HILBERT CLASS POLYNOMIALS AND TRACES OF SINGULAR MODULI

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1. Introduction and Statement of Results

Let $j(z)$ be the modular function for $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$j(z) = \frac{\left(1 + 240 \sum_{n=1}^{\infty} \sum_{v|n} v^3 q^n\right)^3}{q \prod_{n=1}^{\infty} (1-q^n)^{24}} = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots ,$$

where $q = e^{2\pi i z}$. The values of $j(z)$ at imaginary quadratic arguments in the upper half of the complex plane are known as *singular moduli*.

Singular moduli are algebraic integers which play prominent roles in classical and modern number theory (see [C, BCSH]). For example, Hilbert class fields of imaginary quadratic fields are generated by singular moduli. Furthermore, isomorphism classes of elliptic curves with complex multiplication are distinguished by singular moduli.

Throughout, let $d \equiv 0, 3 \pmod 4$ be a positive integer (so that $-d$ is the discriminant of an order in an imaginary quadratic field), and let $H(d)$ be the Hurwitz-Kronecker class number for the discriminant $-d$. Let $\mathcal{Q}_d$ be the set of positive definite integral binary quadratic forms (note. including imprimitive forms, if there are any)

$$Q(x, y) = ax^2 + bxy + cy^2$$

with discriminant $-d = b^2 - 4ac$. For each Q, let τ_Q be the unique complex number in the upper half-plane which is a root of $Q(x, 1) = 0$. The singular modulus $j(\tau_Q)$ depends only on the equivalence class of Q under the action of $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

Define $\omega_Q \in \{1, 2, 3\}$ by

$$\omega_Q = \begin{cases} 2 & \text{if } Q \sim_\Gamma [a, 0, a], \\ 3 & \text{if } Q \sim_\Gamma [a, a, a], \\ 1 & \text{otherwise.} \end{cases} \tag{1.1}$$

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Following Zagier, define the trace of the singular moduli of discriminant $-d$ by

$$\mathrm{Tr}(d) = \sum_{Q\in\mathcal{Q}_d/\Gamma} \frac{(j(\tau_Q)-744)}{\omega_Q}. \tag{1.2}$$

It is natural to seek formulas for $\mathrm{Tr}(d)$, and to investigate its asymptotic behavior. Indeed, the question of asymptotics is closely related to the classical observation that

$$e^{\pi\sqrt{163}} = 262537412640768743.9999999999992\ldots \tag{1.3}$$

is nearly an integer. To make this precise, we recall some classical facts. A primitive positive definite binary quadratic form Q is *reduced* if $|B| \le A \le C$, and $B \ge 0$ if either $|B| = A$ or $A = C$. If $-d < -4$ is a fundamental discriminant (i.e. the discriminant of an imaginary quadratic field), then there are $H(d)$ reduced forms with discriminant $-d$. The set of such reduced forms, say $\mathcal{Q}_d^{\mathrm{red}}$, constitutes a complete set of representatives for $\mathcal{Q}_d/\Gamma$. Moreover, each such reduced form has $1 \le A \le \sqrt{d/3}$ (see page 29 of [C]), and has the property that τ_Q lies in the usual fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$

$$\mathcal{F} = \left\{-\frac{1}{2} \le \mathrm{Re}(z) < \frac{1}{2} \text{ and } |z| > 1\right\} \cup \left\{-\frac{1}{2} \le \mathrm{Re}(z) \le 0 \;\text{ and }\; |z| = 1\right\}. \tag{1.4}$$

Since $J_1(z) = j(z) - 744 = q^{-1} + 196884q + \cdots$, it follows that if $G^{\mathrm{red}}(d)$ is defined by

$$G^{\mathrm{red}}(d) = \sum_{Q=(A,B,C)\in\mathcal{Q}_d^{\mathrm{red}}} e^{\pi Bi/A}\cdot e^{\pi\sqrt{d}/A}, \tag{1.5}$$

then $\mathrm{Tr}(d) - G^{\mathrm{red}}(d)$ is "small." This is illustrated by (1.3) where $H(163) = 1$.

In general, it is natural to study the average value

$$\frac{\mathrm{Tr}(d) - G^{\mathrm{red}}(d)}{H(d)}.$$

If $d = 1931, 2028$ and 2111, then we have

$$\frac{\mathrm{Tr}(d) - G^{\mathrm{red}}(d)}{H(d)} = \begin{cases} 11.981\ldots & \text{if } d = 1931, \\ -24.483\ldots & \text{if } d = 2028, \\ -13.935\ldots & \text{if } d = 2111. \end{cases}$$

Although these examples do not suggest a uniform behavior, a clearer picture emerges when one includes some non-reduced quadratic forms which slightly perturb these values. For each positive integer A, let $\mathcal{Q}_{A,d}^{\mathrm{old}}$ denote the set

$$\mathcal{Q}_{A,d}^{\mathrm{old}} = \{Q = (A,B,C) \;:\; \text{non-reduced with } D_Q = -d \text{ and } |B| \le A\}. \tag{1.6}$$

Define $G^{\mathrm{old}}(d)$ by

$$G^{\mathrm{old}}(d) = \sum_{\substack{\sqrt{d}/2 \le A \le \sqrt{d/3} \\ Q\in\mathcal{Q}_{A,d}^{\mathrm{old}}}} e^{\pi Bi/A}\cdot e^{\pi\sqrt{d}/A}. \tag{1.7}$$

The non-reduced forms Q contributing to $G^{\mathrm{old}}(d)$ are those primitive discriminant $-d$ forms for which τ_Q is in the bounded region obtained by connecting the two endpoints of the lower boundary of $\mathcal{F}$ with a horizontal line.

Direct calculation provides the following suggestive data

$$\frac{\mathrm{Tr}(d) - G^{\mathrm{red}}(d) - G^{\mathrm{old}}(d)}{H(d)} = \begin{cases} -24.672\ldots & \text{if } d = 1931, \\ -24.483\ldots & \text{if } d = 2028, \\ -23.458\ldots & \text{if } d = 2111. \end{cases}$$

As $-d \to -\infty$, these values appear to converge to the constant -24. We informed Duke of these observations, and he has recently proved [D2] a reformulation which implies the following theorem.

