LINEAR RELATIONS BETWEEN MODULAR FORM COEFFICIENTS AND NON-ORDINARY PRIMES

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Abstract. Here we generalize a classical observation of Siegel by determining all the linear relations among the initial Fourier coefficients of a modular form on $SL_2(\mathbb{Z})$. As a consequence, we identify spaces M_k in which there are universal p-divisibility properties for certain p-power coefficients. As a corollary, let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k \cap O_L[[q]]$$

(note: $q := e^{2\pi i z}$) be a normalized Hecke eigenform, and let $k \equiv \delta(k) \pmod{12}$, where $\delta(k) \in \{4, 6, 8, 10, 14\}$. Reproducing earlier results of Hatada and Hida, if p is a prime for which $k \equiv \delta(k) \pmod{p-1}$, and $\mathfrak{p} \subset O_L$ is a prime ideal above p, then we show that

$$a_f(p) \equiv 0 \pmod{\mathfrak{p}}.$$

1. Introduction and statement of results

If $k \geq 4$ is even, then let M_k (respectively S_k) denote the finite dimensional $\mathbb{C}$-vector space of weight k holomorphic modular forms (respectively cusp forms) on $SL_2(\mathbb{Z})$ (see [7] for background on modular forms). As usual, we identify a modular form $f(z)$ by its Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n,$$

where $q := e^{2\pi i z}$. As is customary, let $\Delta(z) \in S_{12}$ be the cusp form

$$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + \cdots , \tag{1.1}$$

and for even $k \geq 4$ let $E_k(z) \in M_k$ be the normalized Eisenstein series

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \left(\sum_{1 \leq d | n} d^{k-1} \right) q^n. \tag{1.2}$$

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The rational numbers B_k are the Bernoulli numbers, and they are defined by the generating function

$$\sum_{k=0}^{\infty} B_k \cdot \frac{t^k}{k!} = \frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \cdots .$$

For convenience, we let $E_0(z) := 1$. Throughout, if $k \geq 4$ is even, then let (for example, see I.2 of [7])

$$d(k) := \dim_{\mathbb{C}}(M_k) = \begin{cases} \lfloor k/12 \rfloor + 1 & \text{if } k \not\equiv 2 \pmod{12}, \\ \lfloor k/12 \rfloor & \text{if } k \equiv 2 \pmod{12}. \end{cases} \tag{1.3}$$

Furthermore, define $\delta(k) \in \{0, 4, 6, 8, 10, 14\}$ by the congruence

$$\delta(k) \equiv k \pmod{12}. \tag{1.4}$$

If N is a non-negative integer, then we let

(1.5)

$$L_{k,N} := \{(c_0, c_1, \ldots, c_{d(k)+N}) \in \mathbb{C}^{d(k)+N+1} \ : \ \sum_{\nu=0}^{d(k)+N} c_\nu a_f(\nu) = 0 \quad \forall\, f(z) = \sum_{n=0}^{\infty} a_f(n) q^n \in M_k\}$$

be the space of linear relations satisfied by the first $d(k)+N+1$ Fourier coefficients of all the forms $f(z) \in M_k$. In his study of Hilbert modular forms, Siegel [8] determined the spaces $L_{k,0}$. To state our results, for each $g(z) \in M_{12N}$, define numbers $b(k, N, g; \nu)$ by

$$\frac{E_{14-\delta(k)}(z)}{\Delta(z)^{d(k)+N}} \cdot g(z) = \sum_{\nu=0}^{d(k)+N} b(k, N, g; \nu) q^{-\nu} \ + \ \sum_{\nu=1}^{\infty} c(k, N, g; \nu) q^{\nu}. \tag{1.6}$$

The numbers $b(k, N, g; \nu)$ are the Fourier coefficients of the "principal part", together with the constant term, of the modular form above. In this notation, we have the following characterization of the $L_{k,N}$.

Theorem 1.1. *The map $\phi_{k,N} : M_{12N} \to L_{k,N}$ defined by*

$$\phi_{k,N}(g(z)) = \{b(k, N, g; \nu) \ : \ \nu = 0, \ldots, d(k) + N\}$$

defines a linear isomorphism between M_{12N} and $L_{k,N}$.

As a corollary to Theorem 1.1, we consider the distribution of non-ordinary primes for normalized Hecke eigenforms. First we recall the following well known problem (see Gouvêa's expository article [1]).

Problem. Suppose that $f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k$ is a normalized Hecke eigenform. A prime p is *non-ordinary* for $f(z)$ if $a_f(p) \equiv 0 \pmod{p}$. Are there infinitely many non-ordinary primes for $f(z)$?

