_2(\mathbb{Z})$. It is well known that $f(z)$ is distinguished by its weight $k$ and its “first few coefficients”. The Riemann-Roch Theorem provides the number of coefficients which are sufficient for distinguishing such a form $f(z)$. Here we obtain a combinatorial extension of this classical fact; specifically, we give universal recursion relations which produce the entire Fourier expansion of any meromorphic modular form on $SL_2(\mathbb{Z})$ in terms of the weight $k$ and its first few coefficients.

To state this result, we require a few preliminaries. If $k \geq 2$ is even, then let $E_k(z)$ denote the usual Eisenstein series

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{1.1}$$
Here $B_k$ denotes the usual $k$th Bernoulli number and $\sigma_{k-1}(n) := \sum_{d \mid n} d^{k-1}$ (note: throughout this note all divisors of integers are taken to be positive). If $k > 2$, then $E_k(z)$ is a weight $k$ modular form on $SL_2(\mathbb{Z})$. Although the Eisenstein series
\[ E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \frac{\sigma_1(n)q^n}{n} \] (1.2)
is not a modular form, it will play an important role. As usual, let $j(z)$ denote the modular function on $SL_2(\mathbb{Z})$ which is holomorphic on $\mathcal{H}$, the upper half of the complex plane, with Fourier expansion
\[ j(z) := q^{-1} + 744 + 196884q + 21493760q^2 + \cdots. \] (1.3)We require a specific sequence of modular functions $j_m(z)$. To define this sequence, let
\[ j_0(z) := 1 \quad \text{and} \quad j_1(z) := j(z) - 744. \] (1.4)
If $m \geq 2$, then define $j_m(z)$ by
\[ j_m(z) := j_1(z) \mid T_0(m), \] (1.5)
where $T_0(m)$ is the usual normalized $m$th weight zero Hecke operator defined by
\[ g(z) \mid T_0(m) = \sum_{d \mid m} \sum_{a \equiv b \pmod{m}} g \left( \frac{az+b}{d} \right). \] (1.6)
Each $j_m(z)$ is a monic polynomial in $j(z)$ of degree $m$. Here we list the first few:
\[
\begin{align*}
  j_0(z) &= 1, \\
  j_1(z) &= j(z) - 744 = q^{-1} + 196884q + \cdots, \\
  j_2(z) &= j(z)^2 - 1488j(z) + 159768 = q^{-2} + 42987520q + \cdots, \\
  j_3(z) &= j(z)^3 - 2232j(z)^2 + 1069956j(z) - 36866976 = q^{-3} + 2592899910q + \cdots.
\end{align*}
\]
Let $\mathfrak{F}$ denote the usual fundamental domain of the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$. By assumption, $\mathfrak{F}$ does not include the cusp at $\infty$. Throughout, let $i := \sqrt{-1}$ and let $\omega := (1 + \sqrt{-3})/2$. If $\tau \in \mathfrak{F}$, then define $e_\tau$ by
\[ e_\tau := \begin{cases} 
  1/2 & \text{if } \tau = i, \\
  1/3 & \text{if } \tau = \omega, \\
  1 & \text{otherwise.}
\end{cases} \] (1.7)
The next result contains the universal polynomial recursion formulae.
Theorem 1. For every positive integer \( n \) define \( F_n(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n] \) by

\[
F_n(x_1, \ldots, x_n) := -\frac{2x_1 \sigma_1(n)}{n} + \sum_{m_1, \ldots, m_{n-1} \geq 0, \sum m_1 + 2m_2 + \cdots + (n-1)m_{n-1} = n} (-1)^{m_1 + \cdots + m_{n-1}} \frac{(m_1 + \cdots + m_{n-1} - 1)!}{m_1! \cdots m_{n-1}!} x_1^{m_1} \cdots x_n^{m_{n-1}}.
\]

If \( f(z) = q^h + \sum_{n=1}^{\infty} a_f(h+n)q^{h+n} \) is a weight \( k \) meromorphic modular form on \( SL_2(\mathbb{Z}) \), then for every positive integer \( n \) we have

\[
a_f(h+n) = F_n(k, a_f(h+1), \ldots, a_f(h+n-1)) - \frac{1}{n} \sum_{\tau \in \mathfrak{H}} e_{\tau \text{ord}_\tau(f)} \cdot j_n(\tau).
\]

The first few polynomials \( F_n \) are

\[
F_1(x) := -2x_1,
F_2(x_1, x_2) := -3x_1 + \frac{x_2^2}{2},
F_3(x_1, x_2, x_3) := -\frac{8x_1}{3} - \frac{x_2^3}{2} + x_2x_3,
F_4(x_1, x_2, x_3, x_4) := -\frac{7x_1}{2} - x_2^2x_3 + x_2x_4 + \frac{x_3^4}{4} + \frac{x_2^2}{2}.
\]