Theorem 1.1. *As $-d$ ranges over negative fundamental discriminants, we have*

$$\lim_{-d \to -\infty} \frac{\mathrm{Tr}(d) - G^{\mathrm{red}}(d) - G^{\mathrm{old}}(d)}{H(d)} = -24.$$

Duke proves this theorem using methods he developed in [D1] concerning the distribution of CM points. He obtains the constant -24 by characterizing limits of this type in terms of values of Atkin's inner product, which may be directly evaluated.

We obtain an exact formula for $\mathrm{Tr}(d)$ in terms of Kloosterman sums and $-24H(d)$, thereby providing a natural explanation for Theorem 1.1. We shall see that Theorem 1.1 is equivalent to bounds for certain specific sums of Kloosterman sums. Our formula for $\mathrm{Tr}(d)$ is the $m = 1$ case of a general family of formulas for "traces" of singular moduli.

Zagier defined [Z1] these general traces using a special sequence of monic polynomials $J_m(x) \in \mathbb{Z}[x]$ (for their properties, see [AKN, BKO, O]). Their generating function, which is equivalent to the denominator formula for the Monster Lie algebra, is given by

(1.8)

$$\begin{aligned}\sum_{m=0}^{\infty} J_m(x) q^m &= \frac{\left(1 + 240 \sum_{n=1}^{\infty} \sum_{v|n} v^3 q^n\right)^2 \left(1 - 504 \sum_{n=1}^{\infty} \sum_{v|n} v^5 q^n\right)}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}} \cdot \frac{1}{j(z) - x} \\ &= 1 + (x - 744)q + (x^2 - 1488x + 159768)q^2 + \cdots.\end{aligned}$$

If $m \geq 1$ and T_m is the usual weight zero mth Hecke operator, then $J_m(x)$ is the degree m polynomial for which

$$J_m(j(z)) = m(j(z) - 744) \mid T_m.$$

Generalizing $\mathrm{Tr}(d)$, for every integer $m \geq 0$ let

$$\mathrm{Tr}_m(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{J_m(j(\tau_Q))}{\omega_Q}. \tag{1.9}$$

Observe that $\mathrm{Tr}_0(d) = H(d)$ and $\mathrm{Tr}_1(d) = \mathrm{Tr}(d)$.

Zagier introduced these traces in relation to his proof of Borcherds' theorem (see [B1, B2]) on the infinite product expansion of modular forms with Heegner divisor. He proved (see Theorem 5 of [Z1]), for each $m \geq 1$, that the generating function for $\mathrm{Tr}_m(d)$ is a *weakly holomorphic modular form* of weight 3/2. To derive exact formulas for these traces, we reformulate Zagier's generating functions using harmonic Poincaré series with singularities at cusps.

Remark. Gross and Zagier [GZ] obtained exact formulas of a different type for norms of differences of suitable singular moduli. Later Dorman [Do1, Do2] obtained further results in this direction.

To state these formulas, we first fix notation. If v is odd, then define ε_v by

$$\varepsilon_v = \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{1.10}$$

Throughout, let $e(w) = e^{2\pi i w}$, and for $k \in \frac{1}{2}\mathbb{Z}$, let $K_k(m,n,c)$ be the generalized Kloosterman sum

$$K_k(m,n,c) = \sum_{v\ (c)^*} \left(\frac{c}{v}\right)^{2k} \varepsilon_v^{2k} e\left(\frac{m\bar{v} + nv}{c}\right). \tag{1.11}$$

In the sum, v runs through the primitive residue classes modulo c, and $\bar{v}$ is the multiplicative inverse of v modulo c.

Theorem 1.2. *If m is a positive integer and $-d < 0$ is a discriminant, then*

$$\mathrm{Tr}_m(d) = -\sum_{n \mid m} nB(n^2, d),$$

where $B(n^2,d)$ is the integer given by

$$B(n^2,d) = 24H(d) - (1+i) \sum_{\substack{c>0 \\ c\equiv 0\ (4)}} (1 + \delta_{\mathrm{odd}}(c/4)) \frac{K_{3/2}(-n^2, d, c)}{n\sqrt{c}} \sinh\left(\frac{4\pi n}{c}\sqrt{d}\right).$$

Here the function δ_{odd} is defined by

$$\delta_{\mathrm{odd}}(v) = \begin{cases} 1 & \text{if } v \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Remark. If $-d < 0$ is a discriminant and $0 < c \equiv 0 \pmod 4$, then it turns out that

$$(1 + \delta_{\mathrm{odd}}(c/4))(1+i)K_{3/2}(-1, d, c) = \sqrt{c}S(d,c),$$

where $S(d,c)$ is the simple exponential sum

$$S(d,c) = \sum_{x^2 \equiv -d \pmod c} e(2x/c)$$

(see Proposition 5 of [D2], and also earlier work of Kohnen in Proposition 5 of [K]). Therefore if $m = 1$, then Theorem 1.2 gives

$$\mathrm{Tr}(d) = -24H(d) + \sum_{\substack{c>0 \\ c\equiv 0\ (4)}} S(d,c)\sinh(4\pi\sqrt{d}/c).$$

Remark. Theorem 1.2 is analogous to the exact formula for the partition function $p(n)$ obtained[1] by Rademacher using the "circle method" [R]. Rather than employing the circle method, we directly construct certain harmonic Poincaré series of weight $3/2$ with singularities at cusps. A similar analysis, where the weight is $-1/2$, provides a direct proof of Rademacher's formula for $p(n)$ (for example, see page 655 of [H]), and is, in fact, a simpler calculation than the one required for Theorem 1.2. To see this, one first notes that the calculations involved in constructing weight k and weight $2-k$ Poincaré series are equally complicated (see Section 3.1). Hence, it suffices to consider those series with weight $k \geq 1$. Moreover, for positive weights $k > 2$, the calculation is straightforward thanks to the absolute convergence of the defining series. If $k = 2$, the situation is nearly as simple since the Poincaré series may be continued using their Fourier expansions. For the half-integral weights $1 \leq k \leq 3/2$, the situation requires spectral theory, and often gives series which are non-holomorphic in z. In fact, it turns out that the presence of the $24H(d)$ term in Theorem 1.2 is directly related to such a non-holomorphic contribution.

