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Singular moduli generating functions for modular curves and surfaces

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1. Introduction and Statement of Results

Let $j(z)$ be the usual modular function for $\mathrm{SL}_2(\mathbb{Z})$

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots,$$

where $q = e^{2\pi i z}$. The values of modular functions such as $j(z)$ at imaginary quadratic arguments in $\mathfrak{h}$, the upper half of the complex plane, are known as *singular moduli.* Singular moduli are algebraic integers which play many roles in number theory. For example, they generate class fields of imaginary quadratic fields, and they parameterize isomorphism classes of elliptic curves with complex multiplication.

This expository article describes the author's recent joint works with Bringmann, Bruinier, Jenkins, and Rouse [**3, 4, 6**] on generating functions for traces of singular moduli. To motivate these results, we begin by comparing the classical evaluations

$$\frac{j\left(\frac{-1+\sqrt{-3}}{2}\right) - 744}{3} = -248, \qquad \frac{j(i) - 744}{2} = 492, \qquad j\left(\frac{1+\sqrt{-7}}{2}\right) - 744 = -4119,$$

with the Fourier coefficients of the modular form

(1.1)
$$g(z) := -\frac{\eta(z)^2 \cdot E_4(4z)}{\eta(2z)\eta(4z)^6} = -q^{-1} + 2 - 248q^3 + 492q^4 - 4119q^7 + 7256q^8 - \cdots,$$

where $E_4(z) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n$ is the usual weight 4 Eisenstein series, and $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's eta-function. The appearance of singular moduli as the initial coefficients of the modular form $g(z)$ is not a coincidence. In a recent groundbreaking paper [**21**], Zagier established that $g(z)$ is indeed the generating function for the "traces" of the $j(z)$ singular moduli. In this important paper, Zagier employs such results to give a new proof of Borcherds' famous theorem

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on the infinite product expansions of integer weight modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with Heegner divisor (for example, see [**1, 2**]).

Here we survey three recent papers inspired by Zagier's work. First we revisit his work from the context of Maass-Poincaré series. This uniform approach gives many of his results as special cases of a single theorem, and, as an added bonus, gives exact formulas for traces of singular moduli. Our first general result (see Theorem 1.1) establishes that the coefficients of certain half-integral weight Maass forms have the property that their coefficients are traces of singular moduli. These works are described in [**3, 6**]. Secondly, we obtain generalizations [**4**] for Hilbert modular surfaces (see Theorem 1.2).

Before we state these results, we first recall some of Zagier's results. For integers λ, let $M^!_{\lambda+\frac{1}{2}}$ be the space of weight $\lambda+\frac{1}{2}$ *weakly holomorphic modular forms* on $\Gamma_0(4)$ satisfying the "Kohnen plus-space" condition. Recall that a meromorphic modular form is weakly holomorphic if its poles (if there are any) are supported at the cusps, and it satisfies Kohnen's plus-space condition if its q-expansion has the form

$$\sum_{(-1)^\lambda n\equiv 0,1 \pmod 4} a(n)q^n. \tag{1.2}$$

Throughout, let $d\equiv 0,3 \pmod 4$ be a positive integer, let $H(d)$ be the Hurwitz-Kronecker class number for the discriminant $-d$, and let $\mathcal{Q}_d$ be the set of positive definite integral binary quadratic forms (note. including imprimitive forms)

$$Q(x,y)=[a,b,c]=ax^2+bxy+cy^2$$

with discriminant $D_Q=-d=b^2-4ac$. For each Q, let τ_Q be the unique root in $\mathfrak{h}$ of $Q(x,1)=0$. The singular modulus $f(\tau_Q)$, for any modular invariant $f(z)$, depends only on the equivalence class of Q under the action of $\Gamma:=\mathrm{PSL}_2(\mathbb{Z})$. If $\omega_Q\in\{1,2,3\}$ is given by

$$\omega_Q:=\begin{cases}2 & \text{if } Q\sim_\Gamma [a,0,a],\\ 3 & \text{if } Q\sim_\Gamma [a,a,a],\\ 1 & \text{otherwise,}\end{cases}$$

then, for a modular invariant $f(z)$, define the trace $\mathrm{Tr}(f;d)$ by

$$\mathrm{Tr}(f;d):=\sum_{Q\in\mathcal{Q}_d/\Gamma}\frac{f(\tau_Q)}{\omega_Q}. \tag{1.3}$$

Theorems 1 and 5 of [**21**] imply the following.

Theorem. (Zagier)
If $f(z)\in\mathbb{Z}[j(z)]$ has a Fourier expansion with constant term 0, then there is a finite principal part $A_f(z)=\sum_{n\le 0}a_f(n)q^n$ for which

$$A_f(z)+\sum_{0<d\equiv 0,3 \pmod 4}\mathrm{Tr}(f;d)q^d\in M^!_{\frac{3}{2}}.$$

Remark. The earlier claim about the modular form $g(z)$ is the $f(z)=J_1(z)=j(z)-744$ case of this theorem.

Remark. Using Poincaré series constructed [**6**] by Bruinier, Jenkins and the author, Duke [**9**] and Jenkins [**13**] have provided new proofs of this theorem by combining earlier results of Niebur [**18**] with facts about Kloosterman-Salié sums.

Zagier gave several generalizations of this result. Here we highlight two of these; the first concerns "twisted traces". For fundamental discriminants D_1, let χ_{D_1} denote the associated genus character for positive definite binary quadratic forms whose discriminants are multiples of D_1. If λ is an integer and D_2 is a non-zero integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$ and $(-1)^\lambda D_1 D_2 < 0$, then define the twisted trace of a modular invariant $f(z)$, say $\mathrm{Tr}_{D_1}(f; D_2)$, by

$$\mathrm{Tr}_{D_1}(f; D_2) := \sum_{Q \in \mathcal{Q}_{|D_1 D_2|}/\Gamma} \frac{\chi_{D_1}(Q) f(\tau_Q)}{\omega_Q}. \tag{1.4}$$

If $f \in \mathbb{Z}[j(z)]$ has a Fourier expansion with constant term 0, then Zagier proved that these traces are coefficients of weight 3/2 forms (see Theorem 6 of [**21**]). The second generalization involves $\mathrm{Tr}(f; d)$ for special non-holomorphic modular functions $f(z)$. In these cases, the corresponding generating functions have weight $\lambda + \frac{1}{2}$, where $\lambda \in \{-6, -4, -3, -2, -1\}$ (see Theorems 10 and 11 of [**21**]).

Remark. Kim [**14, 15**] has established the modularity for traces of singular moduli on certain genus zero congruence subgroups. Using theta lifts, Bruinier and Funke [**7**] (see Theorem 3.1) have recently proven a more general theorem which holds for modular functions on modular curves of arbitrary genus. Their result plays an important role in the proof of Theorem 1.2, our result for Hilbert modular surfaces.

