

# ON THE WORK OF BASIL GORDON

BY

KRISHNASWAMI ALLADI<sup>1</sup> GEORGE E.  
ANDREWS<sup>2</sup> KEN ONO<sup>3</sup> AND RICHARD J. MCINTOSH<sup>4</sup>

ABSTRACT. A review is given of several aspects of the work of Basil Gordon. These include: Rogers-Ramanujan identities, plane partitions, the method of weighted words, modular forms and partition congruences, and the asymptotics of partitions and related  $q$ -series.

## 1. Introduction.

Basil Gordon (along with Bruce Rothschild) edited the Journal of Combinatorial Theory (A) for 32 years. In this paper, we shall consider several of the many contributions made by Basil Gordon to combinatorics apart from his leadership with Bruce Rothschild in building the Journal of Combinatorial Theory (A) into the world's premier combinatorics journal.

An exercise of this nature will undoubtedly fall short of what it might be. This is especially true for someone like Basil Gordon who has obtained deep results in a variety of areas of number theory. His papers purely on algebra and analysis will receive much less attention than they deserve. However, we believe that the topics we cover here represent works by Gordon that have dramatically influenced mathematical research and affected many careers including each of ours.

This tribute to Gordon is organized as follows: Section 2 will look at two early Gordon papers on the Rogers-Ramanujan identities. Section 3 will build on this work to discuss the Alladi-Gordon method of weighted words. Section 4 looks at

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the series of pathbreaking papers on plane partitions. As mentioned earlier, Gordon has done much deep work in analysis often involving questions touching analytic number theory; Section 5 considers some of the asymptotic methods developed in the collaboration of McIntosh and Gordon. Section 6 provides an account of some of the congruence and divisibility theorems which are due to Gordon and some of his collaborators. The final section concludes with a few comments about some other work.

## 2. Rogers-Ramanujan Identities.

Gordon published two absolutely pathbreaking papers [29], [32] in the 1960's that led the way to tremendous subsequent developments. By far the most important paper was [29] which was devoted to the following:

**Theorem 1.** *Let  $A_{k,a}(n)$  denote the number of partitions of  $n$  into parts  $\not\equiv 0, \pm a \pmod{2k+1}$ . Let  $B_{k,a}(n)$  denote the number of partitions of  $n$  of the form  $b_1 + b_2 + \cdots + b_s$ , where  $b_i \geq b_{i+1}$ ,  $b_i - b_{i+k-1} \geq 2$ , and at most  $a-1$  of the  $b_i$  are equal to 1. Then for each  $n \geq 0$  and  $1 \leq a \leq k$ ,*

$$A_{k,a}(n) = B_{k,a}(n).$$

The cases  $k = 2, a = 1, 2$  are the celebrated Rogers-Ramanujan identities. In historical context, this result is absolutely revolutionary. In the mid 1940's, D.H. Lehmer [48] and H.L. Alder [1] had proved theorems showing that certain seemingly natural ways to generalize the Rogers-Ramanujan identities were impossible. Of course, while Lehmer and Alder only closed off some possibilities, the general belief was that it was impossible to extend the Rogers-Ramanujan identities. Indeed, this sentiment was explicitly expressed by H. Rademacher [53] in describing the Rogers-Ramanujan identities:

*“ The unexpected element in all these cases is the association of partitions of a definite type with divisibility properties. The left-side in the identities is trivial. The deeper part is the right side. It can be shown that there can be no corresponding identities for moduli higher than 5. All these appear as wide generalizations of the old Euler theorem in which the minimal difference between the summands is, of course, 1. Euler’s theorem is therefore the nucleus of all such results.”*

Thus Gordon was able to dispel all this pessimism with his fabulously beautiful theorem.

In [32], Gordon was able to fill a gap in the characterization theorem of Alder [1]. Namely, Alder proved that if  $C_d(n)$  denotes the number of partitions of  $n$  of the form  $b_1 + b_2 + \cdots + b_s$ , where  $b_i - b_{i+1} \geq d$  with strict inequality if  $d \mid b_i$ , then for  $d > 3$ , there is no set of integers  $S_d$  such that  $C_d(n)$  always equals the number of partitions of  $n$  into elements of  $S_d$ . Alder’s motivation lay in the fact that

$$S_3 = \{n \mid n \equiv \pm 1 \pmod{6}\}$$

is a valid choice (see Theorem 2 below), but this is the last such  $S_d$ .