Remark. Although we do not offer a proof of Theorem 1.1, here we make some straightforward comments. Using Theorem 1.2, it is not difficult to show that Theorem 1.1 is equivalent to the assertion that

$$\sum_{\substack{c>\sqrt{d/3} \\ c\equiv 0\ (4)}} S(d,c)\sinh\left(\frac{4\pi}{c}\sqrt{d}\right) = o\left(H(d)\right).$$

This follows from the fact the sum over $c \leq \sqrt{d/3}$ is essentially $G^{\mathrm{red}}(d) + G^{\mathrm{old}}(d)$. The sinh factor contributes the size of q^{-1} in the Fourier expansion of a singular modulus, and the summands in the Kloosterman sum provides the corresponding "angles". The contribution $G^{\mathrm{old}}(d)$ arises from the fact that the Kloosterman sum cannot distinguish between reduced and non-reduced forms. In view of Siegel's theorem that $H(d) \gg_\epsilon d^{\frac{1}{2}-\epsilon}$, Theorem 1.1 would follow from a bound for such sums of the form $\ll d^{\frac{1}{2}-\gamma}$, for some $\gamma > 0$. Such bounds are implicit in Duke's proof of Theorem 1.1. Estimates of this type have been established for such sums, but are difficult to establish. These bounds are intimately connected to the problem of bounding coefficients of half-integral weight cusp forms (for example, see works by Duke and Iwaniec [D1, I1]).

Theorem 1.2 has some number theoretic consequences. For instance, suppose that $-d < 0$ is a fundamental discriminant. The singular moduli $j(\tau_Q)$, as Q ranges over $\mathcal{Q}_d$,

[1] This formula perfected earlier work of Hardy and Ramanujan.

are the roots of the *Hilbert class polynomial*

$$H_d(x) = \prod_{Q\in\mathcal{Q}_d/\Gamma} (x - j(\tau_Q)) \in \mathbb{Z}[x], \tag{1.12}$$

an irreducible polynomial which generates the Hilbert class field of $\mathbb{Q}(\sqrt{-d})$. These polynomials are generally quite complicated, and the basic problem of computing them and their roots has a long history (see [Be, Bi, BCSH, C, Do1, Do2, GZ, KY, Wa, We]).

Theorem 1.2 easily leads to exact formulas for any such Hilbert class polynomial. To this end, observe that Theorem 1.2 combined with the coefficients of $J_1(x), \ldots, J_h(x)$, where $h = H(d)$, are sufficient for computing

$$P_m(d) = \sum_{Q\in\mathcal{Q}_d/\Gamma} j(\tau_Q)^m,$$

for $1 \leq m \leq h$. For these m, let $S_m(d)$ denote the usual mth symmetric function of the singular moduli in the set $\{j(\tau_Q) : \ Q \in \mathcal{Q}_d/\Gamma\}$. Obviously, computing the $S_m(d)$ gives $H_d(x)$. These symmetric functions are easily obtained inductively from the $P_m(d)$ by the Newton-Girard relations

$$(-1)^m m S_m(d) + \sum_{k=1}^{m} (-1)^{k+m} P_k(d) S_{m-k}(d) = 0.$$

Consequently, Theorem 1.2 implies the following corollary.

Corollary 1.3. *If $-d < 0$ is a fundamental discriminant, then we have an exact formula for $H_d(x)$ in terms of $-d$ and the coefficients of $J_m(x)$, where $1 \leq m \leq H(d)$.*

Remark. Care is required when implementing the algorithm justifying Corollary 1.3. The formulas for the integers $B(n^2, d)$ do not converge very rapidly.

It is not difficult to obtain convolution identities for the $\mathrm{Tr}_m(d)$ which are somewhat similar to those obtained by Cohen [Co] for generalized class numbers. For example, if $\sigma_3(n) = \sum_{v|n} v^3$, then we obtain the following amusing identity for $\mathrm{Tr}(d) = \mathrm{Tr}_1(d)$.

Theorem 1.4. *If $-d < 0$ is a fundamental discriminant, then*

$$\mathrm{Tr}(d) + 240 \sum_{1\leq k<d/4} \sigma_3(k)\mathrm{Tr}(d-4k) = -264 L(-4, \chi_{-d})$$
$$+ \pi d^{9/4}\sqrt{2} \sum_{\substack{c>0 \\ c\equiv 0\ (4)}} \frac{S(d,c)}{\sqrt{c}} I_{9/2}\left(\frac{4\pi\sqrt{d}}{c}\right) + \begin{cases} -480\sigma_3(d/4) & \text{if } d \equiv 0 \pmod 4, \\ 240\sigma_3\left(\frac{d+1}{4}\right) & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

where $L(s, \chi_{-d})$ is the Dirichlet L-function for the Kronecker character χ_{-d}, and $I_{9/2}(z)$ is the index $9/2$ I-Bessel function (see [AS]*). Moreover, as $-d \to -\infty$, we have*

$$\mathrm{Tr}(d) + 240 \sum_{1\leq k<d/4} \sigma_3(k)\mathrm{Tr}(d-4k) \sim (-1)^d e^{\pi\sqrt{d}} \left(d^2 - \frac{10 d^{3/2}}{\pi} + \frac{45 d}{\pi^2} - \frac{105\sqrt{d}}{\pi^3} + \frac{105}{\pi^4}\right).$$

Remark. The series above is nearly identical to the series for $\mathrm{Tr}(d)$

$$\mathrm{Tr}(d) = -24H(d) + \pi d^{1/4}\sqrt{2} \sum_{\substack{c>0 \\ c\equiv 0\ (4)}} \frac{S(d,c)}{\sqrt{c}}\, I_{1/2}\left(\frac{4\pi}{c}\sqrt{d}\right),$$

which can be obtained from the remark after Theorem 1.2 using the identity $I_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\sinh(z)$. Apart from the $-24H(d)$ summand, the only major differences are that $d^{9/4}$ replaces $d^{1/4}$, and the $I_{9/2}$-Bessel function replaces the $I_{1/2}$-Bessel function. This convolution is simple to implement numerically for the purpose of recursively computing the $\mathrm{Tr}(d)$. This follows from the fact that the sums appearing here converge rapidly.

In Section 2 we recall Zagier's description of the generating functions for the $\mathrm{Tr}_m(d)$. These generating functions are *weakly holomorphic modular forms* of weight 3/2 in "Kohnen's plus-space" on $\Gamma_0(4)$. In Section 3, we describe weakly holomorphic Poincaré series of half-integral weight on $\Gamma_0(4)$, and we construct their projections to Kohnen's plus-space. In the special case of weight 3/2, we relate these projections to Zagier's generating functions. These formulas imply Theorem 1.2.