Generalizing the arguments of Duke and Jenkins alluded to above, we show that the coefficients of certain half-integral weight Maass-Poincaré series are traces of singular moduli. This result includes the results of Zagier described above, and, as an added bonus, gives exact formulas for these traces. To construct these series, let $k := \lambda + \frac{1}{2}$, where λ is an arbitrary integer, and let $M_{\nu,\mu}(z)$ be the usual M-Whittaker function. For $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$, we define

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|).$$

Suppose that $m \geq 1$ is an integer with $(-1)^{\lambda+1} m \equiv 0, 1 \pmod 4$. Define $\varphi_{-m,s}(z)$ by

$$\varphi_{-m,s}(z) := \mathcal{M}_s(-4\pi m y) e(-mx),$$

where $z = x + iy$, and $e(w) := e^{2\pi i w}$. Furthermore, let

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \ : \ n \in \mathbb{Z} \right\}$$

denote the translations within $\mathrm{SL}_2(\mathbb{Z})$. Using this notation, define the Poincaré series

$$\mathcal{F}_\lambda(-m, s; z) := \sum_{A \in \Gamma_\infty \backslash \Gamma_0(4)} (\varphi_{-m,s} \mid_k A)(z) \tag{1.5}$$

for $\mathrm{Re}(s) > 1$. Here $|_k$ denotes the usual half-integral weight k "slash operator" (see Shimura's seminal paper [**20**]). If pr_λ is Kohnen's projection operator (see page 250 of [**17**]) to the weight $\lambda + \frac{1}{2}$ plus-space for $\Gamma_0(4)$, then for $\lambda \notin \{0, 1\}$ define $F_\lambda(-m; z)$ by

$$F_\lambda(-m; z) := \begin{cases} \frac{3}{2} \mathcal{F}_\lambda\left(-m, \frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \geq 2, \\ \frac{3}{2(1-k)\Gamma(1-k)} \mathcal{F}_\lambda\left(-m, 1 - \frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \leq -1. \end{cases} \tag{1.6}$$

Remark. For $\lambda = 0$ or 1 we also have series $F_\lambda(-m;z)$. Their construction requires more care. For $\lambda = 1$ this is carried out in [**6**], and for $\lambda = 0$ see [**3**].

By Theorem 3.5 of [**6**], if $\lambda \geq -6$ with $\lambda \neq -5$, then $F_\lambda(-m;z) \in M^!_{\lambda+\frac{1}{2}}$. For such λ, we denote the corresponding Fourier expansions by

$$F_\lambda(-m;z) = q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_\lambda(-m;n)q^n \in M^!_{\lambda+\frac{1}{2}}. \tag{1.7}$$

For other λ, namely $\lambda = -5$ or $\lambda \leq -7$, it turns out that the $F_\lambda(-m;z)$ are weak Maass forms of weight $\lambda + \frac{1}{2}$ (see Section 2.1). We denote their expansions by

$$F_\lambda(-m;z) = B_\lambda(-m;z) + q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_\lambda(-m;n)q^n, \tag{1.8}$$

where $B_\lambda(-m;z)$ is the "non-holomorphic" part of $F_\lambda(-m;z)$.

Example. If $\lambda = 1$ and $-m = -1$, then we have the modular form in (1.1)

$$-F_1(-1;z) = g(z) = -q^{-1} + 2 - 248q^3 + 492q^4 - 4119q^7 + 7256q^8 - \cdots .$$

Generalizing Zagier's results, we show that the coefficients $b_\lambda(-m;n)$ of the $F_\lambda(-m;z)$ are traces of singular moduli for functions defined by Niebur [**18**]. If $I_s(x)$ denotes the usual I-Bessel function, and if $\lambda > 1$, then let

$$\mathfrak{F}_\lambda(z) := \pi \sum_{A\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(Az)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}(2\pi \mathrm{Im}(Az)) e(-\mathrm{Re}(Az)). \tag{1.9}$$

Remark. For $\lambda = 1$, Niebur's [**18**] shows that $\mathfrak{F}_1(z) = \frac{1}{2}(j(z) - 744)$, where this function is the analytic continuation, as $s \to 1$ from the right, of

$$-12 + \pi \sum_{A\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(Az)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi \mathrm{Im}(Az)) e(-\mathrm{Re}(Az)).$$

Arguing as in [**6, 9, 13**], Bringmann and the author have proved [**3**] the following:

Theorem 1.1. (Bringmann and Ono; Theorem 1.2 of [**3**])
If $\lambda, m \geq 1$ are integers for which $(-1)^{\lambda+1}m$ is a fundamental discriminant (note. which includes 1), then for each positive integer n with $(-1)^\lambda n \equiv 0,1 \pmod 4$ we have

$$b_\lambda(-m;n) = \frac{2(-1)^{[(\lambda+1)/2]} n^{\frac{\lambda}{2}-\frac{1}{2}}}{m^{\frac{\lambda}{2}}} \cdot \mathrm{Tr}_{(-1)^{\lambda+1}m}(\mathfrak{F}_\lambda; n).$$

Remark. A version of Theorem 1.1 holds for integers $\lambda \leq 0$. This follows from a beautiful duality (see Theorem 1.1 of [**3**]) which generalizes an observation of Zagier. Suppose that $\lambda \geq 1$, and that m is a positive integer for which $(-1)^{\lambda+1}m \equiv 0,1 \pmod 4$. For every positive integer n with $(-1)^\lambda n \equiv 0,1 \pmod 4$, this duality asserts that

$$b_\lambda(-m;n) = -b_{1-\lambda}(-n;m).$$

Remark. For $\lambda = 1$, Theorem 1.1 relates $b_1(-m;n)$ to traces and twisted traces of $\mathfrak{F}_1(z) = \frac{1}{2}(j(z) - 744)$. These are Theorems 1 and 6 of Zagier's paper [**21**].

Theorem 1.1 is obtained by reformulating, as traces of singular moduli, exact expressions for the coefficients $b_\lambda(-m;n)$. We shall sketch the proof of this in Section 2. These exact formulas often lead to nice number theoretic consequences. Here we mention one such application which is related to the classical observation that

$$e^{\pi\sqrt{163}} = 262537412640768743.9999999999992\ldots \tag{1.10}$$

is nearly an integer.

To make this precise, we recall some classical facts. A primitive positive definite binary quadratic form Q is *reduced* if $|B| \leq A \leq C$, and $B \geq 0$ if either $|B| = A$ or $A = C$. If $-d < -4$ is a fundamental discriminant, then there are $H(d)$ reduced forms with discriminant $-d$. The set of such reduced forms, say $\mathcal{Q}_d^{\text{red}}$, is a complete set of representatives for $\mathcal{Q}_d/\Gamma$. Moreover, each such reduced form has $1 \leq A \leq \sqrt{d/3}$ (see page 29 of [**8**]), and has the property that τ_Q lies in the usual fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$

$$\mathcal{F} = \left\{-\frac{1}{2} \leq \Re(z) < \frac{1}{2} \text{ and } |z| > 1\right\} \cup \left\{-\frac{1}{2} \leq \Re(z) \leq 0 \ \text{ and } \ |z| = 1\right\}. \tag{1.11}$$

Since $J_1(z) := j(z) - 744 = q^{-1} + 196884q + \cdots$, it follows that if $G^{\text{red}}(d)$ is defined by

$$G^{\text{red}}(d) = \sum_{Q=(A,B,C)\in\mathcal{Q}_d^{\text{red}}} e^{\pi Bi/A} \cdot e^{\pi\sqrt{d}/A}, \tag{1.12}$$

then $\mathrm{Tr}(d) - G^{\text{red}}(d)$ is "small", where $\mathrm{Tr}(d) := \mathrm{Tr}(J_1; d)$. In other words, q^{-1} provides an good approximation for $J_1(z)$ for most points z. This is illustrated by (1.10) where $H(163) = 1$.

