PARTITIONS AND MCKAY NUMBERS FOR S_n

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1. Introduction and Statement of Results

If ℓ is a prime number and G is a finite group, then let $m_\ell(k;G)$ denote the usual ℓ^k-power *McKay number* for G. This integer is the number of irreducible characters, say Ψ, of G with the property that $\mathrm{ord}_\ell(\deg(\Psi)) = k$ (i.e. $\ell^k || \deg(\Psi)$). Here we investigate the arithmetic properties of these integers for the symmetric groups. The McKay numbers $m_\ell(k;S_n)$ are values of certain partition functions (see (1.3)), and this interpretation plays a central role in Olsson's proof of the Alperin and McKay Conjectures for symmetric groups [8]. Here we investigate the number theoretic properties of these partition functions.

A *partition* of an integer n is any non-increasing sequence of positive integers whose sum is n. If

$$\Lambda = \{\lambda_1 \geq \lambda_2 \cdots \geq \lambda_t\}$$

is a partition of n, then we say that $|\Lambda| = n$ and $\Lambda \vdash n$. The Ferrers-Young diagram of Λ is an array of nodes with λ_k nodes in the kth row. We assign numbers to the rows and columns, and coordinates to the nodes, just as we do for a matrix. The (i,j) *hook* is the set of nodes directly below, together with the set of nodes directly to the right of, the (i,j) node, as well as the (i,j) node itself (i.e. the nodes (i,k) with $k \geq j$ together with the nodes (k,j) with $k \geq i$). The *hook number*, denoted by $H(i,j)$, is the total number of nodes on the (i,j) hook. The *hook product* of Λ is

$$H(\Lambda) := \prod_{(i,j)} H(i,j). \tag{1.1}$$

For example, the Ferrers-Young diagram of the partition $\Lambda = \{4,3,1\}$ is

$$\begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ 1 & \bullet_{(1,1)} & \bullet_{(1,2)} & \bullet_{(1,3)} & \bullet_{(1,4)} \\ 2 & \bullet_{(2,1)} & \bullet_{(2,2)} & \bullet_{(2,3)} & \\ 3 & \bullet_{(3,1)} & & & \end{array}.$$

The hooks at $(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(3,1)$, have hook numbers $6,4,3,1,4,2,1,1$, respectively, and so $H(\Lambda) = 576$.

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If ℓ is prime and $a \geq 0$ is an integer, then define the partition function $p_\ell(a;n)$ by

$$p_\ell(a;n) := \#\{\Lambda \vdash n \ : \ \mathrm{ord}_\ell(H(\Lambda)) = a\}. \tag{1.2}$$

By the representation theory of the symmetric group (see [6]), it turns out that

$$p_\ell(a;n) = m_\ell(\mathrm{ord}_\ell(n!) - a; S_n). \tag{1.3}$$

If $p(n)$ denotes the usual partition function (i.e. the number of partitions of n), then we obviously have

$$p(n) = \sum_{a\geq 0} p_\ell(a;n). \tag{1.4}$$

In his famous work on partitions, Ramanujan proved, for every integer n, that (see Chapter 10 of [3])

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5},\\ p(7n+5) &\equiv 0 \pmod{7},\\ p(11n+6) &\equiv 0 \pmod{11},\\ p(25n+24) &\equiv 0 \pmod{25},\\ p(49n+47) &\equiv 0 \pmod{49},\\ p(121n+116) &\equiv 0 \pmod{121}. \end{aligned} \tag{1.5}$$

These congruences are the first cases of three families of congruences modulo arbitrary powers of 5, 7, and 11. In view of (1.4), it is natural to ask whether these congruences descend to congruences for the $p_\ell(a;n)$ functions. Although they are not explicitly stated, a finer analysis of the arguments in the famous paper [4] by Garvan, Kim and Stanton on cranks, and the subsequent paper [5], shows that the congruences in (1.5) with modulus $5, 7, 11, 25$, and 49 do indeed descend. Here we show, in a straightforward way, that these congruences (including the congruence with modulus 121) descend.