It is immediate by the first Rogers-Ramanujan identity ( $k = a = 2$  of Gordon’s theorem), that

$$S_1 = \{n \mid n \equiv \pm 1 \pmod{5}\}.$$

The unanswered question is: does  $S_2$  exist? In [32], Gordon proved that

$$S_2 = \{n \mid n \equiv \pm 1, 4 \pmod{8}\}.$$

While the first published proof of this theorem appeared in 1965, Gordon already knew the result in 1961 when he spoke at the International Congress [28]. This result

had been found earlier by Heinz Göllnitz [26] who did not publish his discovery until 1967 [27]. The theorem has a companion and the two identities together have become known as the Göllnitz-Gordon identities. Gordon's proof is quite different from Göllnitz's and is based on two identities in L.J. Slater's [57] compendium of identities of the Rogers-Ramanujan type. This and other identities in [32] were important in that they drew attention to the immense combinatorial possibilities latent in the extensive  $q$ -series literature on Rogers-Ramanujan type identities.

This early work by Gordon kindled work in what is now a flourishing field. There are extensive surveys of these achievements in [2], [15].

Gordon indicates in [29] his interest in a related identity of Schur [56]. This eventually led to his extensive collaboration with K. Alladi, and this novel and powerful work will be described in the next section.

### 3. The Method of Weighted Words.

The Rogers-Ramanujan identities and Gordon's generalization described in Section 2 presaged an extensive collaboration by Gordon with one of us (Alladi) on how one might effectively study partition identities combinatorially.

This collaboration began in 1989 when Gordon explained his philosophy that various well known Rogers-Ramanujan type identities that could be *refined* (this term is explained below) are to be first formulated abstractly as relations among colored integers expressed in terms of words formed with certain gap conditions, and then the specific partition identity is to be viewed as emerging from this abstract form under certain *dilations* and *translations*. More specifically, the dilations and translations yield the moduli and residue classes pertaining to the partition theorem one begins with. This philosophy gave birth to *the method of weighted words* that Gordon and Alladi first used [10] to generalize and refine Schur's celebrated partition

theorem of 1926 [56], and then to obtain several new companions to Schur's theorem [11]. Subsequently, Alladi, Andrews and Gordon [8], [9], improved the method not only to generalize and refine the deep partition theorem Göllnitz [27], but also that of Capparelli [21], [22], that arose in a study of vertex operators in Lie Algebras.

Perhaps the best example of a partition identity that can be refined is Schur's theorem [56]:

**Theorem 2.** *Let  $S(n)$  denote the number of partitions of  $n$  into distinct parts  $\equiv \pm 1 \pmod{3}$ . Let  $S_1(n)$  denote the number of partitions of  $n$  into parts differing by  $\geq 3$  and such that there are no consecutive multiples of 3. Then*

$$(3.1) \quad S(n) = S_1(n).$$

Gleissberg [25] showed that Theorem 2 could be refined to

$$(3.2) \quad S(n; k) = S_1(n; k),$$

where  $S(n; k)$  and  $S_1(n; k)$  enumerate partitions of the type counted by  $S(n)$  and  $S_1(n)$  with the additional restriction that there are precisely  $k$  parts and the convention that parts  $\equiv 0 \pmod{3}$  are counted twice. Bressoud [20] has given a nice combinatorial proof of Theorem 2 based on this refinement.

Gordon's 1989 observation is that one really ought to think of Schur's theorem in a more general setting as follows: First consider the infinite product

$$(3.3) \quad \prod_{m=1}^{\infty} (1 + aq^m)(1 + bq^m).$$

The generating function of a two parameter refinement of  $S(n)$  can be obtained from the product in (3.3) by the transformations

$$(3.4) \quad \begin{cases} \text{(dilation)} & q \mapsto q^3, \\ \text{(translations)} & a \mapsto aq^{-1}, b \mapsto bq^{-2}, \end{cases}$$

which yields

$$(3.5) \quad \prod_{m=1}^{\infty} (1 + aq^{3m-2})(1 + bq^{3m-1}).$$

Note that this is already stronger than Gleissberg's refinement, because here the powers of  $a$  and  $b$  represent the number of parts of  $S(n)$  in each of the residue classes  $1 \pmod{3}$  and  $2 \pmod{3}$  respectively. The problem then is to have a meaningful series expansion of the product in (3.3) that would lead to a partition function whose parts satisfy difference conditions that would translate to those defining  $S_1(n)$  by applying the transformations (3.4).

To this end, Alladi and Gordon [10] considered integers  $n \geq 2$  that occurred in three colors - two primary colors  $a$  and  $b$ , and one secondary color  $ab$ . The integer 1 will occur only in the two primary colors. Now let  $a_n, b_n$ , and  $ab_n$  denote the integer  $n$  occurring in the color  $a, b$ , and  $ab$  respectively.