It is natural to investigate the "average value"

$$\frac{\mathrm{Tr}(d) - G^{\text{red}}(d)}{H(d)},$$

which for $d = 163$ is $-0.0000000000008\ldots$. Armed with the exact formulas for $\mathrm{Tr}(d)$, it turns out that a uniform picture emerges for a slightly perturbed average, one including some non-reduced quadratic forms. For each positive integer A, let $\mathcal{Q}_{A,d}^{\text{old}}$ denote the set

$$\mathcal{Q}_{A,d}^{\text{old}} = \{Q = (A, B, C) \ : \ \text{non-reduced with } D_Q = -d \text{ and } |B| \leq A\}. \tag{1.13}$$

Define $G^{\text{old}}(d)$ by

$$G^{\text{old}}(d) = \sum_{\substack{\sqrt{d}/2 \leq A \leq \sqrt{d/3} \\ Q\in\mathcal{Q}_{A,d}^{\text{old}}}} e^{\pi Bi/A} \cdot e^{\pi\sqrt{d}/A}. \tag{1.14}$$

The non-reduced forms Q contributing to $G^{\text{old}}(d)$ are those primitive discriminant $-d$ forms for which τ_Q is in the bounded region obtained by connecting the two endpoints of the lower boundary of $\mathcal{F}$ with a horizontal line. The following data is quite suggestive:

$$\frac{\mathrm{Tr}(d) - G^{\text{red}}(d) - G^{\text{old}}(d)}{H(d)} = \begin{cases} -24.672\ldots & \text{if } d = 1931, \\ -24.483\ldots & \text{if } d = 2028, \\ -23.458\ldots & \text{if } d = 2111. \end{cases}$$

Recently, Duke has proved [**9**] a result which implies the following theorem.

Theorem. (Duke [**9**])
As $-d$ ranges over negative fundamental discriminants, we have

$$\lim_{-d\to-\infty} \frac{\mathrm{Tr}(d) - G^{\mathrm{red}}(d) - G^{\mathrm{old}}(d)}{H(d)} = -24.$$

In Section 2 we shall give an explanation of the constant -24 in this theorem. We shall see that it makes a surprising appearance in the exact formulas for $\mathrm{Tr}(d)$.

We shall also describe some generalizations of Theorem 1.1 for Hilbert modular surfaces. Using the groundbreaking work of Hirzebruch and Zagier [**12**] on the intersection theory of Hilbert modular surfaces as a prototype, we consider analogs of Theorem 1.1 for Hilbert modular surfaces defined over $\mathbb{Q}(\sqrt{p})$, where $p \equiv 1 \pmod 4$ is prime. As usual, let $\mathcal{O}_K := \mathbb{Z}\left[\frac{1+\sqrt{p}}{2}\right]$ be the ring of integers of the real quadratic field $K := \mathbb{Q}(\sqrt{p})$. The group $\mathrm{SL}_2(\mathcal{O}_K)$ acts on $\mathfrak{h}\times\mathfrak{h}$ by

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \circ (z_1, z_2) := \left(\frac{\alpha z_1+\beta}{\gamma z_1+\delta}, \frac{\alpha' z_2+\beta'}{\gamma' z_2+\delta'}\right).$$

Here ν' denotes the conjugate of ν in $\mathbb{Q}(\sqrt{p})$. The quotient $X_p := (\mathfrak{h}\times\mathfrak{h})/\mathrm{SL}_2(\mathcal{O}_K)$ is a non-compact surface with finitely many singularities which can be compactified by adding finitely many points (i.e. cusps). Hirzebruch showed [**11**] how to resolve the singularities introduced by adding cusps using cyclic configurations of rational curves. The resulting modular surface Y_p is a nearly smooth compact algebraic surface with quotient singularities supported at those points in $\mathfrak{h}\times\mathfrak{h}$ with a non-trivial isotropy subgroup within $\mathrm{PSL}_2(\mathcal{O}_K)$.

Hirzebruch and Zagier introduced [**12**] a sequence of algebraic curves

$$Z_1^{(p)}, Z_2^{(p)}, \cdots \subset X_p,$$

and studied the generating functions for their intersection numbers. They proved the striking fact that these generating functions are weight 2 modular forms, an observation which allowed them to identify spaces of modular forms with certain homology groups for Y_p. To define these curves, for a positive integer N, consider the points $(z_1, z_2) \in \mathfrak{h}\times\mathfrak{h}$ satisfying an equation of the form

$$Az_1z_2\sqrt{p} + \lambda z_1 - \lambda' z_2 + B\sqrt{p} = 0, \tag{1.15}$$

where $A, B \in \mathbb{Z}$, $\lambda \in \mathcal{O}_K$, and $\lambda\lambda' + ABp = N$. Each such equation defines a curve in $\mathfrak{h}\times\mathfrak{h}$ isomorphic to $\mathfrak{h}$, and their union is invariant under $\mathrm{SL}_2(\mathcal{O}_K)$. The *Hirzebruch-Zagier divisor* $Z_N^{(p)}$ is defined to be the image of this union in X_p.

Remark. If $\left(\frac{N}{p}\right) = -1$, then one easily sees from (1.15) that $Z_N^{(p)}$ is empty.

We let $\widetilde{Z_N^{(p)}}$ denote the closure of $Z_N^{(p)}$ in Y_p. If $(\widetilde{Z_m^{(p)}}, \widetilde{Z_n^{(p)}})$ denotes the intersection number of $\widetilde{Z_m^{(p)}}$ and $\widetilde{Z_n^{(p)}}$ in Y_p, then Hirzebruch and Zagier proved in [**12**], for every positive integer m, that

$$\Phi_m^{(p)}(z) := a_m^{(p)}(0) + \sum_{n=1}^{\infty} (\widetilde{Z_m^{(p)}}, \widetilde{Z_n^{(p)}})q^n \tag{1.16}$$

is a holomorphic weight 2 modular form on $\Gamma_0(p)$ with Nebentypus $\left(\frac{\cdot}{p}\right)$. Here $a_m^{(p)}(0)$ is a simple constant arising from a volume computation. More precisely, $\Phi_m^{(p)}(z)$

is in the *plus space* $M_2^+\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$, the space of holomorphic weight 2 modular forms $F(z) = \sum_{n=0}^{\infty} a(n)q^n$ on $\Gamma_0(p)$ with Nebentypus $\left(\frac{\cdot}{p}\right)$, with the additional property that

$$a(n) = 0 \quad \text{if } \left(\frac{n}{p}\right) = -1. \tag{1.17}$$

Our generalization of Theorem 1.1 to these surfaces is also a generalization of this result of Hirzebruch and Zagier, one which involves forms in $\mathcal{M}_2\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$, the space of weakly holomorphic modular forms of weight 2 on $\Gamma_0(p)$ with Nebentypus $\left(\frac{\cdot}{p}\right)$, and $\mathcal{M}_2^+\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$, the subspace of those forms in $\mathcal{M}_2\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$ that satisfy (1.17).

To explain this, we first note that the "geometric part" of the proof of the modularity of (1.16) gives a concrete description of the intersection points $Z_m^{(p)} \cap Z_n^{(p)}$ in terms of CM points which are the "roots" of $\Gamma_0(m)$ equivalence classes of binary quadratic forms with negative discriminants of the form $-(4mn - x^2)/p$. In this context, it is natural to consider the traces of singular moduli over the points constituting $Z_m^{(p)} \cap Z_n^{(p)}$.