For convenience, if $\ell \in \{5, 7, 11\}$, then let

$$\delta(\ell) := (\ell^2 - 1)/24. \tag{1.6}$$

The congruences in (1.5) may be rewritten, for $m = 1$ and 2, as

$$p(\ell^m n - \delta(\ell)) \equiv 0 \pmod{\ell^m}.$$

Theorem 1.1. *Suppose that $\ell = 5, 7,$ or 11, and that $m = 1$ or 2. If $a \geq 0$ is an integer, then for every positive integer n we have*

$$p_\ell(a; \ell^m n - \delta(\ell)) \equiv 0 \pmod{\ell^m}.$$

Remark. The Ramanujan congruences modulo $5^3, 7^3$ and 11^3 do not descend. For example, although we have the congruence

$$p(125n + 99) \equiv 0 \pmod{125},$$

one easily finds that

$$p_5(14; 99) = 5594200 \equiv 75 \not\equiv 0 \pmod{5^3}.$$

In addition to these Ramanujan congruences, there are universal congruences which hold for all primes ℓ.

Theorem 1.2. *If ℓ is prime, then the following are true.*

(1) *If $0 \leq n < \ell^2$ and $a \geq 1$, then*

$$p_\ell(a; n) \equiv 0 \pmod{\ell}.$$

(2) *If $\ell \nmid a$, then for every non-negative integer n we have*

$$p_\ell(a; n) \equiv 0 \pmod{\ell}.$$

Remark. Theorem 1.2 (1), in the special case when $a = \text{ord}_\ell(n!)$, follows from Olsson's proof [8] of McKay's Conjecture for S_n. This deduction uses the fact that the ℓ-Sylow subgroups, say L, of S_n are abelian when $n < \ell^2$. In these cases, it is not difficult to construct $N_{S_n}(L)$, the normalizer of L within S_n, along with its characters with prime to ℓ degree. The number of such characters is a multiple of ℓ, and their number, as predicted by McKay's Conjecture, equals $m_\ell(0; S_n)$.

Theorems 1.1 and 1.2 follow from a combinatorial study of ℓ-core partitions and the ℓ-adic decomposition of partitions (for example, see Olsson's paper [8]). They are not difficult to prove using a beautiful generating function, due to Nakamura, which is discussed in Section 2.

Nakamura's generating function makes it possible to study the arithmetic properties of the $p_\ell(a; n)$ using the theory of modular forms. Here we give a brief indication of the type of results which one can obtain by employing modular forms.

Define the partition functions $p_\ell^{\text{even}}(n)$ and $p_\ell^{\text{odd}}(n)$ by

$$p_\ell^{\text{even}}(n) := \sum_{a \text{ even}} p_\ell(a; n) \qquad \text{and} \qquad p_\ell^{\text{odd}}(n) := \sum_{a \text{ odd}} p_\ell(a; n). \tag{1.7}$$

When $\ell = 2$, Theorem 1.2 implies the nearly obvious fact that $p_2^{\text{odd}}(n)$ is even for all n. Here we use the elementary theory of modular forms to prove the following recursive refinement of this observation.

Theorem 1.3. *If n is a non-negative integer, then*

$$p_2^{\text{odd}}(n) \equiv 2 \sum_{x,y \geq 1 \text{ odd}} p(n - x^2 - y^2) \pmod{8}.$$

From the proof of Theorem 1.3, it will be clear that one can obtain further such congruences modulo arbitrary powers of 2. However, the complexity of such relations grows rapidly with the power of 2.

There are other consequences of its proof which deserve mention. First we note that

$$\sum_{n=0}^{\infty} \left(p_2^{\text{even}}(n) - p_2^{\text{odd}}(n)\right) q^n = 1 + q - 2q^2 - q^3 - 3q^4 - 5q^5 + 7q^6 + 3q^7 - 6q^8 + 2q^9 + \cdots .$$

Although the numbers $p_2^{\text{even}}(n) - p_2^{\text{odd}}(n)$ are small, it seems to be the case that there are no integers n for which $p_2^{\text{even}}(n) = p_2^{\text{odd}}(n)$. In spite of this expectation, modulo every power of 2 there are Euler-type recurrence congruences, which hold for almost every integer n, relating these quantities.