Remark. Using the weakly holomorphic modular forms constructed in Section 3, Duke obtains a new proof of the modularity of the generating function

$$q^{-1} - 2 - \sum_{0<d\equiv 0,3 \pmod 4} \mathrm{Tr}_1(d)q^d$$

(see the discussion after Theorem 2 of [D2]). This is the first case of Zagier's work on these generating functions, and is the $m = 1$ case of Theorem 2.1 (1).

2. Zagier's generating functions

In [B1, B2], Borcherds determined the infinite product expansion of a wide class of automorphic forms on orthogonal groups. As a special case, his work provides the infinite product expansions of the modular functions

$$H_d(j(z)) = \prod_{Q\in\mathcal{Q}_d} (j(z) - j(\tau_Q)).$$

In a recent paper [Z1], Zagier gave an elementary proof of these cases of Borcherds' theory using modular forms of half-integral weight. As part of this work, Zagier derives an explicit description of the traces $\mathrm{Tr}_m(d)$ as Fourier coefficients of certain meromorphic modular forms of weight 3/2.

Here we briefly recall these functions. For non-negative integers λ, let $M^!_{\lambda+\frac{1}{2}}$ be the infinite dimensional complex vector space of weight $\lambda + \frac{1}{2}$ *weakly holomorphic modular forms* on $\Gamma_0(4)$ satisfying the "Kohnen plus-space" condition. A weight $\lambda + \frac{1}{2}$ meromorphic modular form $f(z)$ on $\Gamma_0(4)$ is in this space provided that its poles (if there are any) are supported at the cusps of $\Gamma_0(4)$, and if its Fourier expansion has the form

$$f(z) = \sum_{(-1)^\lambda n \equiv 0,1 \pmod 4} a(n)q^n. \tag{2.1}$$

Let $M_{\lambda+\frac{1}{2}}$ be the subspace of $M^!_{\lambda+\frac{1}{2}}$ consisting of those forms $f(z)$ which are holomorphic modular forms. Zagier constructs two particular sequences of weakly holomorphic forms.

If $D \equiv 0,1 \pmod 4$ is a positive integer, then let $g_D(z)$ denote the unique element of $M^!_{3/2}$ whose Fourier expansion has the form

$$g_D(z) = q^{-D} + B(D,0) + \sum_{0<d\equiv 0,3 \pmod 4} B(D,d)q^d. \tag{2.2}$$

(The existence and uniqueness of these forms is discussed in Section 4 of [Z1]).

Similarly, for a non-negative integer $d \equiv 0,3 \pmod 4$, let $f_d(z)$ be the unique form in $M^!_{1/2}$ whose expansion has the form

$$f_d(z) = q^{-d} + \sum_{0<D\equiv 0,1 \pmod 4} A(D,d)q^D. \tag{2.3}$$

(Existence and uniqueness follow from Lemma 14.2 of [B1]; see also Section 4 of [Z1]).

It turns out that all of the coefficients $B(D,d)$ and $A(D,d)$ of these forms are integers. If $m \geq 1$, $0 < D \equiv 0,1 \pmod 4$ and $0 \leq d \equiv 0,3 \pmod 4$, then define integers $A_m(D,d)$ and $B_m(D,d)$, using the half-integral weight Hecke operators on $M^!_{\lambda+\frac{1}{2}}$, by

$$\begin{aligned} A_m(D,d) &= \text{ the coefficient of } q^D \text{ in } f_d(z) \mid T_{\frac{1}{2}}(m^2), \\ B_m(D,d) &= \text{ the coefficient of } q^d \text{ in } g_D(z) \mid T_{\frac{3}{2}}(m^2). \end{aligned} \tag{2.4}$$

Theorem 2.1. *(Zagier, [Z1]) The following are true.*

(1) *If $m \geq 1$ and $-d < 0$ is a discriminant, then*

$$\mathrm{Tr}_m(d) = -B_m(1,d).$$

(2) *If $m \geq 1$, $0 < D \equiv 0,1 \pmod 4$, and $0 \leq d \equiv 0,3 \pmod 4$, then*

$$A_m(D,d) = -B_m(D,d).$$

This theorem describes the $\mathrm{Tr}_m(d)$ as the coefficient of the image of $-g_1(z)$ under the action of the Hecke operator $T_{\frac{3}{2}}(m^2)$. One may employ the formulas for these Hecke operators directly, or the relation (see (19) of [Z1])

$$A_m(1,d) = \sum_{n|m} nA(n^2,d),$$

to deduce the following corollary from Theorem 2.1.

Corollary 2.2. *If $-d < 0$ is a discriminant and $m \geq 1$, then*

$$\mathrm{Tr}_m(d) = -\sum_{n|m} nB(n^2, d).$$

Remark. In recent work [AO], Ahlgren and the third author have used Theorem 2.1 and Corollary 2.2 to determine "Ramanujan-type" congruences for the $\mathrm{Tr}_m(d)$.

Example. As usual, let $\eta(z)$ denote Dedekind's eta-function

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

It turns out that $g_1(z) \in M^!_{3/2}$ is given by

$$g_1(z) = \frac{\eta(z)^2 \cdot \left(1 + 240 \sum_{n=1}^{\infty} \sum_{v|n} v^3 q^{4n}\right)}{\eta(2z)\eta(4z)^6} = q^{-1} - 2 + 248q^3 - 492q^4 + \cdots . \tag{2.5}$$

3. Weakly holomorphic Poincaré series

Here we recall and derive results on non-holomorphic Poincaré series of half-integral weight (see also [F], [H], [Br]). For background on the usual holomorphic Poincaré series see [I2, I3], and for detailed information on the special functions appearing here, see references such as [AS] or the Appendix to [I3].

We use these half-integral weight series to construct certain weak Maass forms. Ultimately, we use these forms to describe Zagier's weakly holomorphic modular forms

$$g_D(z) = q^{-D} + B(D, 0) + \sum_{0<d\equiv 0,3 \pmod 4} B(D, d)q^d \in M^!_{\frac{3}{2}}.$$

3.1. **Construction of weakly holomorphic Poincaré series.** As usual, let $\mathbb{H}$ denote the complex upper half-plane, and let z be the standard variable on $\mathbb{H}$. The real (respectively imaginary) part of z shall be denoted by x (respectively y).