To state our result, suppose that $\ell = 1$ or that ℓ is an odd prime with $\left(\frac{\ell}{p}\right) \neq -1$, and let $\Gamma_0^*(\ell)$ be the projective image of the extension of $\Gamma_0(\ell)$ by the Fricke involution $W_\ell = \left(\begin{smallmatrix} 0 & -1 \\ \ell & 0 \end{smallmatrix}\right)$ in $\mathrm{PSL}_2(\mathbb{R})$. Suppose that $f(z) = \sum_{n \gg -\infty} a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(\ell))$, the space of weakly holomorphic modular functions with respect to $\Gamma_0^*(\ell)$, and suppose that $a(0) = 0$. We define the "trace" of $f(z)$ over $Z_\ell^{(p)} \cap Z_n^{(p)}$ by

$$(Z_\ell^{(p)}, Z_n^{(p)})_f^{\mathrm{tr}} := \sum_{\tau \in Z_\ell^{(p)} \cap Z_n^{(p)}} \frac{f(\tau)}{\#\Gamma_0^*(\ell)_\tau}, \tag{1.18}$$

where $\Gamma_0^*(\ell)_\tau$ is the stabilizer of τ in $\Gamma_0^*(\ell)$. We consider their generating functions

$$\Phi_{\ell,f}^{(p)}(z) := A_{\ell,f}^{(p)}(z) + B_{\ell,f}^{(p)}(z) + \sum_{n=1}^{\infty} (Z_\ell^{(p)}, Z_n^{(p)})_f^{\mathrm{tr}} q^n, \tag{1.19}$$

where

$$\begin{aligned} &A_{\ell,f}^{(p)}(z) \\ &:= -\epsilon(\ell) \sum_{m,n \geq 1} m a(-mn) \left(\sum_{\substack{x \in \mathbb{Z} \\ x^2 \equiv m^2 p \pmod{2\ell}}} q^{\frac{x^2 - m^2 p}{4\ell}} + \sum_{\substack{x \in \mathbb{Z} \\ x \equiv m \pmod 2}} q^{\frac{x^2 \ell - m^2 p \ell}{4}} \right), \end{aligned}$$

$$B_{\ell,f}^{(p)}(z) := 2\epsilon(\ell) \sum_{n \geq 1} (\sigma_1(n) + \ell\sigma_1(n/\ell)) a(-n) \sum_{x \in \mathbb{Z}} q^{\ell x^2},$$

and where $\epsilon(\ell) = 1/2$ for $\ell = 1$, and is 1 otherwise. As usual, $\sigma_1(x)$ denotes the sum of the positive divisors of x if x is an integer, and is zero if x is not an integer.

Bringmann, Rouse and the author have shown [**4**] that these generating functions are also modular forms of weight 2. In particular, we obtain a linear map:

$$\Phi^{(p)}_{\ell,\star} : \mathcal{M}_0(\Gamma_0^*(\ell)) \to \mathcal{M}_2\left(\Gamma_0(p\ell^2), \left(\frac{\cdot}{p}\right)\right)$$

(where the map is defined for the subspace of those functions with constant term 0).

Theorem 1.2. (Bringmann, Ono and Rouse; Theorem 1.1 of [**4**])
Suppose that $p \equiv 1 \pmod 4$ is prime, and that $\ell = 1$ or is an odd prime with $\left(\frac{\ell}{p}\right) \neq -1$. If $f(z) = \sum_{n \gg -\infty} a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(\ell))$, with $a(0) = 0$, then the generating function $\Phi^{(p)}_{\ell,f}(z)$ is in $\mathcal{M}_2\left(\Gamma_0(p\ell^2), \left(\frac{\cdot}{p}\right)\right)$.

In Section 3 we combine the geometry of these surfaces with recent work of Bruinier and Funke [**7**] to sketch the proof of Theorem 1.2. In this section we characterize these modular forms $\Phi^{(p)}_{\ell,f}(z)$ when $f(z) = J_1(z) := j(z) - 744$. In terms of the classical Weber functions

$$\mathfrak{f}_1(z) = \frac{\eta(z/2)}{\eta(z)} \quad \text{and} \quad \mathfrak{f}_2(z) = \sqrt{2} \cdot \frac{\eta(2z)}{\eta(z)}, \tag{1.20}$$

we have the following exact description.

Theorem 1.3. (Bringmann, Ono and Rouse; Theorem 1.2 of [**4**])
If $p \equiv 1 \pmod 4$ is prime, then

$$\Phi^{(p)}_{1,J_1}(z) = \frac{\eta(2z)\eta(2pz)E_4(pz)\mathfrak{f}_2(2z)^2\mathfrak{f}_2(2pz)^2}{4\eta(pz)^6} \cdot \left(\mathfrak{f}_1(4z)^4\mathfrak{f}_2(z)^2 - \mathfrak{f}_1(4pz)^4\mathfrak{f}_2(pz)^2\right).$$

Although Theorem 1.3 gives a precise description of the forms $\Phi^{(p)}_{1,J_1}(z)$, it is interesting to note that they are intimately related to Hilbert class polynomials, the polynomials given by

$$H_D(x) = \prod_{\tau \in \mathcal{C}_D} (x - j(\tau)) \in \mathbb{Z}[x], \tag{1.21}$$

where $\mathcal{C}_D$ denotes the equivalence classes of CM points with discriminant $-D$. Each $H_D(x)$ is an irreducible polynomial in $\mathbb{Z}[x]$ which generates a class field extension of $\mathbb{Q}(\sqrt{-D})$. Define $N_p(z)$ as the "multiplicative norm" of $\Phi_{1,J_1}(z)$

$$N_p(z) := \prod_{M \in \Gamma_0(p)\backslash \mathrm{SL}_2(\mathbb{Z})} \Phi^{(p)}_{1,J_1}|M. \tag{1.22}$$

If $N_p^*(z)$ is the normalization of $N_p(z)$ with leading coefficient 1, then we have

$$N_p^*(z) = \begin{cases} \Delta(z)H_{75}(j(z)) & \text{if } p = 5, \\ E_4(z)\Delta(z)^2H_3(j(z))H_{507}(j(z)) & \text{if } p = 13, \\ \Delta(z)^3H_4(j(z))H_{867}(j(z)) & \text{if } p = 17, \\ \Delta(z)^5H_7(j(z))^2H_{2523}(j(z)) & \text{if } p = 29, \end{cases}$$

where $\Delta(z) = \eta(z)^{24}$ is the usual Delta-function. These examples illustrate a general phenomenon in which $N_p^*(z)$ is essentially a product of certain Hilbert class polynomials.

To state the general result, define integers $a(p)$, $b(p)$, and $c(p)$ by

$$a(p) := \frac{1}{2}\left(\left(\frac{3}{p}\right)+1\right), \tag{1.23}$$

$$b(p) := \frac{1}{2}\left(\left(\frac{2}{p}\right)+1\right), \tag{1.24}$$

$$c(p) := \frac{1}{6}\left(p-\left(\frac{3}{p}\right)\right), \tag{1.25}$$

and let $\mathcal{D}_p$ be the negative discriminants $-D \neq -3, -4$ of the form $\frac{x^2-4p}{16f^2}$ with $x, f \geq 1$.

Theorem 1.4. (Bringmann, Ono and Rouse; Theorem 1.3 of [**4**]) *Assume the notation above. If $p \equiv 1 \pmod 4$ is prime, then*

$$N_p^*(z) = (E_4(z)H_3(j(z)))^{a(p)} \cdot H_4(j(z))^{b(p)} \cdot \Delta(z)^{c(p)} \cdot H_{3\cdot p^2}(j(z)) \cdot \prod_{-D\in\mathcal{D}_p} H_D(j(z))^2.$$

The remainder of this survey is organized as follows. In Section 2 we compute the coefficients of the Maass-Poincaré series $F_\lambda(-m; z)$, and we sketch the proof of Theorem 1.1 by employing facts about Kloosterman-Salié sums. Moreover, we give a brief discussion of Duke's theorem on the "average values"

$$\frac{\mathrm{Tr}(d) - G^{\mathrm{red}}(d) - G^{\mathrm{old}}(d)}{H(d)}.$$

In Section 3 we sketch the proof of Theorems 1.2, 1.3 and 1.4.

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2. Maass-Poincaré series and the proof of Theorem 1.1

In this section we sketch the proof of Theorem 1.1. We first recall the construction of the forms $F_\lambda(-m; z)$, and we then give exact formulas for the coefficients $b_\lambda(-m; n)$. The proof then follows from classical observations about Kloosterman-Salié sums and their reformulation as Poincaré series.