Theorem 1.4. *For every positive integer j, almost every non-negative integer n has the property that*

$$\sum_{k\in\mathbb{Z}} (-1)^k \left(p_2^{\text{even}}(n-\omega(k)) - p_2^{\text{odd}}(n-\omega(k))\right) \equiv 0 \pmod{2^j},$$

where $\omega(k) := \frac{3k^2+k}{2}$.

Remark. Here we make precise the statement of Theorem 1.4. For each positive integer j, there is a positive $\alpha_j > 0$ for which the number of non-negative integers $n \leq X$ not satisfying the congruence is $O\left(\frac{X}{\log(X)^{\alpha_j}}\right)$.

In Section 3 we prove Theorem 1.1 and Theorem 1.2, and in Section 4 we prove Theorems 1.3 and 1.4. In Section 5 we present examples illustrating Theorems 1.1 and 1.2, and we raise natural questions.

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2. A generating function

Here we recall a beautiful theorem of Nakamura (see Theorem 3.5 of [7]) which provides an infinite product formula for

$$F_\ell(x;q) = \sum_{n=0}^{\infty} A_\ell(x;n)q^n := \sum_{\Lambda} x^{\text{ord}_\ell(H(\Lambda))} q^{|\Lambda|} = \sum_{n=0}^{\infty}\sum_{a=0}^{\infty} p_\ell(a;n)x^a q^n. \tag{2.1}$$

Here the sum is over all partitions Λ.

Theorem 2.1. *If ℓ is prime and if $[k]_\ell := (\ell^k - 1)/(\ell - 1)$, then*

$$F_\ell(x;q) = \prod_{k=0}^{\infty}\prod_{n=1}^{\infty} \frac{(1 - x^{\ell n[k]_\ell} q^{\ell^{k+1}n})^{\ell^{k+1}}}{(1 - x^{n[k]_\ell} q^{\ell^k n})^{\ell^k}}.$$

Corollary 2.2. *Suppose that n is a non-negative integer.*

(1) *There are polynomials $a_\ell(x;j) \in \mathbb{Z}[x]$ for which*
$$A_\ell(x;n) = p(n) + \sum_{j=1}^{\lfloor n/\ell \rfloor} p(n-\ell j) a_\ell(x;j).$$

(2) *There are polynomials $b_\ell(x;j)$ and $c_\ell(x;j)$ in $\mathbb{Z}[x]$ for which*
$$A_\ell(x;n) = p(n) + \sum_{\substack{j,k\geq 0,\\ (j,k)\neq(0,0)}} p(n-\ell j - \ell^2 k) b_\ell(x;j) c_\ell(x;k).$$
Moreover, if $\ell \nmid j$, then $b_\ell(x;j) \equiv 0 \pmod{\ell}$ (i.e. $b_\ell(x;j) \in \ell\mathbb{Z}[x]$).

Proof. Using Euler's generating function
$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$
and by separating the $k=0$ denominator from the product in Theorem 2.1, we have
$$\begin{aligned} F_\ell(x;q) &= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \prod_{k=1}^{\infty}\prod_{n=1}^{\infty} \frac{1}{(1-x^{n[k]_\ell} q^{\ell^k n})^{\ell^k}} \prod_{k=0}^{\infty}\prod_{n=1}^{\infty} (1-x^{\ell n[k]_\ell} q^{\ell^{k+1} n})^{\ell^{k+1}} \\ &= \left[\sum_{n=0}^{\infty} p(n)q^n\right] \cdot \prod_{k=0}^{\infty}\prod_{n=1}^{\infty} \frac{(1-x^{\ell n[k]_\ell} q^{\ell^{k+1} n})^{\ell^{k+1}}}{(1-x^{n[k+1]_\ell} q^{\ell^{k+1} n})^{\ell^{k+1}}} \qquad (2.2) \\ &= \left[\sum_{n=0}^{\infty} p(n)q^n\right] \cdot \left[1 + \sum_{n=1}^{\infty} a_\ell(x;n) q^{\ell n}\right]. \end{aligned}$$
The last equality is obtained by observing that all the powers of q in the preceding infinite product are positive multiples of ℓ, and by expanding the denominators here using geometric series. This proves (1).