We begin by recalling some facts on modular forms of half-integral weight (see [S]). Let $\mathfrak{G}$ be the group of pairs $(A, \phi(z))$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and ϕ is a holomorphic function on $\mathbb{H}$ satisfying

$$|\phi(z)| = (\det A)^{-1/4}|cz + d|^{1/2}.$$

The group law is given by

$$(A_1, \phi_1(z))(A_2, \phi_2(z)) = (A_1A_2, \phi_1(A_2z)\phi_2(z)).$$

Throughout, suppose that $k \in \frac{1}{2}\mathbb{Z}$. The group $\mathfrak{G}$ acts on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$(f \mid_k (A, \phi))(z) = \phi(z)^{-2k} f(Az).$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, let

$$j(\gamma, z) = \left(\frac{c}{d}\right)\varepsilon_d^{-1}\sqrt{cz+d}$$

be the automorphy factor for the classical Jacobi theta function $\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2}$. Here $\sqrt{z}$ is the principal branch of the holomorphic square root, and ε_d is given by (1.10).

The map

$$\gamma \mapsto \tilde{\gamma} = (\gamma, j(\gamma, z))$$

defines a group homomorphism $\Gamma_0(4) \to \mathfrak{G}$. For convenience, if $\gamma \in \Gamma_0(4)$, we write $f \mid_k \gamma$ instead of $f \mid_k \tilde{\gamma}$.

Let $m \in \mathbb{Z}$ and $\varphi^0 : \mathbb{R}_{>0} \to \mathbb{C}$ be a function satisfying

$$\varphi^0(y) = O(y^\alpha), \qquad y \to 0, \tag{3.1}$$

for some $\alpha \in \mathbb{R}$. Then $\varphi(z) = \varphi^0(y)e(mx)$ is a function on $\mathbb{H}$, which is invariant under the action of the subgroup $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right);\ n \in \mathbb{Z}\}$ of $\Gamma_0(4)$. We consider the Poincaré series (at the cusp ∞ of weight k for the group $\Gamma_0(4)$)

$$F(z, \varphi) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} (\varphi \mid_k \gamma)(z). \tag{3.2}$$

By comparing with an Eisenstein series, one shows that this series converges locally uniformly absolutely for $k > 2 - 2\alpha$. Hence, it is a $\Gamma_0(4)$-invariant function on $\mathbb{H}$.

Let $c \in P^1(\mathbb{Q})$ be a cusp, and suppose that $(A, \phi) \in \mathfrak{G}$ is chosen so that $A \in \mathrm{SL}_2(\mathbb{Z})$ and $A\infty = c$. As usual, we say that a function $f(z)$ has *moderate growth* at c if there is a $C \in \mathbb{R}$ for which, as $y \to \infty$, we have

$$\left(f \mid_k (A, \phi)\right)(z) = O(y^C).$$

Proposition 3.1. *Let φ be as above and assume that $k > 2 - 2\alpha$. Near the cusp at ∞, the function $F(z, \varphi) - \varphi(z)$ has moderate growth. Near the other cusps, $F(z, \varphi)$ has moderate growth. If $F(z, \varphi)$ is twice continuously differentiable, then it has the locally uniformly absolutely convergent Fourier expansion*

$$F(z, \varphi) = \varphi(z) + \sum_{n \in \mathbb{Z}} a(n, y)e(nx),$$

where

$$a(n, y) = \sum_{\substack{c>0 \\ c \equiv 0\ (4)}}^{\infty} c^{-k} K_k(m, n, c) \int_{-\infty}^{\infty} z^{-k} \varphi^0\left(\frac{y}{c^2|z|^2}\right) e\left(-\frac{mx}{c^2|z|^2} - nx\right) dx. \tag{3.3}$$

Remark. By the Weil bound for Kloosterman sums (for example, see Lemma 3.2 of [Bre] or Section 4.3 of [I2]), the series (3.3) actually converges absolutely for $k > 3/2 - 2\alpha$.

Proof of Proposition 3.1. The assertion is obtained by standard arguments. For completeness, we sketch how the Fourier expansion is calculated. By definition, we have

$$a(n,y) = \int_0^1 \left(F(z,\varphi) - \varphi(z)\right) e(-nx)\, dx.$$

Inserting the definition of $F(z,\varphi)$ and applying Poisson summation, we obtain

$$a(n,y) = \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)/\Gamma_\infty \\ c(\gamma)>0}} \int_{-\infty}^{\infty} (\varphi\mid_k \gamma)(z) e(-nx)\, dx.$$

Using $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{a}{c} - \frac{1}{c^2(z+d/c)}$ and $\varphi(z) = \varphi^0(y)e(mx)$, we find that $a(n,y)$ is equal to

$$\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)/\Gamma_\infty \\ c(\gamma)>0}} \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{2k} c^{-k} \int\limits_{-\infty}^{\infty} (z+d/c)^{-k} \varphi\left(\frac{a}{c} - \frac{1}{c^2(z+d/c)}\right) e(-nx) dx$$

$$= \sum_{c>0} \sum_{d\ (c)^*} \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{2k} e\left(\frac{m\bar{d}+nd}{c}\right) c^{-k} \int\limits_{-\infty}^{\infty} z^{-k} \varphi^0\left(\frac{y}{c^2|z|^2}\right) e\left(-\frac{mx}{c^2|z|^2} - nx\right) dx.$$

This yields the assertion. □

Particularly important Poincaré series are obtained by letting φ^0 be certain Whittaker functions. Let $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ be the usual Whittaker functions as defined on page 190 of Chapter 13 in [AS]. Following [Br] Chapter 1.3, for $s\in\mathbb{C}$ and $y\in\mathbb{R}-\{0\}$ we define

$$\mathcal{M}_s(y) = |y|^{-k/2} M_{k/2\,\mathrm{sgn}(y),\, s-1/2}(|y|), \tag{3.4}$$

$$\mathcal{W}_s(y) = |y|^{-k/2} W_{k/2\,\mathrm{sgn}(y),\, s-1/2}(|y|). \tag{3.5}$$

The functions $\mathcal{M}_s(y)$ and $\mathcal{W}_s(y)$ are holomorphic in s. Later we will be interested in certain special s-values. For $y>0$, we have

$$\mathcal{M}_{k/2}(-y) = y^{-k/2} M_{-k/2,\, k/2-1/2}(y) = e^{y/2}, \tag{3.6}$$

$$\mathcal{W}_{1-k/2}(y) = y^{-k/2} W_{k/2,\, 1/2-k/2}(y) = e^{-y/2}. \tag{3.7}$$

If m is a non-zero integer, then the function

$$\varphi_{m,s}(z) = \mathcal{M}_s(4\pi m y) e(mx)$$

is an eigenfunction of the weight k hyperbolic Laplacian

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{3.8}$$

and has eigenvalue $s(1-s)+(k^2-2k)/4$. It satisfies $\varphi_{m,s}(z)=O(y^{\mathrm{Re}(s)-k/2})$ as $y\to 0$. Consequently, the corresponding Poincaré series

$$F_m(z,s)=F(z,\varphi_{m,s}) \tag{3.9}$$

converges for $\mathrm{Re}(s)>1$, and it defines a $\Gamma_0(4)$-invariant eigenfunction of the Laplacian. In particular $F_m(z,s)$ is real analytic.