The \( n = 1 \) case of Theorem 1 implies that

\[
a_f(h+1) = 60k - 744h - \sum_{\tau \in \mathfrak{H}} e_{\tau \text{ord}_\tau(f)} \cdot j(\tau).
\]

Example 1. Since

\[
\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n = q - 24q^2 + 252q^3 - \cdots,
\]

the unique normalized weight 12 cusp form on \( SL_2(\mathbb{Z}) \), is nonvanishing in \( \mathfrak{H} \), Theorem 1 implies that

\[
\tau(n+1) = F_n(12, \tau(2), \ldots, \tau(n)).
\]

Here we also examine the infinite product expansion of modular forms. In \([B1, B2]\), Borcherds investigated the infinite product expansion of those modular forms possessing
a Heegner divisor (note: Zagier [Z] has recently obtained a more elementary proof). For such forms \( f(z) \), he provided a striking description for the infinite product expansion. For example, if the integers \( c(n) \) are the exponents in the infinite product expansion of the classical Eisenstein series \( E_4(z) \)

\[
E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n = (1 - q)^{-240}(1 - q^2)^{26760} \cdots = \prod_{n=1}^{\infty} (1 - q^n)^{c(n)},
\]

his theorem implies that there is a weight 1/2 meromorphic modular form

\[
G(z) = \sum_{n \geq -3} b(n)q^n = q^{-3} + 4 - 240q + 26760q^4 + \cdots - 4096240q^9 + \cdots
\]
on \( \Gamma_0(4) \) with the property that \( c(n) = b(n^2) \) for every positive integer \( n \).

In more recent work [Br-O], Bruinier and the author have obtained a different type of result for a wider class of forms using \( p \)-adic modular forms. However, these results do not provide formulae for the exponents in the infinite product expansion of a generic modular form. The next result provides such formulae.

**Theorem 2.** Suppose that \( f(z) = \sum_{n=h}^{\infty} a_f(n)q^n \) is a weight \( k \) meromorphic modular form on \( SL_2(\mathbb{Z}) \) for which \( a_f(h) = 1 \), and let \( c(n) \) denote the complex numbers for which

\[
f(z) = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)}.
\]

If \( n \) is a positive integer, then

\[
c(n) = 2k + \frac{1}{n} \sum_{\tau \in \mathbb{R}} e_{\tau} \text{ord}_{\tau}(f) \sum_{d|n} \mu(n/d)j_d(\tau).
\]

**Example 2.** Let \( f(z) = \Delta(z) = q - 24q^2 + 252q^3 - \cdots \) be the unique normalized cusp form of weight 12 on \( SL_2(\mathbb{Z}) \). Since it is nonvanishing on \( \mathcal{H} \), Theorem 2 easily gives the classical infinite product formula

\[
\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.
\]

**Example 3.** Here we revisit the infinite product for \( E_4(z) \). Since the divisor of \( E_4(z) \) is supported at a simple zero at \( \tau = \omega \), Theorem 2 implies that

\[
E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n = (1 - q)^{-240}(1 - q^2)^{26760} \cdots = \prod_{n=1}^{\infty} (1 - q^n)^{c(n)},
\]
where
\[ c(n) = 8 + \frac{1}{3n} \sum_{d|n} \mu(n/d)j_d(\omega). \]

By (2.4), it turns out that
\[ \sum_{n=0}^{\infty} j_n(\omega)q^n = \frac{E_6(z)}{E_4(z)} = 1 - 744q + 159768q^2 - 36866976q^3 + \cdots. \]

Using these terms, Theorem 2 gives
\[ c(1) = 8 + j_1(\omega)/3 = 8 - 744/3 = -240, \]
\[ c(2) = 8 + (j_2(\omega) - j_1(\omega))/6 = 26760, \]
\[ c(3) = 8 + (j_3(\omega) - j_1(\omega))/9 = -4096240. \]

In §2 we sketch the proof of a fundamental formula describing the action of Ramanujan’s Theta-operator (see also [Br-K-O]), and in §3 we use this result to prove Theorems 1 and 2.

2. AN EXPLICIT FORMULA FOR RAMANUJAN’ S Θ-OPERATOR

Recall that Ramanujan’s Theta-operator is defined by
\[ \Theta \left( \sum_{n=h}^{\infty} a(n)q^n \right) := \sum_{n=h}^{\infty} na(n)q^n. \]  

We refer to Θ as Ramanujan’s operator since he observed that [R]
\[ \Theta(E_4) = (E_4E_2 - E_6)/3 \quad \text{and} \quad \Theta(E_6) = (E_6E_2 - E_8)/2. \]  

If \( f(z) = \sum_{n=h}^{\infty} a_f(n)q^n \) is a weight \( k \) meromorphic modular form on \( SL_2(\mathbb{Z}) \), then
\[ \Theta(f) = (\tilde{f} + kfE_2)/12, \]  

where \( \tilde{f} \) is a meromorphic modular form of weight \( k + 2 \) on \( SL_2(\mathbb{Z}) \). Here we give an explicit description of the form \( \tilde{f} \) appearing above in terms of the modular functions \( j_m(z) \).