To prove (2), we argue as above, by separating the $k=0$ factor from (2.2), and we obtain
$$\begin{aligned} F_\ell(x;q) &= \left[\sum_{n=0}^{\infty} p(n)q^n\right] \cdot \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})^{\ell}}{(1-x^n q^{\ell n})^{\ell}} \cdot \prod_{k=1}^{\infty}\prod_{n=1}^{\infty} \frac{(1-x^{\ell n[k]_\ell} q^{\ell^{k+1} n})^{\ell^{k+1}}}{(1-x^{n[k+1]_\ell} q^{\ell^{k+1} n})^{\ell^{k+1}}} \\ &= \left[\sum_{n=0}^{\infty} p(n)q^n\right] \cdot \left[1 + \sum_{n=1}^{\infty} b_\ell(x;n) q^{\ell n}\right] \cdot \left[1 + \sum_{n=1}^{\infty} c_\ell(x;n) q^{\ell^2 n}\right]. \end{aligned}$$
To see that $b_\ell(x;n) \equiv 0 \pmod{\ell}$, for those n coprime to ℓ, observe that
$$1 + \sum_{n=1}^{\infty} b_\ell(x;n) q^{\ell n} = \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})^{\ell}}{(1-x^n q^{\ell n})^{\ell}} \equiv \prod_{n=1}^{\infty} \frac{(1-q^{\ell^2 n})}{(1-x^{\ell n} q^{\ell^2 n})} \pmod{\ell}.$$
$\square$

3. Proofs of Theorem 1.1 and 1.2

We begin by showing that the Ramanujan congruences (1.5) imply Theorem 1.1.

Proof of Theorem 1.1. Suppose that $b > a$ is any positive integer, and let $\omega := e^{2\pi i/b}$. Define the q-series $\mathfrak{F}_\ell(a,b;q)$ by

$$\mathfrak{F}_\ell(a,b;q) := \frac{1}{b}\sum_{s=1}^{b} \omega^{-as} F_\ell(\omega^s;q). \tag{3.1}$$

By definition (2.1), we have

$$\mathfrak{F}_\ell(a,b;q) = \frac{1}{b}\sum_{\Lambda}\left(\sum_{s=1}^{b}\omega^{s(\mathrm{ord}_\ell(H(\Lambda))-a)}\right) q^{|\Lambda|}.$$

Since we have

$$\sum_{s=1}^{b}\omega^{sj} = \begin{cases} b & \text{if } b \mid j, \\ 0 & \text{otherwise,}\end{cases} \tag{3.2}$$

it follows that

$$\mathfrak{F}_\ell(a,b;q) = \sum_{n=0}^{\infty} \mathfrak{p}_\ell(a,b;n) q^n \in Z[[q]],$$

where

$$\mathfrak{p}_\ell(a,b;n) := \#\{\Lambda \vdash n \ : \ \mathrm{ord}_\ell(H(\Lambda)) \equiv a \pmod{b}\}. \tag{3.3}$$

Combining Corollary 2.2 (1) with (2.1), (3.1) and (3.3), we find that

$$\begin{aligned}\mathfrak{p}_\ell(a,b;n) &= \frac{1}{b}\sum_{s=1}^{b}\omega^{-as}A_\ell(\omega^s;n)\\ &= \frac{1}{b}\sum_{s=1}^{b}\omega^{-as}\left[p(n) + \sum_{j=1}^{\lfloor n/\ell\rfloor} p(n-\ell j) a_\ell(\omega^s;j)\right]\\ &= \frac{1}{b}\cdot p(n)\sum_{s=1}^{b}\omega^{-as} + \frac{1}{b}\sum_{j=1}^{\lfloor n/\ell\rfloor} p(n-\ell j)\sum_{s=1}^{b}\omega^{-as}a_\ell(\omega^s;j).\end{aligned} \tag{3.4}$$