2.1. Maass-Poincaré series. Here we give more details on the Poincaré series $F_\lambda(-m; z)$ (see [**5, 3, 6, 10**] for more on such series). Suppose that λ is an integer, and that $k := \lambda + \frac{1}{2}$. For each $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4)$, let

$$j(A, z) := \left(\frac{\gamma}{\delta}\right)\epsilon_\delta^{-1}(\gamma z + \delta)^{\frac{1}{2}}$$

be the factor of automorphy for half-integral weight modular forms. If $f : \mathfrak{h} \to \mathbb{C}$ is a function, then for $A \in \Gamma_0(4)$ we let

$$(f \mid_k A)(z) := j(A, z)^{-2\lambda-1} f(Az). \tag{2.1}$$

As usual, let $z = x + iy$, and for $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$, we let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{2.2}$$

where $M_{\nu,\mu}(z)$ is the standard M-Whittaker function which is a solution to the differential equation

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) u = 0.$$

If m is a positive integer, and $\varphi_{-m,s}(z)$ is given by

$$\varphi_{-m,s}(z) := \mathcal{M}_s(-4\pi m y) e(-mx),$$

then recall from the introduction that

$$\mathcal{F}_\lambda(-m, s; z) := \sum_{A \in \Gamma_\infty \backslash \Gamma_0(4)} (\varphi_{-m,s} \mid_k A)(z). \tag{2.3}$$

It is easy to verify that $\varphi_{-m,s}(z)$ is an eigenfunction, with eigenvalue

$$s(1-s) + (k^2 - 2k)/4, \tag{2.4}$$

of the weight k hyperbolic Laplacian

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Since $\varphi_{-m,s}(z) = O\left(y^{\mathrm{Re}(s) - \frac{k}{2}}\right)$ as $y \to 0$, it follows that $\mathcal{F}_\lambda(-m, s; z)$ converges absolutely for $\mathrm{Re}(s) > 1$, is a $\Gamma_0(4)$-invariant eigenfunction of the Laplacian, and is real analytic.

Special values, in s, of these series provide examples of half-integral weight weak Maass forms. A *weak Maass form of weight k* for the group $\Gamma_0(4)$ is a smooth function $f : \mathfrak{h} \to \mathbb{C}$ satisfying the following:

(1) For all $A \in \Gamma_0(4)$ we have
$$(f \mid_k A)(z) = f(z).$$
(2) We have $\Delta_k f = 0$.
(3) The function $f(z)$ has at most linear exponential growth at all the cusps.

In particular, the discussion above implies that the special s-values at $k/2$ and $1 - k/2$ of $\mathcal{F}_\lambda(-m, s; z)$ are weak Maass forms of weight $k = \lambda + \frac{1}{2}$ when the series is absolutely convergent. If $\lambda \notin \{0, 1\}$ and $m \geq 1$ is an integer for which $(-1)^{\lambda+1} m \equiv 0, 1 \pmod 4$, then this implies that the Kohnen projections $F_\lambda(-m; z)$, from the introduction, are weak Maass forms of weight $k = \lambda + \frac{1}{2}$ on $\Gamma_0(4)$ in Kohnen's plus space.

If $\lambda = 1$ and m is a positive integer for which $m \equiv 0, 1 \pmod 4$, then define $F_1(-m; z)$ by

$$F_1(-m; z) := \frac{3}{2} \mathcal{F}_1 \left(-m, \frac{3}{4}; z\right) \mid \mathrm{pr}_1 + 24 \delta_{\square, m} G(z). \tag{2.5}$$

The function $G(z)$ is given by the Fourier expansion

$$G(z) := \sum_{n=0}^{\infty} H(n) q^n + \frac{1}{16\pi\sqrt{y}} \sum_{n=-\infty}^{\infty} \beta(4\pi n^2 y) q^{-n^2},$$

where $H(0) = -1/12$ and

$$\beta(s) := \int_1^\infty t^{-\frac{3}{2}} e^{-st} dt.$$

Proposition 3.6 of [**6**] establishes that each $F_1(-m;z)$ is in $M^!_{\frac{3}{2}}$.

Remark. The function $G(z)$ plays an important role in the work of Hirzebruch and Zagier [**12**] which is intimately related to Theorems 1.2, 1.3 and 1.4.

Remark. An analogous argument is used to define the series $F_0(-m;z) \in M^!_{\frac{1}{2}}$.

2.2. Exact formulas for the coefficients $b_\lambda(-m;n)$. Here we give exact formulas for the $b_\lambda(-m;n)$, the coefficients of the holomorphic parts of the Maass-Poincaré series $F_\lambda(-m;z)$. These coefficients are given as explicit infinite sums in half-integral weight Kloosterman sums weighted by Bessel functions. To define these Kloosterman sums, for odd δ let

$$\epsilon_\delta := \begin{cases} 1 & \text{if } \delta \equiv 1 \pmod 4, \\ i & \text{if } \delta \equiv 3 \pmod 4. \end{cases} \tag{2.6}$$

If λ is an integer, then we define the $\lambda + \frac{1}{2}$ weight Kloosterman sum $K_\lambda(m,n,c)$ by

$$K_\lambda(m,n,c) := \sum_{v \pmod c)^*} \left(\frac{c}{v}\right) \epsilon_v^{2\lambda+1} e\left(\frac{m\bar{v}+nv}{c}\right). \tag{2.7}$$

In the sum, v runs through the primitive residue classes modulo c, and $\bar{v}$ denotes the multiplicative inverse of v modulo c. In addition, for convenience we define $\delta_{\square,m} \in \{0,1\}$ by

$$\delta_{\square,m} := \begin{cases} 1 & \text{if } m \text{ is a square}, \\ 0 & \text{otherwise}. \end{cases} \tag{2.8}$$

Finally, for integers c define $\delta_{\mathrm{odd}}(c)$ by

$$\delta_{\mathrm{odd}}(c) := \begin{cases} 1 & \text{if } c \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Theorem 2.1. *Suppose that λ is an integer, and suppose that m is a positive integer for which $(-1)^{\lambda+1}m \equiv 0,1 \pmod 4$. Furthermore, suppose that n is a non-negative integer for which $(-1)^\lambda n \equiv 0,1 \pmod 4$.*

(1) *If $\lambda \geq 2$, then $b_\lambda(-m;0) = 0$, and for positive n we have*

$$b_\lambda(-m;n) = (-1)^{[(\lambda+1)/2]}\pi\sqrt{2}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}}(1-(-1)^\lambda i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} (1+\delta_{\mathrm{odd}}(c/4))\frac{K_\lambda(-m,n,c)}{c} \cdot I_{\lambda-\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

(2) *If $\lambda \leq -1$, then*

$$b_\lambda(-m;0) = (-1)^{[(\lambda+1)/2]}\pi^{\frac{3}{2}-\lambda}2^{1-\lambda}m^{\frac{1}{2}-\lambda}(1-(-1)^\lambda i) \times \frac{1}{(\frac{1}{2}-\lambda)\Gamma(\frac{1}{2}-\lambda)} \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} (1+\delta_{\mathrm{odd}}(c/4))\frac{K_\lambda(-m,0,c)}{c^{\frac{3}{2}-\lambda}},$$

and for positive n we have

$$b_\lambda(-m;n) = (-1)^{[(\lambda+1)/2]}\pi\sqrt{2}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}}(1-(-1)^\lambda i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} (1+\delta_{\mathrm{odd}}(c/4))\frac{K_\lambda(-m,n,c)}{c} \cdot I_{\frac{1}{2}-\lambda}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

(3) *If $\lambda = 1$, then $b_1(-m;0) = -2\delta_{\square,m}$, and for positive n we have*

$$b_1(-m;n) = 24\delta_{\square,m}H(n) - \pi\sqrt{2}(n/m)^{\frac{1}{4}}(1+i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} (1+\delta_{\mathrm{odd}}(c/4))\frac{K_1(-m,n,c)}{c} \cdot I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