Next in order to discuss partitions, an ordering of the colored integers was required and the one chosen was

$$(3.6) \quad \text{Scheme 1 :} \quad a_1 < b_1 < ab_2 < a_2 < b_2 < ab_3 < a_3 < b_3 < \dots$$

The reason for this choice is that (3.4) implies that

$$(3.7) \quad a_j = 3j - 2, \quad b_j = 3j - 1, \quad ab_j = 3j - 3,$$

and so the  $a_j$  and  $b_j$  for  $j \geq 1$  and  $ab_j$  for  $j \geq 2$  represent all positive integers in the natural order in the residue classes  $1, 2$  and  $3 \pmod{3}$  respectively. The transformations in (3.4) are called the *standard transformations* for Schur's theorem.

Next consider partitions  $\pi : e_1 + e_2 + \dots + e_\nu$ , where the  $e_i$  are symbols from Scheme 1 such that the gap between the symbols is  $\geq 1$  with the added restriction

that the gap between  $e_\ell$  and  $e_{\ell+1}$  is  $> 1$  if

$$(3.8) \quad \begin{cases} e_\ell \text{ is an } a\text{-part and } e_{\ell+1} \text{ is a } b\text{-part,} \\ \text{or if } e_\ell \text{ is an } ab\text{-part.} \end{cases}$$

Call such a partition  $\pi$  a Type 1 partition. Thus Type 1 partitions are certain words formed with these symbols with conditions on the weights (subscripts). Hence the name *the method of weighted words*. The definition of Type 1 partitions might seem a bit artificial, but under the standard transformations the gap conditions (3.8) translate to the difference conditions of the partition function  $S_1(n)$  in Theorem 2. More importantly, Type 1 partitions are interesting because their generating function has the following elegant form as proved in [10].

**Theorem 3.** *Let*

$$G = G(i, j, k) = \sum_k q^{\sigma(\pi)}$$

*be the generating function of all Type 1 partitions  $\pi$  for which  $\nu_a(\pi) = i$ ,  $\nu_b(\pi) = j$  and  $\nu_{ab}(\pi) = k$ . Then*

$$G = \frac{q^{T_{i+j+k} + T_k}}{(q)_i (q)_j (q)_k}.$$

To prove this result, first consider the generating function  $H(i, j, k)$  of all minimal Type 1 partitions with  $i$   $a$ -parts,  $j$   $b$ -parts and  $k$   $ab$ -parts. It is possible to show that

$$(3.9) \quad H = H(i, j, k) = q^{T_{i+j+k} + T_k} \begin{bmatrix} i + j + k \\ i, j, k \end{bmatrix}$$

by induction on the length  $i + j + k$  of the word, and standard recurrences for  $q$ -multinomial coefficients. The formula for the generating function  $G$  follows from (3.9) because of the relation

$$(3.10) \quad G(i, j, k) = \frac{H(i, j, k)}{(q)_{i+j+k}}.$$

Once  $G$  was determined, it was possible to obtain the following analytic identity which yields a generalized and refined form of Schur's theorem:

$$(3.11) \quad \sum_{r,s} a^r b^s \sum_{0 \leq m \leq \min(r,s)} \frac{q^{T_{r+s-m} + T_m}}{(q)_{r-m} (q)_{s-m} (q)_m} = \sum_{r,s} \frac{a^r b^s q^{T_r + T_s}}{(q)_r (q)_s} = (-aq)_\infty (-bq)_\infty.$$

In view of this, (3.11) is called the *key identity* for Schur's theorem. It was the first time Schur's theorem was formulated in terms of an analytic identity. Also, it was the first time a fundamental connection between Schur's theorem and the  $q$ -multinomial coefficients was established.

This approach to Schur's theorem had several important consequences, one of which was to generate five companion partition identities. We now describe this briefly.

In 1971, while using a computer to search for Rogers-Ramanujan type identities, Andrews [14] found the following companion to Theorem 2:

**Theorem 4.** *Let  $S_2(n)$  denote the number of partitions of  $n$  in the form  $b_1 + b_2 + \dots$ , such that  $b_i - b_{i+1} \geq 3, 2$  or  $5$  depending on  $b_i \equiv 1, 2$  or  $3 \pmod{3}$ . Then*

$$S_1(n) = S_2(n)$$

Andrews' proof of Theorem 4 was similar to his proof of Theorem 2. But the exact connections between Theorems 2 and 4 were not clear. The approach by the method of weighted words shed light on the precise connections between the two results.