To state these results, for every point \( \tau \in \mathcal{H} \) let \( H_\tau(z) := \sum_{n=0}^{\infty} j_n(\tau)q^n \). Asai, Kaneko, and Ninomiya [Th. 3, A-K-N] proved that
\[ H_\tau(z) = \sum_{n=0}^{\infty} j_n(\tau)q^n = \frac{E_4^2(z)E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z) - j(\tau)}. \]  

In particular, for every \( \tau \) it turns out that \( H_\tau(z) \) is a weight 2 meromorphic modular form. The utility of (2.4) was already known; for example, it can be used to prove the famous denominator formula for the monster Lie algebra.

In [Br-K-O], the following result is proved.
Theorem 2.1. If \( f = \sum_{n=h}^{\infty} a_f(n)q^n \) is a nonzero weight \( k \) meromorphic modular form on \( SL_2(\mathbb{Z}) \) for which \( a_f(h) = 1 \), then

\[
\frac{\Theta(f)}{f} = \frac{kE_2}{12} - f_\Theta,
\]

where \( f_\Theta \) is defined by

\[
f_\Theta := \sum_{\tau \in \mathfrak{F}} e_{\tau} \text{ord}_\tau(f) H_\tau(z).
\]

For convenience, we recall the following straightforward fact (see [Prop. 2.1, Br-K-O]) which explains the connection between Theorem 2.1 and the combinatorial properties of the infinite product exponents of a given modular form.

Proposition 2.2. Let \( f = \sum_{n=h}^{\infty} a_f(n)q^n \) be a meromorphic function in a neighborhood of \( q = 0 \), and suppose that \( a_f(h) = 1 \). Then there are uniquely determined complex numbers \( c(n) \) such that

\[
f = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)},
\]

where the product converges in a small neighborhood of \( q = 0 \). Moreover, the following identity is true

\[
\frac{\Theta(f)}{f} = h - \sum_{n=1}^{\infty} \sum_{d|n} c(d) dq^n.
\]

Sketch of the Proof of Theorem 2.1. As before, let \( \mathfrak{F} \) denote the standard fundamental domain for the action of \( SL_2(\mathbb{Z}) \) on \( \mathcal{H} \). We cut off \( \mathcal{F} \) by a horizontal line \( \mathcal{L} := \{ iC - t : -\frac{1}{2} \leq t \leq \frac{1}{2} \} \) where \( C > 0 \) is chosen so large that all poles and zeros of \( f \), apart from those at the cusp at infinity, are contained in \( \{ z \in \mathcal{H} : \text{Im}(z) < C \} \).

For simplicity, suppose that \( f(z) \) has no zero or pole on the boundary \( \partial \mathcal{F} \) except possibly \( i \) or \( \omega \), and let \( \gamma \) be the closed path with positive orientation consisting of \( \mathcal{L} \) and \( \gamma_1 \) where \( \gamma_1 \) is the part of \( \partial \mathfrak{F} \) below \( \mathcal{L} \) modified in the usual way (i.e. as in the standard proof of the “\( k/12 \)-valence formula”).

We integrate \( \frac{1}{2\pi i} \frac{f'(z)}{f(z)} J_n(z) \) along \( \gamma \). By the residue theorem, taking into account that \( J_n(z) \) is holomorphic on \( \mathcal{H} \), this integral is equal to

\[
\sum_{\tau \in \mathfrak{F} - \{ \omega, i \}} \text{ord}_\tau(f) J_n(\tau).
\]
On the other hand, the integral can be evaluated separately along the different pieces of $\gamma$. If we let $r$ tend to zero, we then find that

$$\sum_{\tau \in \mathcal{D} - \{\omega, i\}} \text{ord}_\tau(f) j_n(\tau) =$$

$$-\frac{1}{3} \text{ord}_\omega(f) j_n(\omega) - \frac{1}{2} \text{ord}_i(f) j_n(i) + \frac{1}{2\pi i} \int_\rho \frac{F'(q)}{F(q)} J_n(q) dq - \frac{k}{2\pi i} \int_\sigma \frac{j_n(z)}{z} dz. \quad (2.5)$$

Here $F(q) = f(z)$ as before and $J_n(q) := j_n(z)$. Furthermore, $\rho$ is a small circle around $q = 0$ with negative orientation and not containing any pole or zero of $F(q)$ except possibly 0, and $\sigma$ is the part of the unit circle in the upper half-plane that connects $\omega$ and $i$, with positive orientation.