Similarly, Corollary 2.2 (2) implies that
(3.5)

$$\mathfrak{p}_\ell(a,b;n) = \frac{1}{b}\cdot p(n)\sum_{s=1}^{b}\omega^{-as} + \frac{1}{b}\sum_{\substack{j,k\geq 0\\ (j,k)\neq(0,0)}} p(n-\ell j-\ell^2 k)\sum_{s=1}^{b}\omega^{-as}b_\ell(\omega^s;j)c_\ell(\omega^s;k).$$

Replacing n by $\ell n - \delta(\ell)$ in (3.4), we obtain

$$\mathfrak{p}_\ell(a,b;\ell n-\delta(\ell)) = \frac{1}{b}\cdot p(\ell n-\delta(\ell))\sum_{s=1}^{b}\omega^{-as}+\frac{1}{b}\sum_{j=1}^{n-1}p(\ell n-\ell j-\delta(\ell))\sum_{s=1}^{b}\omega^{-as}a_\ell(\omega^s;j).$$

Since each $a_\ell(x;j) \in \mathbb{Z}[x]$, (3.2) and (1.5) together imply that

$$\mathfrak{p}_\ell(a,b;\ell n-\delta(\ell)) \equiv 0 \pmod{\ell}. \tag{3.6}$$

Replacing n by $\ell^2 n - \delta(\ell)$ in (3.5), we find that

$$\mathfrak{p}_\ell(a,b;\ell^2 n-\delta(\ell)) = \frac{1}{b}\cdot p(\ell^2 n-\delta(\ell))\sum_{s=1}^{b}\omega^{-as}$$
$$+\frac{1}{b}\sum_{\substack{j,k\geq 0\\(j,k)\neq(0,0)}} p(\ell^2 n-\ell j-\ell^2 k-\delta(\ell))\sum_{s=1}^{b}\omega^{-as}b_\ell(\omega^s;j)c_\ell(\omega^s;k).$$

By (1.5), the fact that $b_\ell(x;j) \equiv 0 \pmod{\ell}$ for those j coprime to ℓ, and (3.2), it follows that

$$\mathfrak{p}_\ell(a,b;\ell^2 n-\delta(\ell)) \equiv 0 \pmod{\ell^2}. \tag{3.7}$$

For any given integer n, the finiteness of the number of partitions of size n implies that every sufficiently large b has the property that $\mathfrak{p}_\ell(a,b;n) = p_\ell(a;n)$. In view of (3.6) and (3.7), this completes the proof of Theorem 1.1.

□

Proof of Theorem 1.2. Combining the proof of Corollary 2.2 (2) with the fact that $(1-X)^\ell \equiv (1-X^\ell) \pmod{\ell}$, we find that

$$F_\ell(x;q) \equiv \left[\sum_{n=0}^{\infty}p(n)q^n\right]\cdot\prod_{n=1}^{\infty}\frac{(1-q^{\ell^2 n})}{(1-x^{\ell n}q^{\ell^2 n})}\cdot\prod_{k=1}^{\infty}\prod_{n=1}^{\infty}\frac{(1-x^{\ell^{k+2}n[k]_\ell}q^{\ell^{2k+2}n})}{(1-x^{\ell^{k+1}n[k+1]_\ell}q^{\ell^{2k+2}n})} \pmod{\ell}$$
$$\equiv \left[\sum_{n=0}^{\infty}p(n)q^n\right]\cdot\prod_{n=1}^{\infty}\frac{(1-q^{\ell^2 n})}{(1-x^{\ell n}q^{\ell^2 n})}\cdot\left[1+\sum_{n=1}^{\infty}d_\ell(x^\ell;n)q^{\ell^4 n}\right] \pmod{\ell}.$$