An ordering was needed for the colored integers, and for certain reasons given above, Scheme 1 was chosen. Any other ordering could have been chosen, for instance,

$$\text{Scheme 2 :} \quad a_1 < ab_2 < b_1 < a_2 < ab_3, b_2 < \dots$$

If we used this scheme and defined Type 2 partitions similarly, we would then see that Type 2 partitions, under the standard transformations, become the partitions enumerated by  $S_2(n)$ . This then yields a bijective correspondence between the partitions enumerated by  $S_1(n)$  and  $S_2(n)$  (see [11]). Having noted that Theorem 4 emerges by merely using a different ordering of the symbols, it was natural to ask the question whether other companions to Theorem 2 emerge from other orderings, and indeed they do. More precisely, there are six orderings of the starting symbols

$$a_1, b_1, ab_2,$$

and so there are six schemes. Scheme 1 corresponds to Theorem 2 and Scheme 2 to Theorem 4. In [11] there are four new companions to Schur's theorem (Theorem 2) that were not detected by the computer search. What is more, it is shown in [11] that Schur's theorem and its five companions correspond to the six fundamental recurrences for the  $q$ -multinomial coefficients

$$\begin{bmatrix} i + j + k \\ i, j, k \end{bmatrix}.$$

Thus the method of weighted words not only provided a generalization of Schur's theorem, but also a fresh insight by yielding the companions. In addition, the method established fundamental connections with the  $q$ -multinomial coefficients.

Gordon with Alladi decided next to attempt to understand the deep partition theorem of Göllnitz [27] (Theorem 5 below) from the point of view of the method of weighted words.

**Theorem 5.** *Let  $C(n)$  denote the number of partitions of  $n$  into distinct parts  $\equiv 2, 4$  or  $5 \pmod{6}$ .*

Let  $D(n)$  denote the number of partitions of  $n$  in the form  $h_1 + h_2 + \cdots + h_\nu$ , such that  $h_\nu \neq 1$  or  $3$ ,  $h_i - h_{i+1} \geq 6$  with strict inequality if  $h_i \equiv 6, 7$  or  $9 \pmod{6}$ .

Then

$$C(n) = D(n).$$

The idea here was to start with the triple product

$$(3.12) \quad \prod_{m=1}^{\infty} (1 + aq^m)(1 + bq^m)(1 + cq^m)$$

and to think of a three parameter refinement of the partition function  $D(n)$  in Theorem 5 as emerging from (3.12) under the transformations

$$(3.13) \quad \begin{cases} \text{(dilation)} & q \mapsto q^6, \\ \text{(translations)} & a \mapsto aq^{-4}, b \mapsto bq^{-2}, c \mapsto cq^{-1}. \end{cases}$$

The real problem here was to determine a meaningful series expansion for this triple product, one that would have a combinatorial interpretation as the generating function of partitions given by colored integers satisfying certain gap conditions, and these partitions under the transformations (3.13) should correspond to the partitions enumerated by  $C(n)$  in Theorem 5. The experience Gordon and Alladi had with Schur's theorem helped to determine such a partition function.

From this point on, the theory blossomed. More than a dozen papers (see [3]-[12]) have been the outgrowth of this beautiful insight by Gordon. Among these perhaps the most surprising is the intricate and difficult extension [7] of Göllnitz's theorem.

Why is Göllnitz's theorem so much more difficult to prove than Schur's theorem? One reason is that whereas Schur's theorem involves the complete alphabet of non-zero colors that can be formed with two primary colors  $a, b$ , Göllnitz's theorem involves only an incomplete list of colors formed with three primary colors

$a, b, c$  because the ternary color  $abc$  is dropped. In his 1971 computer search for identities of Rogers-Ramanujan type, Andrews [14] asked whether there exists an identity that goes beyond Göllnitz's theorem. Actually, the precise formulation of this problem came only after the method of weighted words, namely whether there exists a partition theorem emerging from the expansion of the infinite quadruple product

$$(3.14) \quad \prod_{m=1}^{\infty} (1 + aq^m)(1 + bq^m)(1 + cq^m)(1 + dq^m)$$

that would reduce to (3.12) when we set  $d = 0$ . Gordon and Alladi worked initially on this problem but without success. This problem was solved only recently by Alladi, Andrews and Berkovich, and the full proof of the new partition identity and the key identity corresponding to the expansion of the quadruple product (3.14) is given in [7]. Although Gordon was not an author in this paper, this difficult problem of Andrews was solved using Gordon's philosophy - namely, the method of weighted words.