Although there are strong results on the more general problem for very special modular forms on congruence subgroups $\Gamma_0(M)$ (e.g. such as CM cusp forms, and weight 2 newforms associated to elliptic curves over $\mathbb{Q}$), little is known. Using Theorem 1.1, we obtain elementary results related to this question. The following theorem applies for all forms when $p = 2$ and 3, and requires that $\delta(k) \neq 0$ for primes $p \geq 5$.

Theorem 1.2. *Let p be prime, and suppose that $f(z) = \sum_{n=0}^{\infty} a_f(n) q^n \in M_k \cap O_L[[q]]$, where O_L denotes the algebraic integers of a number field L.*

(1) *If* $p \in \{2,3\}$*, and* $b \geq 1$ *is an integer for which* $12p^b - 2 \geq k$*, then*

$$a_f(p^b) \equiv 0 \pmod{p}.$$

(2) *Suppose that* $p \geq 5$*, and that* $\delta(k) \in \{4,6,8,10,14\}$*. If* $b \geq 1$ *is an odd integer, and* $a \geq 0$ *is an integer for which*

$$k = (\delta(k) - 2)p^b + 2 - a(p-1),$$

then

$$a_f(p^b) \equiv -(24 + \alpha_k)a_f(0) \pmod{p},$$

where

$$\alpha_k := \frac{-2(14 - \delta(k))}{B_{14-\delta(k)}} \in \mathbb{Z}.$$

Remark. Theorem 1.2 does not include cases where $p \geq 5$ is prime and $\delta(k) = 0$. The condition on k in the statement of Theorem 1.2 (2) never holds when $\delta(k) = 0$. More to the point, the conclusion of Theorem 1.2 (2) does not always hold. For example, $p = 13$ is an ordinary prime for $\Delta(z)$.

Theorem 1.2 allows us to re-prove some results of Hatada [2] (in the case where $p = 2$ and 3) and Hida [3, 4, 5] (for primes $p \geq 5$) on non-ordinary primes.

Corollary 1.3. *Let* p *be prime, and suppose that* $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k$ *is a normalized Hecke eigenform. Let* L_f *be the number field generated by the coefficients of* $f(z)$*, and let* $\mathfrak{p} \subset O_{L_f}$ *be any prime ideal above* p*.*

(1) *If* $p = 2$ *or* 3*, then*

$$a_f(p) \equiv 0 \pmod{\mathfrak{p}}.$$

(2) *If* $p \geq 5$ *is prime,* $\delta(k) \in \{4,6,8,10,14\}$ *and* $k \equiv \delta(k) \pmod{p-1}$*, then*

$$a_f(p) \equiv 0 \pmod{\mathfrak{p}}.$$

Remark. For primes $p \geq 5$, it is well known that (see page 164 of [7])

$$E_{p-1}(z) \equiv 1 \pmod{p}.$$

Consequently, a prime $p \geq 5$ is non-ordinary for an eigenform $f(z) \in S_k$ if and only if it is non-ordinary for every "mod p" eigenform

$$f(z) \cdot E_{p-1}(z)^a,$$

where a is a non-negative integer. Since $\delta(k)$ often does not equal $\delta(k + a(p-1))$, we are able to apply Corollary 1.3 and find all of the small non-ordinary primes for eigenforms as in Hida's work. For example, the primes underlined in the table (see page 203 of [1]) below are examples of Corollary 1.3 applied to low weight eigenforms $f(z)$ and their congruent companions $f(z) \cdot E_{p-1}(z)^a$.

Eigenform $f(z)$	Primes $p \leq 10^6$ for which $a_f(p) \equiv 0 \pmod{p}$
$\Delta(z)$	2, 3, 5, 7, 2411
$\Delta(z)E_4(z)$	2, 3, 5, 7, 11, 13, 59, 15271, 187441
$\Delta(z)E_6(z)$	2, 3, 5, 7, 11, 13
$\Delta(z)E_8(z)$	2, 3, 5, 7, 11, 13, 17, 3371, 64709
$\Delta(z)E_{10}(z)$	2, 3, 5, 7, 13, 17, 19
$\Delta(z)E_{14}(z)$	2, 3, 5, 7, 11, 13, 17, 19, 23

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2. PROOF OF THEOREM 1.1

The proof is a generalization of a method of Siegel [8] where the $N = 0$ case is treated.