Proposition 3.2. *If m is a negative integer, then the Poincaré series $F_m(z,s)$ has the Fourier expansion*

$$F_m(z,s)=\mathcal{M}_s(4\pi my)e(mx)+\sum_{n\in\mathbb{Z}}c(n,y,s)e(nx),$$

where the coefficients $c(n,y,s)$ are given by

$$\begin{cases} \dfrac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s-k/2)}\left|\dfrac{n}{m}\right|^{\frac{k-1}{2}}\displaystyle\sum_{\substack{c>0\\ c\equiv 0\ (4)}}\dfrac{K_k(m,n,c)}{c}\,J_{2s-1}\left(\dfrac{4\pi}{c}\sqrt{|mn|}\right)\mathcal{W}_s(4\pi ny), & n<0,\\ \dfrac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s+k/2)}\left|\dfrac{n}{m}\right|^{\frac{k-1}{2}}\displaystyle\sum_{\substack{c>0\\ c\equiv 0\ (4)}}\dfrac{K_k(m,n,c)}{c}\,I_{2s-1}\left(\dfrac{4\pi}{c}\sqrt{|mn|}\right)\mathcal{W}_s(4\pi ny), & n>0,\\ \dfrac{4^{1-k/2}\pi^{1+s-k/2}i^{-k}|m|^{s-k/2}y^{1-s-k/2}\Gamma(2s-1)}{\Gamma(s+k/2)\Gamma(s-k/2)}\displaystyle\sum_{\substack{c>0\\ c\equiv 0\ (4)}}\dfrac{K_k(m,0,c)}{c^{2s}}, & n=0.\end{cases}$$

Here $J_\nu(z)$ and $I_\nu(z)$ denote the usual Bessel functions as defined in Chapter 9 of [AS]. *The Fourier expansion defines an analytic continuation of $F_m(z,s)$ to* $\mathrm{Re}(s)>3/4$.

Proof. In view of Proposition 3.1, it suffices to compute the integral

$$\begin{aligned}&\int_{-\infty}^{\infty} z^{-k}\mathcal{M}_s\left(4\pi m\frac{y}{c^2|z|^2}\right)e\left(-\frac{mx}{c^2|z|^2}-nx\right)dx\\ &=(4\pi|m|y)^{-k/2}c^k i^{-k}\\ &\quad\times\int_{-\infty}^{\infty}\left(\frac{y-ix}{y+ix}\right)^{-k/2}M_{-k/2,\,s-1/2}\left(\frac{4\pi|m|y}{c^2(x^2+y^2)}\right)e\left(-\frac{mx}{c^2(x^2+y^2)}-nx\right)dx.\end{aligned}$$

The latter integral equals the integral I in [Br] (1.40). It is evaluated on [Br] p. 33, and inserting it above yields the asserted formula for $c(n,y,s)$. ∎

By noticing that the Poincaré series $F_m(z,s)$ occur as the Fourier coefficients of the automorphic resolvent kernel (also referred to as automorphic Green function) for $\Gamma_0(4)$ (see [H] and [F] §3), and by using the spectral expansion of the automorphic resolvent kernel, one finds that $F_m(z,s)$ has a meromorphic continuation in s to the whole complex plane. For $\mathrm{Re}(s)>1/2$, it has simple poles at points of the discrete spectrum of Δ_k (see [F] Corollary 3.6).

For the special s-values $k/2$ and $1-k/2$, the function $F_m(z,s)$ is annihilated by Δ_k. As we will see below, it is actually often holomorphic in z. Consequently, these special values are of particular interest.

If $k \leq 1/2$ then $F_m(z,s)$ is holomorphic in s near $s = 1-k/2$, and one can consider $F_m(z, 1-k/2)$. For instance, this was done in [Br] Chapter 1.3 in a slightly different setting. However, here we are mainly interested in the case that $k \geq 3/2$ (later in particular in $k = 3/2$). Then $F_m(z,s)$ is holomorphic in s near $s = k/2$.

A *weak Maass form* of weight k for the group $\Gamma_0(4)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ such that

$$f \mid_k \gamma = f,$$

for all $\gamma \in \Gamma_0(4)$, and $\Delta_k f = 0$, with at most linear exponential growth at all cusps (see [BF] Section 3). If such an f is actually holomorphic on $\mathbb{H}$, it is called a *weakly holomorphic modular form.* It is then meromorphic at the cusps.

Theorem 3.3. *Assume the notation above.*

(1) *If $k \geq 2$, then $F_m(z, k/2)$ is a weakly holomorphic modular form of weight k for the group $\Gamma_0(4)$. The Fourier expansion at the cusp ∞ is given by*

$$F_m(z,k/2) = e(mz) + \sum_{n>0} c(n,y,k/2)e(nx),$$

where for $n > 0$ we have

$$c(n,y,k/2) = 2\pi i^{-k} \left|\frac{n}{m}\right|^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c\equiv 0\ (4)}} \frac{K_k(m,n,c)}{c} I_{k-1}\left(\frac{4\pi}{c}\sqrt{|mn|}\right) e^{-2\pi n y}. \tag{3.10}$$

At the other cusps, $F_m(z,k/2)$ is holomorphic (and actually vanishes).