#### 4. Plane Partitions.

Gordon's work on plane partitions is truly remarkable. This topic originated in the work of P.A. MacMahon at the turn of the twentieth century. MacMahon's work was little understood at the time. Consequently decades passed with little attention being paid to this subject.

In the early 1960's, Gordon started his investigations that foreshadowed his subsequent pathbreaking work with his student Lorne Houten. His first paper [30] was entitled, Two New Representations of the Partition Function, and contained two theorems on two-rowed plane partitions. Now a plane partition of  $n$  (as opposed to ordinary or linear partitions as described in the previous section) is a left justified



strictly along each row. The second paper determines an infinite product representation for  $B_k(q)$ .

$$B_k(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-\lfloor \frac{k}{2} \rfloor} \prod_{m=1}^{\infty} (1 - q^{2m-1})^{-2\lfloor \frac{k}{2} \rfloor} \prod_{\nu=1}^{k-2} (1 - q^\nu)^{\lfloor \frac{k-\nu}{2} \rfloor}.$$

Even more importantly, the Pfaffian method, which has been used so successfully in subsequent researches on plane partitions, is introduced in this second paper. In plane partition enumeration, we must sometimes calculate the sum of minors of a matrix, and the next result enables one to write this compactly as a Pfaffian, something that is not at all clear at the outset. Their main result on Pfaffians is

**Theorem 8.** *Let  $\{a_n\}$ ,  $-\infty < n < \infty$ , be a sequence of independent variables. Put  $s = \sum a_n$  and  $c_\nu = \sum_m a_m a_{m+\nu}$ . If  $n > 0$ , let*

$$d_n = c_0 + 2(c_1 + \cdots + c_{n-1}) + c_n;$$

*define  $d_0 = 0$  and  $d_{-n} = -d_n$ . Let  $g_1, \dots, g_k$  be given integers. Then for even  $k$ , the infinite series*

$$\sum_{h_1 > \cdots > h_k} \det(a_{g_i + h_j})$$

*is equal to the Pfaffian of the skew-symmetric  $k \times k$  matrix  $D_k = (d_{g_j - g_i})$ .*

*For odd  $k$ , it is equal to the Pfaffian of the  $(k+1) \times (k+1)$  matrix*

$$D'_k = \begin{pmatrix} 0 & s \\ -s & D_{k-1} \end{pmatrix}$$

*obtained by bordering  $D_{k-1}$  with a row of  $s$ 's, a column of  $-s$ 's, and a zero.*

The proof runs nearly six pages. In recent years, the role of Pfaffians in combinatorics is widespread. Both S. Okada [51] and S. Stembridge [59] have used the

Pfaffian method to solve deep problems on plane partitions. Both Okada and Stembridge use Theorem 8 together with another theorem for a sum of minors where one also sums over the  $g_i$ 's. We note in passing that Theorem 8 was also rediscovered in [47]; more recently Ishikawa and Wakayama [46] found a generalization which itself has found numerous applications in various aspects of enumerative combinatorics.

Paper III is devoted to the asymptotics of  $b_k(n)$  and follows the methods of E.M. Wright.

The fourth paper has received less subsequent study than the others in this series which is unfortunate. Its primary focus is

$$C_k(q) = \sum_{n=0}^{\infty} C_k(n)q^n,$$

where  $C_k(n)$  is the number of  $k$ -rowed partitions of  $n$  whose nonzero parts decrease strictly along each column. Quite surprising the closed form found for  $C_k(q)$  not only involves not only familiar, classical infinite products but also the false theta function

$$\sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}.$$

Paper V is again pathbreaking. In it, Gordon and Houten begin the study of plane partitions under certain symmetry conditions. The results are striking and indeed foreshadow Gordon's [33] eventual proof of the Bender-Knuth conjecture. While [33] appeared in 1983, the work was done by Gordon prior to 1971 and is described by Stanley in his survey on plane partitions [58].

A gap of nearly 8 years exists between the appearance of paper V and paper VI. Paper VI undertakes the examination of plane partitions where difference conditions are specified between parts. This is a really tough problem, and it is quite surprising that they were able to find results relating instances of the two-dimensional problem

to the related one-dimensional generating functions. The final paper (unnumbered) in the Gordon-Houten series [40] appeared in 1980 now treating multirowed partitions with totally distinct parts.