By Proposition 2.2, we have that

$$qF'(q) F(q) = \frac{\Theta(f)}{f} = h - \sum_{n=1}^\infty \sum_{d|n} c(d) dq^n,$$

where $h$ is the order of $F$ at $q = 0$. Therefore we find that

$$\frac{1}{2\pi i} \int_\rho \frac{F'(q)}{F(q)} J_n(q) dq = \sum_{d|n} c(d) d. \quad (2.6)$$

Since (2.5) holds for $f = \Delta$, and since $\Delta$ has no zeros on $\mathcal{H}$, we obtain from (2.5) that

$$\frac{1}{2\pi i} \int_\sigma \frac{j_n(z)}{z} dz = 2\sigma_1(n). \quad (2.7)$$

To prove that

$$\frac{\Theta(f)}{f} = \frac{kE_2}{12} - f_\Theta,$$

where $f_\Theta$ has the claimed form, one now simply argues coefficient by coefficient with (1.2), (2.5), (2.6) and (2.7).

□

3. PROOF OF THEOREMS 1 AND 2

Proof of Theorem 1. By Theorem 2.1, if $n$ is a positive integer, then

$$\sum_{d|n} c(d) d = 2k\sigma_1(n) + \sum_{\tau \in \mathcal{D}} e_{\tau \text{ord}_\tau(f)} j_n(\tau). \quad (3.1)$$
Therefore, to prove Theorem 1 it suffices to obtain a closed formula for

\[ b(n) = \sum_{d \mid n} c(d)d \]  \hspace{1cm} (3.2)

in terms of the coefficients of \( f(z) \). In particular, a straightforward calculation reveals that it suffices to show that if \( n \geq 1 \), then

\[ b(n) = n \sum_{m_1, \ldots, m_n \geq 0, \atop m_1 + 2m_2 + \cdots + nm_n = n} (-1)^{m_1 + \cdots + m_n} \times \frac{(m_1 + \cdots + m_n - 1)!}{m_1! \cdots m_n!} \cdot a_f(h + 1)^{m_1} \cdots a_f(h + n)^{m_n}. \]  \hspace{1cm} (3.3)

To prove (3.3), observe that

\[ 0 = b(n) + b(n - 1)a_f(h + 1) + b(n - 2)a_f(h + 2) + \cdots + b(1)a_f(h + n - 1) + na_f(h + n), \]

and then use the fact that

\[ 0 = s_n - s_{n-1}\sigma_1 + s_{n-2}\sigma_2 - \cdots + (-1)^{n-1}s_1\sigma_{n-1} + (-1)^n n\sigma_n. \]

Here the \( \sigma_i \) are the elementary symmetric functions in \( X_1, \ldots, X_n \) and the \( s_i \) are the power functions in these variables (i.e. \( s_i := X_1^i + \cdots + X_n^i \)). By evaluating these identities at \( (X_1, \ldots, X_n) = (\lambda(1, n), \ldots, \lambda(n, n)) \) where the \( \lambda(j, n) \) are the roots of the polynomial

\[ X^n + a_f(h + 1)X^{n-1} + a_f(h + 2)X^{n-2} + \cdots + a_f(h + n), \]

one obtains (3.3). This final calculation requires the fact that

\[ s_i = i \sum_{m_1, \ldots, m_n \geq 0, \atop m_1 + 2m_2 + \cdots + nm_n = i} (-1)^{m_2 + m_4 + \cdots} \frac{(m_1 + m_2 + \cdots + m_n - 1)!}{m_1! m_2! \cdots m_n!} \cdot \sigma_1^{m_1} \cdots \sigma_n^{m_n}. \]

\[ \square \]

Proof of Theorem 2. By (3.1), it follows that

\[ F(n) := \sum_{d \mid n} c(d)d = 2k\sigma_1(n) + \sum_{\tau \in \mathcal{D}} e_{\tau \text{ord}_{\tau}}(f)j_n(\tau). \]
By Möbius Inversion, it follows that

\[ c(n) = \frac{1}{n} \sum_{d|n} \mu(n/d)F(d) \]

\[ = \frac{1}{n} \sum_{d|n} \mu(n/d) \left( 2k\sigma_1(d) + \sum_{\tau \in \mathfrak{F}} e_{\tau} \text{ord}_{\tau}(f)j_d(\tau) \right). \]

The formula for \( c(n) \) follows from the elementary identity

\[ n = \sum_{d|n} \mu(n/d)\sigma_1(d). \]

□

References


Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706
E-mail address: ono@math.wisc.edu