Proof of Theorem 1.1. Let us first show that

$$\sum_{\nu=0}^{d(k)+N} b(k, N, g; \nu) a_f(\nu) = 0 \tag{2.1}$$

for all $g \in M_{12N}$ and all $f(z) = \sum_{\nu=0}^{\infty} a_f(\nu) q^\nu \in M_k$. If we let

$$G(z) := \frac{E_{14-\delta(k)}}{\Delta(z)^{d(k)+N}} \cdot g(z),$$

then (2.1) is equivalent to the assertion that the constant term of the series Gf is zero. The dimension formula (1.3) implies that

$$\frac{fg}{E_{\delta(k)} \Delta^{d(k)+N-1}}$$

is a meromorphic modular function on $SL_2(\mathbb{Z})$ of weight zero. Since $k \equiv \delta(k) \pmod{12}$, we find from the valence formula (for example, see page 6 of [7]) that $f/E_{\delta(k)}$ is holomorphic on $\mathcal{H}$. Therefore it follows that

$$\frac{fg}{E_{\delta(k)} \Delta^{d(k)+N-1}}$$

is a polynomial in the Hauptmodul $j(z)$:

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + \cdots .$$

On the other hand, an easy calculation (for example, see equation (8) of [8]) reveals that

$$-\frac{1}{2\pi i} \frac{d}{dz} j = \frac{E_{14}}{\Delta}.$$

Moreover, we have that

$$j^m \frac{d}{dz} j = \frac{1}{m+1} \frac{d}{dz} j^{m+1} \qquad (m \in \mathbb{Z},\ m \geq 0).$$

Since the constant term in the Fourier expansion of $\frac{d}{dz} j^{m+1}$ is zero, by linearity it follows that

$$Gf = \frac{fg}{E_{\delta(k)} \Delta^{d(k)+N-1}} \cdot \frac{E_{14}}{\Delta}$$

has constant term zero. This confirms (2.1) since

$$E_{14-\delta(k)}(z) E_{\delta(k)}(z) = E_{14}(z).$$

Suppose that $f(z) = \sum_{n=0}^{\infty} a_f(n) q^n \in M_k$. The map $\phi_{k,N}$ is clearly linear, and it is injective, since a modular form of weight $\ell \leq 2$ (and vanishing at infinity if $\ell = 0$) is identically zero. To complete the proof, recall the well-known fact (see part I, chap. III, sect. 4 of [7]) that the $d(k)$ functionals $\{a_f(0), a_f(1), \ldots, a_f(d(k)-1)\}$ form a basis of the dual space M_k^*. Therefore, it follows that

$$\dim L_{k,N} = N + 1 = \dim M_{12N}.$$

This proves the theorem. □

Remark. If $k \equiv 2 \pmod 4$, then $E_{14-\delta(k)}$ is either 1, E_4 or E_8 and hence has positive Fourier coefficients. Therefore, taking $N = 0$, we obtain a linear relation

$$\sum_{\nu=0}^{d(k)} c_\nu a_f(\nu) = 0$$

between the first $d(k)+1$ Fourier coefficients of modular forms in M_k where all the c_ν are strictly positive (this was observed in [8]). In particular, this implies that for $k \equiv 2 \pmod 4$ the first sign change of the Fourier coefficients of a non-zero cusp form $f \in M_k$ with real Fourier coefficients already occurs among the first $d(k)+1$ coefficients (and this bound is sharp, too, as is easily seen). If $k \equiv 0 \pmod 4$, the above reasoning breaks down. To our knowledge, an answer to the corresponding question on the first sign change remains open in these cases.

For some general results about sign changes of cusp forms on rather general subgroups of $SL_2(\mathbb{R})$, we refer to [6].

Remark. A similar result as stated in the Theorem 1.1 can certainly be proved for modular forms on genus zero subgroups of $SL_2(\mathbb{Z})$ (and in particular for half-integral weight modular forms of level 4).

3. Proof of Theorem 1.2 and Corollary 1.3

We begin by restating one of the main conclusions of Theorem 1.1 in a convenient form.

Theorem 3.1. *If $f(z) \in M_k$ and $g(z) \in M_{12N}$, then the constant term of*

$$\frac{E_{14-\delta(k)}(z) g(z)}{\Delta(z)^{d(k)+N}} \cdot f(z)$$

is zero.