There are other papers by Gordon that should be mentioned in this section. Multipartite partitions formed the topic of [31], and therein Gordon proved a conjecture of E.M. Wright [61], [62] and extended a theorem of Carlitz [24]. Also in the early 1960's, Gordon and Cheema [23] laid the groundwork for future combinatorial studies of plane partitions restricting themselves to the two and three row case (cf. Atkin [17]).

It should be stressed that these achievements are truly wonderful. They have inspired many to study plane partitions deeply and seriously. Apart from MacMahon who founded the subject, no one has been more influential or had a greater impact on the study of plane partitions than Basil Gordon.

## 5. Asymptotic Methods.

Many of Gordon's insights can be seen through the work of some of his doctoral students. This is especially the case in the area of asymptotics which played a major role in the dissertation work of several of Gordon's graduate students, including Brenner and McIntosh. Brenner [19] used the Mellin inversion formula and methods of Meinardus [50] to prove that if  $p(n|S)$  is the number of partitions of  $n$  into elements of a set  $S$ , then as  $t \rightarrow 0^+$ ,

$$(5.1) \quad \sum_{n=0}^{\infty} p(n|S)e^{-nt} \sim \exp \left\{ \frac{\pi^2 A}{6t} + D'(0) - D(0) \ln t \right\}$$

provided

$$D(s) = \sum_{\nu \in S} \nu^{-s},$$

where  $s$  is a complex variable, can be analytically continued to some half plane  $\operatorname{Re} s \geq -c$  with  $0 < c < 1$ , and has a pole of residue  $A$  at  $s = 1$ .

In his dissertation, McIntosh obtained asymptotic formulas for binomial sums by estimating the sum of the terms in the neighbourhood of the greatest term. Gordon encouraged McIntosh to modify his method to obtain the complete asymptotic expansion of the  $q$ -series

$$\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n},$$

where  $a$ ,  $b$  and  $c$  are constants,  $q = e^{-t}$  and  $t \rightarrow 0^+$ . Under certain conditions on the constants  $a$  and  $b$ , a complete asymptotic expansion was obtained by McIntosh. The asymptotic expansion of  $q$ -factorials can be expressed in terms of Bernoulli polynomials and polylogarithm functions defined by

$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}.$$

In [49] the following theorem was proved.

**Theorem 9.** *Let  $a$ ,  $b$ ,  $c$ , and  $q$  be real numbers with  $a > 0$ ,  $b > 0$  and  $|q| < 1$ . Let  $z$  denote the positive root of  $az^{2b} + z - 1 = 0$ . When  $q = e^{-t}$  and  $t \rightarrow 0^+$  we have for each nonnegative integer  $p$ ,*

$$(5.2) \quad \log \sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n} = \{ Li_2(1-z) + b \log^2 z \} t^{-1} \\ + c \log z - \frac{1}{2} \log \{ z + 2b(1-z) \} + \sum_{k=1}^p R_k(b, c, z) t^k + O(t^{p+1}),$$

where  $R_1, R_2, \dots, R_p$  are rational functions of  $b$ ,  $c$  and  $z$ .

Among the more tantalizing items in the legacy of Ramanujan are the mock theta functions. These functions resemble the classical theta functions; the unit circle is

their natural boundary, and their asymptotic behavior near the unit circle is as elegant as that of the classical theta functions such as the Dedekind eta-function. Ramanujan discovered these functions a few months before he died. He briefly described 17 of these functions in a letter to G. H. Hardy. This topic has that grand enigmatic quality that was sure to attract Gordon's attention.

Asymptotics has played a major role in the search by Gordon and McIntosh [45] for new mock theta functions. Gordon and McIntosh have observed that most of the known mock theta functions are related to a  $q$ -series of the type

$$F(\lambda, r, q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\lambda n(n+1)}}{1 - q^{n+r}},$$

where  $\lambda = 1$  or  $\frac{3}{2}$  and  $r$  is a rational number. For  $\lambda = 1$  and  $\lambda = \frac{3}{2}$  the modular transformation formula for  $F(\lambda, r, q)$  was obtained by McIntosh using asymptotic and numerical methods. The transformation formula for  $F(\frac{3}{2}, r, q)$  was proved by Gordon using contour integration. Most recently, McIntosh modified Gordon's proof to obtain the following transformation formula for  $F(\lambda, r, q)$  with  $\lambda = \frac{k}{2}$ , where  $k$  is any positive integer. If we let  $q = e^{-\alpha}$  with  $\alpha > 0$  and  $q_1 = e^{-\pi^2/\alpha}$ , then