The last equality is obtained by observing that all the powers of q (resp. x) in the preceding doubly infinite product are positive multiples of ℓ^4 (resp. ℓ), and by expressing every denominator using geometric series. It is clear that the $d_\ell(x;n)$ are polynomials in $\mathbb{Z}[x]$, and so Theorem 1.2 follows easily. □

4. Modular forms and the proof of Theorem 1.3 and 1.4

As usual, let $M_k(\Gamma_0(N))$ denote the space of holomorphic integer weight k modular forms on the congruence subgroup $\Gamma_0(N)$ (for example, see [10]). Here we prove Theorems 1.3 and 1.4 using facts about the Dedekind eta-function

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \tag{4.1}$$

where $q := e^{2\pi iz}$. For our purposes, we simply need to recall that

$$\frac{\eta(z)^2}{\eta(2z)} = 1-2q+2q^4-\cdots = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} \tag{4.2}$$

is a holomorphic modular form of weight 1/2 on $\Gamma_0(16)$ (for example, see Chapter 1 of [10]).

Theorem 4.1. *For every integer $j \geq 3$, there is a modular form*

$$F_j(z) = \sum_{n=0}^{\infty} a_j(n)q^n \in M_{w(j)}(\Gamma_0(2^{j+3})),$$

where $w(j) = 2^{j-1}(2^{j-2}-1)$, with integer coefficients for which

$$F_j(z) \equiv \prod_{n=1}^{\infty}(1-q^n)\cdot\sum_{n=0}^{\infty}\left(p_2^{\text{even}}(n)-p_2^{\text{odd}}(n)\right)q^n \pmod{2^j}.$$

Proof. By Theorem 2.1, we have

$$\begin{aligned} F_2(-1;q) &= \left[\sum_{n=0}^{\infty}p(n)q^n\right]\cdot\prod_{k=0}^{\infty}\prod_{n=1}^{\infty}\frac{(1-q^{2^{k+1}n})^{2^{k+1}}}{(1-(-1)^n q^{2^{k+1}n})^{2^{k+1}}}\\ &= \left[\sum_{n=0}^{\infty}p(n)q^n\right]\cdot\prod_{k=0}^{\infty}\prod_{n=0}^{\infty}\frac{(1-q^{2^{k+1}(2n+1)})^{2^{k+1}}}{(1+q^{2^{k+1}(2n+1)})^{2^{k+1}}}\end{aligned}$$

Since $\prod_{n=0}^{\infty}\frac{(1-q^{2n+1})}{(1+q^{2n+1})} = \prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-q^{4n})}{(1-q^{2n})^3}$, it follows that

$$\prod_{n=1}^{\infty}(1-q^n)\cdot\sum_{n=0}^{\infty}\left(p_2^{\text{even}}(n)-p_2^{\text{odd}}(n)\right)q^n = \prod_{k=0}^{\infty}f(2^{k+1}z)^{2^{k+1}}, \tag{4.3}$$

where

$$f(z) = \frac{\eta(z)^2\eta(4z)}{\eta(2z)^3}. \tag{4.4}$$

Since $(1-X)^2 \equiv (1-X^2) \pmod 2$, it follows that $f(z) \equiv 1 \pmod 2$. Moreover, if m is a non-negative integer, then a straightforward inductive argument implies that

$$f(2^m z)^{2^m} \equiv 1 \pmod{2^{m+1}}.$$

Combining this observation with (4.3) and (4.4), we find that

$$\prod_{n=1}^{\infty}(1-q^n)\cdot\sum_{n=0}^{\infty}\left(p_2^{\text{even}}(n)-p_2^{\text{odd}}(n)\right)q^n \equiv \prod_{k=0}^{j-3} f(2^{k+1}z)^{2^{k+1}} \pmod{2^j}.$$

Now define $f_j(z)$ by

$$f_j(z) = f(z)\cdot\left[\frac{\eta(2z)^2}{\eta(4z)}\right]^{2^{j-1}}. \tag{4.5}$$