Proof of Theorem 1.2. First we prove Theorem 1.2 (2). Define $g(z)$ by

$$g(z) := E_{14-\delta(k)}(z)^{p^b-1} \cdot E_{p-1}(z)^a. \tag{3.1}$$

The fact that $b \geq 1$ is odd implies that $g(z) \in M_{12N}$, where

$$N = \frac{(p^b-1)(14-\delta(k)) + a(p-1)}{12} \in \mathbb{Z}_+. \tag{3.2}$$

To see this, observe that the given representation of k implies that

$$a(p-1) = -(k-2) + (\delta(k)-2)p^b \equiv (p-1)(k-2) \pmod{12}.$$

Formula (3.2) combined with (1.3) and the given representation of k implies that

$$d(k) + N = p^b.$$

Theorem 3.1, combined with the fact (see page 164 of [7]) that

$$E_{p-1}(z) \equiv 1 \pmod p$$

shows that the constant term of

$$\begin{aligned}\frac{E_{14-\delta(k)}(z)^{p^b}E_{p-1}(z)^a}{\Delta(z)^{p^b}}\cdot f(z) &\equiv \frac{E_{14-\delta(k)}(p^bz)}{\Delta(p^bz)}\cdot f(z)\\ &\equiv \left(q^{-p^b}+24+324q^{p^b}+\cdots\right)\left(1+\alpha_k q^{p^b}+\cdots\right)\cdot f(z)\\ &\equiv \left(q^{-p^b}+(24+\alpha_k)+\cdots\right)\cdot\left(\sum_{n=0}^{\infty}a_f(n)q^n\right) \pmod p\end{aligned}$$

is zero modulo p. The second line in the displayed formula above is obtained by explicitly computing the first three terms of $1/\Delta(p^bz)$. The conclusion of Theorem 1.2 (2) follows immediately.

To prove Theorem 1.2 (1), one argues as in the $\delta(k)=14$ and $N=0$ case above. In this case, we have

$$E_{14-\delta(k)}(z)=E_0(z)=1.$$

One simply replaces $E_{p-1}(z)^a$ by

$$E_{12p^b+2-k}(z)\in M_{12p^b+2-k}$$

in (3.1). Here we require that $12p^b+2-k\geq 4$. The congruence for $E_{p-1}(z)$ is replaced by the universal congruence

$$E_k(z)\equiv 1 \pmod{24}.$$

□

Proof of Corollary 1.3. Suppose that $f(z)=\sum_{n=1}^{\infty}a_f(n)q^n\in S_k$ is a normalized Hecke eigenform. We begin by establishing, in each case, that there is a positive integer b for which

$$a_f(p^b)\equiv 0 \pmod p. \tag{3.3}$$

If $p=2$ or 3, then Theorem 1.2 (1) implies (3.3) since $a_f(0)=0$. If $p\geq 5$ is prime, then there are integers $1\leq b\equiv 1 \pmod 2$ and $a\geq 0$ for which

$$k=(\delta(k)-2)p^b+2-a(p-1)=(\delta(k)-2)(p-1+1)^b+2-a(p-1)\equiv\delta(k) \pmod{p-1}.$$

By Theorem 1.2 (2), since $a_f(0)=0$, we obtain (3.3).

The definition of the Hecke operators implies, for every non-negative integer n, that

$$a_f(p)a_f(p^n)=a_f(p^{n+1})+p^{k-1}a_f(p^{n-1})\equiv a_f(p^{n+1}) \pmod p.$$

By induction, we have that

$$a_f(p^b)\equiv a_f(p)^b \pmod p.$$

Corollary 1.3 follows immediately from the truth of (3.3).

□

References

[1] F. Q. Gouvêa, *Non-ordinary primes: A story*, Exp. Math. **6**, (1997), pages 195-205.
[2] K. Hatada, *Eigenvalues of Hecke operators on $SL_2(\mathbb{Z})$*, Math. Ann. **239** (1979), pages 75-96.
[3] H. Hida, *Galois representations into $GL_2(\mathbb{Z}_p[[x]])$ attached to ordinary cusp forms*, Invent. Math. **85** (1986), pages 545-613.
[4] H. Hida, *Iwasawa modules attached to congruences of cusp forms*, Ann. Sci. École Norm. Sup., (4) **19** (1986), pages 231-273.
[5] H. Hida, *Theory of p-adic Hecke algebras and Galois representations*, Sugaka Expositions **2** (1989), pages 75-102.

[6] M. Knopp, W. Kohnen, and W. Pribitkin, *On the signs of Fourier coefficients of cusp forms*, The Ramanujan J. **7** (2003), pages 269-277.
[7] S. Lang, *Introduction to modular forms*, Springer-Verlag, Berlin, 1976.
[8] C. L. Siegel, *Berechnung von Zetafunktionen an ganzzahligen Stellen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. KL, II, (1969), pages 87-102.

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