$$(5.4) \quad q^{\lambda r(1-r)} F(\lambda, r, q) = \frac{\pi}{\alpha} \csc(\pi r) G(\lambda, r, q_1^4) - \sum_{m=1}^{2\lambda-1} \theta_1\left(\frac{m\pi}{2\lambda}, q_1^{\frac{1}{\lambda}}\right) L(\lambda, m, r, \alpha),$$

where

$$G(\lambda, r, q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^{2\lambda n} (2 - 2 \cos 2\pi r) q^{n(\lambda n + \frac{1}{2})}}{1 - 2q^n \cos 2\pi r + q^{2n}},$$

the Jacobi theta function  $\theta_1$  is defined by

$$\theta_1(z, q) = 2q^{\frac{1}{4}} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n+1)z,$$

and the Mordell integral  $L$  is given by

$$L(\lambda, m, r, \alpha) = \int_0^{\infty} e^{-\lambda \alpha x^2} \frac{\cosh(2\lambda r - m)\alpha x}{\cosh \lambda \alpha x} dx.$$

## 6. Congruences for coefficients of modular forms and partition functions.

Although Gordon did not publish many papers on the congruence properties of partition functions and modular form coefficients, he made fundamental contributions which have inspired a recent resurgence in the area (for example, see Chapter 5 of [52]). In particular, the third author owes him a great debt for introducing him to the beautiful Serre and Swinnerton-Dyer theory of  $\ell$ -adic Galois representations which has been at the forefront of contemporary arithmetic algebraic geometry and number theory.

It is clear that Ramanujan's work on the partition function  $p(n)$  and the tau-function  $\tau(n)$  motivated much of Gordon's work in the area. As usual, let  $p(n)$  denote the number of partitions of an integer  $n$ . Euler demonstrated that the generating function for  $p(n)$  is given by the delightfully simple infinite product

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots .$$

After a moment's reflection on the combinatorial definition of the partition function and the form of this generating function, we have no particular reason to believe that it possesses any interesting arithmetic properties. There is nothing, for example, which would lead us to think that  $p(n)$  should exhibit a preference to be even rather than odd. A natural suspicion, therefore, might be that the values of  $p(n)$  are distributed evenly modulo 2. A quick computation of the first 10,000 values confirms this suspicion; of these 10,000 values exactly 4,996 are even and 5,004 are odd. Surprisingly the equi-distribution modulo 2 is still an open question.

However, something quite different happens modulo 5, 7 and 11. For example, we find that many more than the expected one-fifth of the first 10,000 values of  $p(n)$

are divisible by 5. What is the explanation for this aberration? The answer follows from Ramanujan's ground-breaking discoveries on the arithmetic of  $p(n)$ . Here is his own account [54].

*“I have proved a number of arithmetic properties of  $p(n)$ ...in particular that*

$$p(5n + 4) \equiv 0 \pmod{5},$$

$$p(7n + 5) \equiv 0 \pmod{7}.$$

*...I have since found another method which enables me to prove all of these properties and a variety of others, of which the most striking is*

$$p(11n + 6) \equiv 0 \pmod{11}.$$

*There are corresponding properties in which the moduli are powers of 5, 7, or 11... It appears that there are no equally simple properties for any moduli involving primes other than these three.”*

Ramanujan proved these congruences in a series of papers (the proofs of the congruences modulo 5 and 7 are quite ingenious but are not terribly difficult, while the proof of the congruence modulo 11 is much harder). He also sketched proofs of extensions of these congruences; for example,

$$p(25n + 24) \equiv 0 \pmod{25},$$

$$p(49n + 47) \equiv 0 \pmod{49}.$$

Thanks to works of Ramanujan, Atkin, and Watson (see [16, 18, 61]), we now know the following theorem.

**Theorem 10.** *If  $n$  is an integer and  $k \geq 1$ , then*

$$\begin{aligned} p(5^k n + \delta_{5,k}) &\equiv 0 \pmod{5^k} \\ p(7^k n + \delta_{7,k}) &\equiv 0 \pmod{7^{\lfloor k/2 \rfloor + 1}} \\ p(11^k n + \delta_{11,k}) &\equiv 0 \pmod{11^k} \end{aligned}$$

where  $\delta_{p,k} := 1/24 \pmod{p^k}$ .

Gordon has always been quite fond of the partition function  $Q(n)$ , which counts the number of partitions of an integer  $n$  into distinct parts. Like the ordinary partition function  $p(n)$ ,  $Q(n)$  has a simple infinite product generating function

$$\sum_{n=0}^{\infty} Q(n)x^n = \prod_{n=1}^{\infty} (1 + x^n) = 1 + x + x^2 + 2x^3 + 2x^4 + \dots$$

Thanks to Gordon and Hughes, and Rødseth [41, 55], it is known that  $Q(n)$  also satisfies some striking Ramanujan-type congruences. In their work they established two beautiful families of such congruences including the following theorem.