(4) *If $\lambda = 0$, then $b_0(-m;0) = 0$, and for positive n we have*

$$b_0(-m;n) = -24\delta_{\square,n}H(m) + \pi\sqrt{2}(m/n)^{\frac{1}{4}}(1-i) \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} (1+\delta_{\mathrm{odd}}(c/4))\frac{K_0(-m,n,c)}{c} \cdot I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Remark. For positive integers m and n, the formulas for $b_\lambda(-m;n)$ are nearly uniform in λ. In fact, this uniformity may be used to derive a nice duality (see Theorem 1.1 of [**3**]) for these coefficients. More precisely, suppose that $\lambda \geq 1$, and that m is a positive integer for which $(-1)^{\lambda+1}m \equiv 0,1 \pmod 4$. For every positive integer n with $(-1)^\lambda n \equiv 0,1 \pmod 4$, this duality asserts that

$$b_\lambda(-m;n) = -b_{1-\lambda}(-n;m).$$

The proof of Theorem 2.1 requires some further preliminaries. For $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$, we let

$$\mathcal{W}_s(y) := |y|^{-\frac{k}{2}} W_{\frac{k}{2}\operatorname{sgn}(y),\, s-\frac{1}{2}}(|y|), \tag{2.9}$$

where $W_{\nu,\mu}$ denotes the usual W-Whittaker function. For $y > 0$, we have the relations:

$$\mathcal{M}_{\frac{k}{2}}(-y) = e^{\frac{y}{2}}, \tag{2.10}$$

$$\mathcal{W}_{1-\frac{k}{2}}(y) = \mathcal{W}_{\frac{k}{2}}(y) = e^{-\frac{y}{2}}, \tag{2.11}$$

and

$$\mathcal{W}_{1-\frac{k}{2}}(-y) = \mathcal{W}_{\frac{k}{2}}(-y) = e^{\frac{y}{2}}\Gamma\left(1-k,y\right), \tag{2.12}$$

where

$$\Gamma(a,x) := \int_x^\infty e^{-t}t^a\frac{dt}{t}$$

is the incomplete Gamma function. For $z \in \mathbb{C}$, the functions $M_{\nu,\mu}(z)$ and $M_{\nu,-\mu}(z)$ are related by the identity

$$W_{\nu,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\nu)}M_{\nu,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\nu)}M_{\nu,-\mu}(z).$$

From these facts, we easily find, for $y > 0$, that

$$\mathcal{M}_{1-\frac{k}{2}}(-y) = (k-1)e^{\frac{y}{2}}\Gamma(1-k,y) + (1-k)\Gamma(1-k)e^{\frac{y}{2}}. \tag{2.13}$$

SKETCH OF THE PROOF OF THEOREM 2.1. For simplicity, suppose that $\lambda \notin \{0,1\}$, and suppose that m is a positive integer for which $(-1)^{\lambda+1}m \equiv 0,1 \pmod 4$. Computing the Fourier expansion requires the integral

$$\int_{-\infty}^{\infty} z^{-k}\mathcal{M}_s\left(-4\pi m \frac{y}{c^2|z|^2}\right) e\left(\frac{mx}{c^2|z|^2} - nx\right) dx,$$

which may be found on page 357 of [**10**]. This calculation implies that $\mathcal{F}_\lambda(-m,s;z)$ has a Fourier expansion of the form

$$\mathcal{F}_\lambda(-m,s;z) = \mathcal{M}_s(-4\pi my)e(-mx) + \sum_{n\in\mathbb{Z}} c(n,y,s)e(nx).$$

If $J_s(x)$ is the usual Bessel function of the first kind, then the coefficients $c(n,y,s)$ are given as follows. If $n<0$, then

$$c(n,y,s) := \frac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s-\frac{k}{2})}\left|\frac{n}{m}\right|^{\frac{\lambda}{2}-\frac{1}{4}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} \frac{K_\lambda(-m,n,c)}{c} J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right)\mathcal{W}_s(4\pi ny).$$

If $n>0$, then

$$c(n,y,s) := \frac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s+\frac{k}{2})}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} \frac{K_\lambda(-m,n,c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\mathcal{W}_s(4\pi ny).$$

Lastly, if $n=0$, then

$$c(0,y,s) := \frac{4^{\frac{3}{4}-\frac{\lambda}{2}}\pi^{\frac{3}{4}+s-\frac{\lambda}{2}} i^{-k} m^{s-\frac{\lambda}{2}-\frac{1}{4}} y^{\frac{3}{4}-s-\frac{\lambda}{2}}\Gamma(2s-1)}{\Gamma(s+\frac{k}{2})\Gamma(s-\frac{k}{2})} \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} \frac{K_\lambda(-m,0,c)}{c^{2s}}.$$

The Fourier expansion defines an analytic continuation of $\mathcal{F}_\lambda(-m,s;z)$ to $\mathrm{Re}(s) > 3/4$. For $\lambda \geq 2$, the presence of the Γ-factor above implies that the Fourier coefficients $c(n,y,s)$ vanish for negative n. Therefore, $\mathcal{F}_\lambda(-m,\frac{k}{2};z)$ is a weakly holomorphic modular form on $\Gamma_0(4)$. Applying Kohnen's projection operator (see page 250 of [**17**]) to these series gives Theorem 2.1 (1).

As we have seen, if $\lambda \leq -1$, then $\mathcal{F}_\lambda(-m, 1-\frac{k}{2};z)$ is a weak Maass form of weight $k = \lambda + \frac{1}{2}$ on $\Gamma_0(4)$. Using (2.12) and (2.13), we find that its Fourier expansion has the form

$$\begin{aligned}(2.14)\quad & \mathcal{F}_\lambda\left(-m, 1-\frac{k}{2};z\right) \\ &= (k-1)\left(\Gamma(1-k,4\pi my) - \Gamma(1-k)\right)q^{-m} + \sum_{n\in\mathbb{Z}} c(n,y)e(nz),\end{aligned}$$

where the coefficients $c(n,y)$, for $n<0$, are given by

$$2\pi i^{-k}(1-k)\left|\frac{n}{m}\right|^{\frac{\lambda}{2}-\frac{1}{4}}\Gamma(1-k,4\pi|n|y)\cdot \sum_{\substack{c>0 \\ c\equiv 0 \pmod 4}} \frac{K_\lambda(-m,n,c)}{c} J_{\frac{1}{2}-\lambda}\left(\frac{4\pi}{c}\sqrt{|mn|}\right).$$

For $n \geq 0$, (2.11) allows us to conclude that the $c(n, y)$ are given by

$$\begin{cases} 2\pi i^{-k}\Gamma(2-k)(n/m)^{\frac{\lambda}{2}-\frac{1}{4}} \sum\limits_{\substack{c>0 \\ c\equiv 0 \pmod 4}} \dfrac{K_\lambda(-m,n,c)}{c} \cdot I_{\frac{1}{2}-\lambda}\left(\dfrac{4\pi}{c}\sqrt{mn}\right), & n>0, \\ 4^{\frac{3}{4}-\frac{\lambda}{2}}\pi^{\frac{3}{2}-\lambda} i^{-k} m^{\frac{1}{2}-\lambda} \sum\limits_{\substack{c>0 \\ c\equiv 0 \pmod 4}} \dfrac{K_\lambda(-m,0,c)}{c^{\frac{3}{2}-\lambda}}. & n=0. \end{cases}$$

One easily checks that the claimed formulas for $b_\lambda(-m;n)$ are obtained from these formulas by applying Kohnen's projection operator pr_λ. $\square$

REMARK. In addition to those $\lambda \geq 0$, if $\lambda \in \{-6, -4, -3, -2, -1\}$, then the functions $F_\lambda(-m;z)$ are in $M^!_{\lambda+\frac{1}{2}}$, and their q-expansions are of the form

$$F_\lambda(-m;z) = q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n\equiv 0,1 \pmod 4}} b_\lambda(-m;n)q^n. \tag{2.15}$$

This claim is equivalent to the vanishing of the non-holomorphic terms appearing in the proof of Theorem 2.1 for these λ. This vanishing is proved in Section 2 of [**3**].