(2) *If $k = 3/2$, then $F_m(z, k/2)$ is a weak Maass form of weight k for the group $\Gamma_0(4)$. The Fourier coefficients $c(n,y,k/2)$ with positive index n are still given by* (3.10)*. Near the cusp ∞ the function $F_m(z,k/2) - e(mz)$ is bounded. Near the other cusps the function $F_m(z,k/2)$ is bounded.*

Proof. If $k \geq 2$, the assertion immediately follows from Propositions 3.1 and 3.2. For the computation of the Fourier expansion, we notice that the sums over c in the formula of Proposition 3.2 converge absolutely by the Weil bound for Kloosterman sums. (For $k > 2$, one can actually argue more directly by noticing that $F_m(z,k/2) = F(z, e(mz))$ converges absolutely.)

If $k = 3/2$, then Proposition 3.1 and the discussion preceding the theorem imply that $F_m(z,k/2)$ is a weak Maass form and has the claimed growth near the cusps. The formula for the Fourier coefficients with positive index follows by analytic continuation using the fact that (3.10) converges. Convergence of such series is well known if one replaces the $I_{1/2}$-Bessel function by the $J_{1/2}$-Bessel function (for example, see [D1, I1]). Since $J_{1/2}(1/x) \sim I_{1/2}(1/x)$ as $x \to +\infty$, the convergence of (3.10) follows. $\square$

Remark. Notice that if $k = 3/2$, $m = -a^2$, and $n = -b^2$, where $a, b \in \mathbb{Z}\backslash\{0\}$, then the series over c occurring in the formula for $c(n, y, s)$ in Proposition 3.2 diverges at $s = k/2$. The corresponding singularity cancels with the zero of $\Gamma(s - k/2)^{-1}$.

There is an anti-linear differential operator ξ_k that takes weak Maass forms of weight k to weakly holomorphic modular forms of weight $2-k$ (see Proposition 3.2 of [BF]). If $f(z)$ is a weak Maass form of weight k, then by definition

$$\xi_k(f)(z) = 2iy^k \overline{\tfrac{\partial}{\partial \bar{z}} f(z)}. \tag{3.11}$$

In addition, this operator has the property that $\ker(\xi_k)$ is the subset of weight k weak Maass forms which are weakly holomorphic modular forms (see Proposition 3.2 of [BF]). Consequently, if $k \geq 2$, then Theorem 3.3 implies that $\xi_k(F_m(z, k/2)) = 0$. However, the situation is quite different when $k = 3/2$.

Proposition 3.4. *If $k = 3/2$ and $c(0)$ is chosen so that the constant coefficient of the Poincaré series $F_m(z, k/2)$ is given by*

$$c(0, y, k/2) = c(0)y^{1-k},$$

then

$$\xi_k(F_m(z, k/2)) = \frac{1}{2}c(0)\theta(z).$$

Proof. The assertion of Theorem 3.3 on the growth at the cusps of $F_m(z, k/2)$ implies that $\xi_k(F_m(z, k/2))$ is actually a holomorphic modular form of weight $2 - k = 1/2$ for the group $\Gamma_0(4)$. Hence it has to be a multiple of $\theta(z)$. By comparing the constant terms one obtains the factor of proportionality. □

3.2. **Projection to Kohnen's plus-space.** It is our aim to relate the generating functions for Zagier's traces of singular moduli (see Section 2) to the Poincaré series of weight $k = 3/2$. More generally, we shall describe all of Zagier's function $g_D(z) \in M^!_{3/2}$ in terms of the Poincaré series $F_{-D}(z, k/2)$.

One easily checks that $g_D(z)$ has, in general, poles at all cusps of $\Gamma_0(4)$, while the singularities of $F_m(z, k/2)$ are supported at the cusp infinity. Consequently, we also have to consider Poincaré series at the other cusps of $\Gamma_0(4)$ and take suitable linear combinations. This can be done in a quite conceptual (and automatic) way by applying a projection operator to the Kohnen plus-space.

Throughout this subsection, assume that $k \in \mathbb{Z}+\frac{1}{2}$, $k \geq 3/2$, and let $\lambda = k-\frac{1}{2}$. Kohnen (see p. 250 of [K]) constructed a projection operator pr from the space of modular forms of weight k for $\Gamma_0(4)$ to the subspace of those forms satisfying the plus-space condition. It is defined by

$$f \mid_k \mathrm{pr} = \frac{1}{3}f + (-1)^{[(\lambda+1)/2]}\frac{1}{3\sqrt{2}}\sum_{\nu\ (4)} f \mid_k B\tilde{A}_\nu, \tag{3.12}$$

where

$$B = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{(2\lambda+1)\pi i/4} \right), \qquad A_\nu = \begin{pmatrix} 1 & 0 \\ 4\nu & 1 \end{pmatrix},$$

and

$$[r] = \max\{n \in \mathbb{Z} : \ n \leq r\}.$$

It is normalized such that $\mathrm{pr}^2 = \mathrm{pr}$. Although the fact that pr is the projection to the plus-space is only proved for holomorphic modular forms in Kohnen's paper, his argument follows *mutatis mutandis* for non-holomorphic forms.

We define the Poincaré series of weight k and index m for $\Gamma_0(4)$ by

$$F_m^+(z) = \frac{3}{2} F_m(z, k/2) \mid_k \mathrm{pr}, \tag{3.13}$$

where $F_m(z, k/2)$ is the special value at $s = k/2$ of the weight k series defined in (3.9).

Theorem 3.5. *Assume that m is a negative integer and $(-1)^\lambda m \equiv 0, 1 \pmod 4$. The Poincaré series $F_m^+(z)$ is a weak Maass form of weight k for the group $\Gamma_0(4)$ satisfying the plus-space condition.*

(1) *If $k > 3/2$, then $F_m^+(z) \in M^!_{\lambda+\frac{1}{2}}$, and it has a Fourier expansion of the form*

$$F_m^+(z) = q^m + \sum_{\substack{n>0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} c^+(n) q^n,$$

where

$$c^+(n) = (-1)^{[(\lambda+1)/2]} (1 - (-1)^\lambda i) \pi \sqrt{2} \left| \frac{n}{m} \right|^{\lambda/2 - 1/4} \times \sum_{\substack{c>0 \\ c \equiv 0\ (4)}} (1 + \delta_{\mathrm{odd}}(c/4)) \frac{K_k(m, n, c)}{c} I_{\lambda - 1/2}\left(\frac{4\pi}{c} \sqrt{|mn|} \right), \tag{3.14}$$

for $n > 0$. At the other cusps, $F_m^+(z)$ is holomorphic (and actually vanishes). Here δ_{odd} is as defined in Theorem 1.2.