**Theorem 11.** *If  $k \geq 0$  is an integer, then for every integer  $n$  with  $24n \equiv -1 \pmod{5^{2k+1}}$  we have  $Q(n) \equiv 0 \pmod{5^k}$ .*

In related work, Gordon and Hughes [42] obtained similar Ramanujan-type congruences for  $r_s(n)$ , the number of representations of an integer  $n$  as a sum of  $s$  squares.

Partition functions modulo small primes  $\ell$  can often be quite simple. For example, Euler's Pentagonal Number Theorem implies that

$$\sum_{n=0}^{\infty} Q(n)x^n \equiv \prod_{n=1}^{\infty} (1 - x^n) \pmod{2} = \sum_{k=-\infty}^{\infty} (-1)^k x^{(3k^2+k)/2}.$$

In particular,  $Q(n)$  is even for almost all  $n$ .

Motivated by this simple observation, Gordon and Ono investigated the partition functions  $b_t(n)$ , which count the number of partitions of an integer  $n$  whose parts are not multiples of  $t$ , with the aim of obtaining a general theorem which includes this elementary observation. (Note: one easily sees that  $Q(n) = b_2(n)$ ).

Like  $p(n)$  and  $Q(n)$  before, the generating function for  $b_t(n)$  is a simple infinite product:

$$\sum_{n=0}^{\infty} b_t(n)x^n = \prod_{n=1}^{\infty} \frac{(1 - x^{tn})}{(1 - x^n)}.$$

Using the transformation properties of the classical Dedekind eta-function

$$\eta(z) = x^{1/24} \prod_{n=1}^{\infty} (1 - x^n),$$

where  $x := e^{2\pi iz}$ , it turns out that these generating functions are essentially Fourier expansions of weight 0 modular functions. Moreover, in special cases these Fourier expansions turn out to be the reduction of integer weight cusp forms.

Over the last 35 years, there has been a tremendous amount of activity on the arithmetic properties of coefficients of integer weight cusp forms. Indeed, Wiles' proof of Fermat's Last Theorem depends on a deep correspondence between  $L$ -functions of elliptic curves and Fourier expansions of weight 2 cusp forms. Central to his arguments are families of Galois representations associated to the division points of an elliptic curve, and Galois representations associated to integer weight cusp forms.

Thanks to deep results of Eichler, Shimura, Serre, and Deligne in generality, one can now view coefficients of integer weight cusp forms as linear combinations of "traces of Frobenius" of Galois representations. In the special situations alluded to above, the  $\ell$ -adic behavior of the  $b_t(n)$  are literally dictated by these representations. Making ample use of these observations, Gordon and the third author employed a

powerful theorem of Serre, which essentially only depends on the mere existence of these  $\ell$ -adic Galois representations and the Chebotarev Density Theorem, to prove the following theorem [44].

**Theorem 12.** *Suppose that  $t = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ , where the  $p_j$  are distinct primes. If  $p_i$  is a prime for which  $p_i^{\alpha_i} \geq \sqrt{t}$ , then for every positive integer  $j$  we have*

$$\lim_{N \rightarrow +\infty} \frac{\#\{0 \leq n < N : b_t(n) \equiv 0 \pmod{p_i^j}\}}{N} = 1.$$

## 7. Conclusion.

Of the almost 75 papers written by Basil Gordon, we have, in this tribute, only referred to 22, fewer than one third. He has also published on classical analysis, coding theory, group theory, number theory, and tilings, to name a few of his mathematical interests.

Among the most well-known of these other contributions are the GMW (Gordon-Mills-Welch) difference sets [43]. This is a famous construction of difference sets for a certain set of parameters for which until recently only another (relatively simple) family of difference sets was known (the Singer difference sets).

As was said at the beginning, this survey is really just a sample of the work of a brilliant, innovative, witty, kind and generous mathematician, Basil Gordon.

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Department of Mathematics, University of Florida, Gainesville, FL  
alladi@math.ufl.edu

Department of Mathematics, The Pennsylvania State University, University Park,  
PA 16802 andrews@math.psu.edu

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706  
ono@math.wisc.edu

Department of Mathematics, University of Regina, Regina, Saskatchewan, Canada,  
mcintosh@math.uregina.ca