2.3. Sketch of the proof of Theorem 1.1. Here we sketch the proof of Theorem 1.1. Armed with Theorem 2.1, this proof reduces to classical facts relating half-integral weight Kloosterman sums to Salié sums. To define these sums, suppose that $0 \neq D_1 \equiv 0, 1 \pmod 4$. If λ is an integer, $D_2 \neq 0$ is an integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$, and N is a positive multiple of 4, then define the generalized Salié sum $S_\lambda(D_1, D_2, N)$ by

(2.16)
$$S_\lambda(D_1,D_2,N) := \sum_{\substack{x \pmod N \\ x^2\equiv(-1)^\lambda D_1D_2 \pmod N}} \chi_{D_1}\left(\frac{N}{4}, x, \frac{x^2-(-1)^\lambda D_1D_2}{N}\right) e\left(\frac{2x}{N}\right),$$

where $\chi_{D_1}(a, b, c)$, for a binary quadratic form $Q = [a, b, c]$, is given by

(2.17)
$$\chi_{D_1}(a,b,c) := \begin{cases} 0 & \text{if } (a,b,c,D_1)>1, \\ \left(\frac{D_1}{r}\right) & \text{if } (a,b,c,D_1)=1 \text{ and } Q \text{ represents } r \text{ with } (r,D_1)=1. \end{cases}$$

REMARK. If $D_1 = 1$, then χ_{D_1} is trivial. Therefore, if $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$, then

$$S_\lambda(1,D_2,N) = \sum_{\substack{x \pmod N \\ x^2\equiv(-1)^\lambda D_2 \pmod N}} e\left(\frac{2x}{N}\right).$$

Half-integral weight Kloosterman sums are essentially equal to such Salié sums, a fact which plays a fundamental role throughout the theory of half-integral weight modular forms. The following proposition is due to Kohnen (see Proposition 5 of [**17**]).

PROPOSITION 2.2. Suppose that N is a positive multiple of 4. If λ is an integer, and D_1 and D_2 are non-zero integers for which $D_1, (-1)^\lambda D_2 \equiv 0, 1 \pmod 4$, then

$$N^{-\frac{1}{2}}(1-(-1)^\lambda i)(1+\delta_{\mathrm{odd}}(N/4)) \cdot K_\lambda((-1)^\lambda D_1, D_2, N) = S_\lambda(D_1,D_2,N).$$

As a consequence, we may rewrite the formulas in Theorem 2.1 using Salié sums. The following proposition, well known to experts, then describes these Salié sums as Poincaré-type series over CM points.

Proposition 2.3. Suppose that λ is an integer, and that D_1 is a fundamental discriminant. If D_2 is a non-zero integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$ and $(-1)^\lambda D_1 D_2 < 0$, then for every positive integer a we have

$$S_\lambda(D_1, D_2, 4a) = 2 \sum_{Q \in \mathcal{Q}_{|D_1 D_2|}/\Gamma} \frac{\chi_{D_1}(Q)}{\omega_Q} \sum_{\substack{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{Im}(A\tau_Q) = \frac{\sqrt{|D_1 D_2|}}{2a}}} e\left(-\mathrm{Re}\left(A\tau_Q\right)\right).$$

Proof. For every integral binary quadratic form

$$Q(x, y) = ax^2 + bxy + cy^2$$

of discriminant $(-1)^\lambda D_1 D_2$, let $\tau_Q \in \mathfrak{h}$ be as before. Clearly τ_Q is equal to

$$\tau_Q = \frac{-b + i\sqrt{|D_1 D_2|}}{2a}, \tag{2.18}$$

and the coefficient b of Q solves the congruence

$$b^2 \equiv (-1)^\lambda D_1 D_2 \pmod{4a}. \tag{2.19}$$

Conversely, every solution of (2.19) corresponds to a quadratic form with an associated CM point thereby providing a one-to-one correspondence between the solutions of

$$b^2 - 4ac = (-1)^\lambda D_1 D_2 \qquad (a, b, c \in \mathbb{Z}, a, c > 0)$$

and the points of the orbits

$$\bigcup_Q \left\{ A\tau_Q : \ A \in \mathrm{SL}_2(\mathbb{Z})/\Gamma_{\tau_Q} \right\},$$

where Γ_{τ_Q} denotes the isotropy subgroup of τ_Q in $\mathrm{SL}_2(\mathbb{Z})$, and where Q varies over the representatives of $\mathcal{Q}_{|D_1 D_2|}/\Gamma$. The group Γ_∞ preserves the imaginary part of such a CM point τ_Q, and preserves (2.19). However, it does not preserve the middle coefficient b of the corresponding quadratic forms modulo $4a$. It identifies the congruence classes $b, b+2a \pmod{4a}$ appearing in the definition of $S_\lambda(D_1, D_2, 4a)$. Since $\chi_{D_1}(Q)$ is fixed under the action of Γ_∞, the corresponding summands for such pairs of congruence classes are equal. Proposition 2.3 follows since $\#\Gamma_{\tau_Q} = 2\omega_Q$, and since both Γ_{τ_Q} and Γ_∞ contain the negative identity matrix. □

Sketch of the proof of Theorem 1.1. Here we prove the cases where $\lambda \geq 2$. The argument when $\lambda = 1$ is identical. For $\lambda \geq 2$, Theorem 2.1 (1) implies that

$$b_\lambda(-m; n) = (-1)^{[(\lambda+1)/2]} \pi \sqrt{2} (n/m)^{\frac{\lambda}{2} - \frac{1}{4}} (1 - (-1)^\lambda i) \times \sum_{\substack{c > 0 \\ c \equiv 0 \pmod 4}} (1 + \delta_{\mathrm{odd}}(c/4)) \frac{K_\lambda(-m, n, c)}{c} \cdot I_{\lambda - \frac{1}{2}}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Using Proposition 2.2, where $D_1 = (-1)^{\lambda+1} m$ and $D_2 = n$, for integers $N = c$ which are positive multiples of 4, we have

$$c^{-\frac{1}{2}} (1 - (-1)^\lambda i)(1 + \delta_{\mathrm{odd}}(c/4)) \cdot K_\lambda(-m, n, c) = S_\lambda((-1)^{\lambda+1} m, n, c).$$

These identities, combined with the change of variable $c = 4a$, give

$$b_\lambda(-m;n) = \frac{(-1)^{[(\lambda+1)/2]}\pi}{\sqrt{2}}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}} \sum_{a=1}^{\infty} \frac{S_\lambda((-1)^{\lambda+1}m, n, 4a)}{\sqrt{a}} \cdot I_{\lambda-\frac{1}{2}}\left(\frac{\pi\sqrt{mn}}{a}\right).$$

Using Proposition 2.3, this becomes

$$b_\lambda(-m;n) = \frac{2(-1)^{[(\lambda+1)/2]}\pi}{\sqrt{2}}(n/m)^{\frac{\lambda}{2}-\frac{1}{4}} \sum_{Q\in\mathcal{Q}_{nm}/\Gamma} \frac{\chi_{(-1)^{\lambda+1}m}(Q)}{\omega_Q}$$
$$\sum_{a=1}^{\infty} \sum_{\substack{A\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{Im}(A\tau_Q)=\frac{\sqrt{mn}}{2a}}} \frac{I_{\lambda-\frac{1}{2}}(2\pi\mathrm{Im}(A\tau_Q))}{\sqrt{a}} \cdot e(-\mathrm{Re}(A\tau_Q)).$$