(2) *If $k = 3/2$, then the Fourier coefficients $c^+(n)$ with positive index n are still given by (3.14). Near the cusp at ∞, the function $F_m^+(z) - e(mz)$ decays as y^{1-k}. Near the other cusps the function $F_m^+(z)$ decays as y^{1-k}.*

Proof. Using Theorem 3.3, the projection of $F_m(z, k/2)$ to the plus-space can be calculated in exactly the same way as the projection of the usual holomorphic Poincaré series (see Proposition 4 of [K]). □

To obtain the Fourier expansion completely for $k = 3/2$, we need to compute $\xi_k(F_m^+)$. This can be done most easily by comparing the non-holomorphic part of F_m^+ with the non-holomorphic part of Zagier's Eisenstein series $G(z)$ of weight 3/2 (see [Z2]). This Eisenstein series can be constructed by taking the special value at $s = k/2$ of the

Eisenstein series $E(z,s) = F(z, y^{s-k/2})$ of weight $3/2$ for $\Gamma_0(4)$, and by computing its projection to the plus-space. So we have that

$$G(z) = \frac{3}{2} E(z, k/2) \mid_k \mathrm{pr}\,.$$

The Fourier expansion of $G(z)$ was determined by Zagier, and it is given by

$$G(z) = \sum_{n=0}^{\infty} H(n) q^n + \frac{1}{16\pi\sqrt{y}} \sum_{n=-\infty}^{\infty} \beta(4\pi n^2 y) q^{-n^2}, \tag{3.15}$$

where $H(0) = \zeta(-1) = -\frac{1}{12}$, and $\beta(s) = \int_1^\infty t^{-3/2} e^{-st} dt$.

Proposition 3.6. *If $k = 3/2$ and m is a negative integer with $-m \equiv 0, 1 \pmod 4$, then the following are true.*

(1) *If $-m$ is the square of a non-zero integer, then*

$$F_m^+(z) + 24G(z) \in M_{3/2}^!.$$

(2) *If $-m$ is not a square, then $F_m^+(z) \in M_{3/2}^!$.*

Proof. Clearly, the assertion of Proposition 3.4 also holds for F_m^+. The function $\xi_{3/2}(F_m^+)$ is a multiple of θ. On the other hand, direct computation reveals that

$$\xi_{3/2}(G(z)) = -\frac{1}{16\pi}\theta(z).$$

Hence there is a constant r such that $f = F_m^+ + rG$ is annihilated by $\xi_{3/2}$. For this choice of r, we have that $f \in M_{3/2}^!$.

To determine r, we use the remark at the end of §5 of [Z1]. As a consequence of the residue theorem on compact Riemann surfaces, the constant term in the q-expansion of fg has to vanish for any $g \in M_{1/2}^!$. We apply this for $g = \theta$. Since $f = q^m + rH(0) + O(q)$, we obtain the assertion. □

3.3. **Reformulation of Zagier's functions and the proof of Theorem 1.2.** Here we derive exact formulas for the coefficients of weakly holomorphic modular forms in $g_D(z) \in M_{3/2}^!$. For every positive integer D with $D \equiv 0, 1 \pmod 4$, let $g_D(z) \in M_{3/2}^!$ be the modular form defined by (2.2). Recall that its Fourier expansion is given by

$$g_D(z) = q^{-D} + B(D, 0) + \sum_{0 < n \equiv 0,3 \pmod 4} B(D, n) q^n.$$

Using Proposition 3.6, we obtain the following formula for the coefficients of each $g_D(z)$.

Theorem 3.7. *Let D be a positive integer with $D \equiv 0, 1 \pmod 4$. Then the Fourier coefficient $B(D, n)$ with positive index n, where $n \equiv 0, 3 \pmod 4$, is given by*

$$B(D, n) = 24\delta_{\square, D} H(n) - (1+i) \sum_{\substack{c>0 \\ c \equiv 0\ (4)}} (1 + \delta_{\mathrm{odd}}(c/4)) \frac{K_{3/2}(-D, n, c)}{\sqrt{cD}} \sinh\left(\frac{4\pi}{c}\sqrt{Dn}\right).$$

Here $\delta_{\Box,D} = 1$ if D is a square, and $\delta_{\Box,D} = 0$ otherwise.

Proof. In view of Proposition 3.6, we obviously have that

$$g_D(z) = F^+_{-D}(z) + 24\delta_{\Box,D}G(z).$$

The assertion follows from Theorem 3.5, and the fact that $I_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\sinh(z)$ (see [AS] formula (10.2.13)). $\Box$

Proof of Theorem 1.2. By letting $D = n^2$ in Theorem 3.7, we find that

$$B(n^2, d) = 24H(d) - (1+i)\sum_{\substack{c>0\\ c\equiv 0\ (4)}}(1+\delta_{\mathrm{odd}}(c/4))\frac{K_{3/2}(-n^2,d,c)}{n\sqrt{c}}\sinh\left(\frac{4\pi n}{c}\sqrt{d}\right).$$

Theorem 1.2 follows immediately from Corollary 2.2. $\Box$

3.4. **Proof of Theorem 1.4.** Let $F^+_{-1}(z) \in M^!_{11/2}$ be the $m = -1$ Poincaré series in Theorem 3.5

$$F^+_{-1}(z) = q^{-1} + 312q^3 - 1632q^4 + \cdots.$$

Furthermore, let $H_5(z) \in M^+_{11/2}$ be the weight 11/2 Cohen-Eisenstein series

$$H_5(z) = \sum_{n=0}^{\infty} h_5(n)q^n = -\frac{1}{132}\left(1 - 88q^3 - 330q^4 + \cdots\right).$$

If $-d < 0$ is fundamental, then Cohen proved [Co] that $h_5(d) = L(-4, \chi_{-d})$.

If $E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$ is the usual weight 4 Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$, then it turns out that

$$g_1(z)E_4(4z) = F^+_{-1}(z) + 264H_5(z).$$

By the remark after Theorem 1.2, and the fact that $K_{3/2}(-1,d,c) = K_{11/2}(-1,d,c)$ for every $0 < c \equiv 0 \pmod 4$, the claimed identity follows from Theorems 2.1 and 3.5.

The claimed asymptotic follows easily from the $c = 4$ summand after rewriting the $I_{9/2}$-Bessel function in terms of sinh and cosh.

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