Arguing as above, we find that $f_j(z)\equiv f(z) \pmod{2^j}$. By construction, we have

$$f_j(z) = \frac{\eta(z)^2}{\eta(2z)}\cdot\left[\frac{\eta(2z)^2}{\eta(4z)}\right]^{2^{j-1}-1} \in M_{2^{j-2}}(\Gamma_0(32),\chi_2),$$

where $\chi_2 = \left(\frac{2}{\bullet}\right)$. This follows easily from the fact that $f_j(z)$ is a product of the weight 1/2 modular forms which are essentially the theta functions given in (4.2). The theorem follows by letting

$$F_j(z) := \prod_{k=0}^{j-3} f_j(2^{k+1}z)^{2^{k+1}}.$$

□

Remark. The weight $w(j)$ in Theorem 4.1 is not minimal. This is a consequence of the fact that we chose to prove Theorem 4.1 using a classical theta function identity. A proof involving a careful analysis of the poles of $f(z)$ would provide lower weight choices for $F_j(z)$.

Theorem 1.4 follows easily from Theorem 4.1.

Proof of Theorem 1.4. Suppose that $F(z)=\sum_{n=0}^{\infty}A(n)q^n\in\mathbb{Z}[[q]]$ is a holomorphic integer weight modular form. If M is an integer, then a famous theorem of Serre [11] asserts that

$$A(n)\equiv 0 \pmod{M}$$

for almost every integer n (i.e. in the sense of arithmetic density). By combining this result, in the case of the modular forms $F_j(z)$ in Theorem 4.1, with Euler's identity

$$\prod_{n=1}^{\infty}(1-q^n)=\sum_{k=-\infty}^{\infty}(-1)^k q^{\omega(k)},$$

we obtain Theorem 1.4.

□

Proof of Theorem 1.3. Let $F_4(z) \in M_{24}(\Gamma_0(128))$ be the modular form in the statement of Theorem 4.1. The first few terms of $F_4(z)$ are

$$F_4(z) = 1 - 4q^2 - 32q^4 + 144q^6 + \cdots .$$

By Theorem 4.1, it follows that

$$\sum_{n=0}^{\infty} \left(p_2^{\text{even}}(n) - p_2^{\text{odd}}(n)\right) q^n \equiv \left[\sum_{n=0}^{\infty} p(n)q^n\right] \cdot F_4(z) \pmod{16}. \tag{4.6}$$

A computer calculation reveals that $F_4(z) \pmod{16}$ agrees with the first 500 terms of the Fourier expansion of the modular form

$$G_4(z) = E_4(z)^6 + 12\eta(8z)^2\eta(16z)^2E_6(z)E_4(z)^4 \pmod{16}.$$

Here $E_4(z)$ and $E_6(z)$ are the usual Eisenstein series

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sum_{d\mid n} d^3q^n,$$

$$E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sum_{d\mid n} d^5q^n.$$

The modular form $G_4(z)$ is also in $M_{24}(\Gamma_0(128))$ (for example, see Chapter 1 of [10]). A theorem of Sturm [12] implies that these modular forms are congruent modulo 16 since their q-expansions modulo 16 agree for the first 385 terms. Hence it follows that

$$F_4(z) \equiv G_4(z) \equiv 1 + 12\eta(8z)^3 \cdot \frac{\eta(16z)^2}{\eta(8z)} \pmod{16}.$$

By Jacobi's theta function identities (for example, see Chapter 1 of [10]), we have

$$\eta(8z)^3 = \sum_{x=0}^{\infty}(-1)^x(2x+1)q^{(2x+1)^2},$$

$$\frac{\eta(16z)^2}{\eta(8z)} = \sum_{x=0}^{\infty} q^{(2x+1)^2}.$$

Since $12(-1)^{(x-1)/2}x \equiv 12 \pmod{16}$ for all odd x, (4.6) implies that

$$\sum_{n=0}^{\infty} \left(p_2^{\text{even}}(n) - p_2^{\text{odd}}(n)\right) q^n \equiv \left[\sum_{n=0}^{\infty} p(n)q^n\right] \left[1 + 12\sum_{x,y\geq 1 \text{ odd}} q^{x^2+y^2}\right] \pmod{16}.$$

Theorem 1.3 follows since

$$p(n) = p_2^{\text{even}}(n) + p_2^{\text{odd}}(n).$$

□

5. Examples and concluding remarks

Here we present several examples illustrating Theorems 1.1 and 1.2.

Example. First we illustrate Theorem 1.2. Here we list the first few terms of the series

$$F_\ell(x;q) = \sum_{n=0}^{\infty}\sum_{a=0}^{\infty} p_\ell(a;n)x^a q^n$$

for $\ell = 2$ and 3. For $\ell = 2$ we have

$$\begin{aligned} F_2(x;q) &= 1 + q + 2xq^2 + (2x+1)q^3 + (x^2+4x^3)q^4 + (2x+x^2+4x^3)q^5 + \cdots \\ &\equiv 1 + q + q^3 + x^2q^4 + x^2q^5 + q^6 + x^2q^7 + (x^2+1)q^{10}\cdots \pmod 2. \end{aligned} \tag{5.1}$$

For $\ell = 3$ we have

(5.2)

$$\begin{aligned} F_3(x;q) &= 1 + q + 2q^2 + 3xq^3 + (2+3x)q^4 + (1+6x)q^5 + (2+9x^2)q^6 + \cdots \\ &\equiv 1 + \cdots + 2q^4 + q^5 + 2q^6 + q^8 + (x^3+2)q^9 + (x^3+2)q^{10} + \cdots \pmod 3. \end{aligned}$$

Examples (5.1) and (5.2) illustrate the two phenomena described in Theorem 1.2. If $0 \le n < \ell^2$ and $a \ge 1$, then

$$p_\ell(a;n) \equiv 0 \pmod{\ell}.$$

This is illustrated by the absence of any powers of x in the first few terms of the reductions of these series modulo ℓ. The second phenomenon is that

$$p_\ell(a;n) \equiv 0 \pmod{\ell}$$

whenever $\ell \nmid a$. This is illustrated by the fact that the only powers of x in $F_2(x;q) \pmod 2$ (resp. $F_3(x;q) \pmod 3$) are powers of x^2 (resp. x^3).

Example. To illustrate Theorem 1.1, we exhibit the first few terms of $F_5(x;q)$:

$$F_5(x;q) = 1 + q + \cdots + (2+5x)q^5 + \cdots + (5+25x)q^9 + \cdots + (10+25x+100x^2)q^{14} + \cdots.$$

All the coefficients of the polynomial coefficients of q^{5n+4} are multiples of 5.

We conclude with a number of natural questions.

Question 1. By the works of the author and Ahlgren [1, 2, 9], it is known that there are "Ramanujan type" congruences for $p(n)$ for every modulus M coprime to 6. Here are some examples for the primes $17 \le M \le 31$:

$$\begin{aligned} &p(48037937n + 1122838) \equiv 0 \pmod{17}, \\ &p(1977147619n + 815655) \equiv 0 \pmod{19}, \\ &p(14375n + 3474) \equiv 0 \pmod{23}, \\ &p(348104768909n + 43819835) \equiv 0 \pmod{29}, \\ &p(4063467631n + 30064597) \equiv 0 \pmod{31}. \end{aligned}$$

Do any of these congruences descend to congruences for the functions $p_\ell(a;n)$?

Question 2. What is the order of magnitude of $p_\ell(a,n)$?

Question 3. Is it true that

$$p_2^{\text{even}}(n) \neq p_2^{\text{odd}}(n)$$

for every integer n?

Question 4. What is the asymptotic behavior of

$$p_2^{\text{even}}(n) - p_2^{\text{odd}}(n)$$

as $n \to +\infty$?

Question 5. Investigate the properties of $p_\ell^{\text{even}}(n) - p_\ell^{\text{odd}}(n)$ for odd primes ℓ.

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