The definition of $\mathfrak{F}_\lambda(z)$ in (1.9), combined with the obvious change of variable relating $1/\sqrt{a}$ to $\mathrm{Im}(A\tau_Q)^{\frac{1}{2}}$, gives

$$b_\lambda(-m;n) = \frac{2(-1)^{[(\lambda+1)/2]}n^{\frac{\lambda}{2}-\frac{1}{2}}}{m^{\frac{\lambda}{2}}} \cdot \pi \sum_{Q\in\mathcal{Q}_{nm}/\Gamma} \frac{\chi_{(-1)^{\lambda+1}m}(Q)}{\omega_Q}$$
$$\sum_{A\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(A\tau_Q)^{\frac{1}{2}} \cdot I_{\lambda-\frac{1}{2}}(2\pi\mathrm{Im}(A\tau_Q))e(-\mathrm{Re}(A\tau_Q))$$
$$= \frac{2(-1)^{[(\lambda+1)/2]}n^{\frac{\lambda}{2}-\frac{1}{2}}}{m^{\frac{\lambda}{2}}} \cdot \mathrm{Tr}_{(-1)^{\lambda+1}m}(\mathfrak{F}_\lambda; n).$$

□

2.4. The "24 Theorem". Here we explain the source of -24 in the limit

$$\lim_{-d\to-\infty} \frac{\mathrm{Tr}(d) - G^{\mathrm{red}}(d) - G^{\mathrm{old}}(d)}{H(d)} = -24. \tag{2.20}$$

Combining Theorems 1.1 and 2.1 with Proposition 2.2, we find that

$$\mathrm{Tr}(d) = -24H(d) + \sum_{\substack{c>0 \\ c\equiv 0\ (4)}} S(d,c)\sinh(4\pi\sqrt{d}/c),$$

where $S(d,c)$ is the Salié sum

$$S(d,c) = \sum_{x^2\equiv -d \pmod{c}} e(2x/c).$$

The constant -24 arises from (2.5). It is not difficult to show that the "24 Theorem" is equivalent to the assertion that

$$\sum_{\substack{c>\sqrt{d/3} \\ c\equiv 0\ (4)}} S(d,c)\sinh\left(\frac{4\pi}{c}\sqrt{d}\right) = o\left(H(d)\right).$$

This follows from the fact the sum over $c \le \sqrt{d/3}$ is essentially $G^{\mathrm{red}}(d) + G^{\mathrm{old}}(d)$. The sinh factor contributes the size of q^{-1} in the Fourier expansion of a singular modulus, and the summands in the Kloosterman sum provides the corresponding "angles". The contribution $G^{\mathrm{old}}(d)$ arises from the fact that the Kloosterman sum cannot distinguish between reduced and non-reduced forms. In view of Siegel's theorem that $H(d) \gg_\epsilon d^{\frac{1}{2}-\epsilon}$, (2.20) follows from a bound for such sums of the form

$\ll d^{\frac{1}{2}-\gamma}$, for some $\gamma > 0$. Such bounds are implicit in Duke's proof of this result [**9**].

3. Traces on Hilbert modular surfaces

In this section we sketch the proofs of Theorems 1.2, 1.3 and 1.4. In the first subsection we recall the arithmetic of the intersection points on the relevant Hilbert modular surfaces, and in the second subsection we recall recent work of Bruinier and Funke concerning traces of singular moduli on more generic modular curves. In the last subsection we sketch the proofs of the theorems.

3.1. Intersection points on Hilbert modular surfaces. Here we provide (for $\ell = 1$ or an odd prime with $\left(\frac{\ell}{p}\right) \neq -1$) an interpretation of $Z_\ell^{(p)} \cap Z_n^{(p)}$ as a union of $\Gamma_0^*(\ell)$ equivalence classes of CM points. As before, for $-D \equiv 0, 1 \pmod 4$ with $D > 0$, we let $\mathcal{Q}_D$ be the set of all (not necessarily primitive) binary quadratic forms

$$Q(x, y) = [a, b, c](x, y) := ax^2 + bxy + cy^2$$

with discriminant $b^2 - 4ac = -D$. To each such form Q, we let the CM point τ_Q be as before. For $\ell = 1$ or an odd prime and $D > 0$, $-D \equiv 0, 1 \pmod 4$ we define $\mathcal{Q}_D^{[\ell]}$ to be the subset of $\mathcal{Q}_D$ with the additional condition that $\ell | a$. It is easy to show that $\mathcal{Q}_D^{[\ell]}$ is invariant under $\Gamma_0^*(\ell)$.

If that $\ell = 1$ or ℓ is an odd prime with $\left(\frac{\ell}{p}\right) \neq -1$, then there is a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$ with norm ℓ. Define

$$\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{p}) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(K) : \alpha, \delta \in \mathcal{O}_K, \gamma \in \mathfrak{p}, \beta \in \mathfrak{p}^{-1} \right\}.$$

In this case there is a matrix $A \in \mathrm{GL}_2^+(K)$ such that $A^{-1}\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{p})A = \mathrm{SL}_2(\mathcal{O}_K)$. Define

$$\phi : (\mathfrak{h} \times \mathfrak{h})/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{p}) \to (\mathfrak{h} \times \mathfrak{h})/\mathrm{SL}_2(\mathcal{O}_K)$$

by

$$\phi((z_1, z_2)) := (Az_1, A'z_2).$$

Let Γ be the stabilizer of $\{(z, z) : z \in \mathfrak{h}\} \subseteq \mathfrak{h} \times \mathfrak{h}$ in $\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{p})$. Then $\Gamma = \Gamma_0(\ell)$ if $\ell \neq p$ and $\Gamma = \Gamma_0^*(\ell)$ if $\ell = p$. The image of $\{(z, z) : z \in \mathfrak{h}\}$ under ϕ is $Z_\ell^{(p)}$. Hence, we have a natural map $\psi : \mathfrak{h}/\Gamma \to Z_\ell^{(p)}$. By the work of Hirzebruch and Zagier [**12**], if $\ell = 1$ or an odd prime with $\left(\frac{\ell}{p}\right) \neq -1$, and $n \geq 1$, then we may define

$$Z_\ell^{(p)} \cap Z_n^{(p)} := \bigcup_{\substack{x \in \mathbb{Z} \\ x^2 < 4\ell n \\ x^2 \equiv 4\ell n \pmod p}} \left\{ \tau_Q : Q \in \mathcal{Q}_{(4\ell n - x^2)/p}^{[\ell]} \,/\, \Gamma_0^*(\ell) \right\}. \tag{3.1}$$

Here the repetition of x and $-x$ indicates that $Z_\ell^{(p)} \cap Z_n^{(p)}$ is a multiset where a CM point τ_Q occurs twice if $Q \in \mathcal{Q}_{(4\ell n - x^2)/p}^{[\ell]}$ for $x \neq 0$. In addition, if $\ell > 1$ and $\ell | n$, then we include

$$\bigcup_{\substack{x \in \mathbb{Z} \\ x^2 < 4n/\ell \\ x^2 \equiv 4n/\ell \pmod p}} \left\{ \tau_Q : Q \in \mathcal{Q}_{(4n/\ell - x^2)/p}^{[\ell]} \,/\, \Gamma_0^*(\ell) \right\},$$

where each point with non-zero x is taken with multiplicity 2ℓ, and a point where $x=0$ is taken with multiplicity ℓ.

To justify our definition we argue as follows. Hirzebruch and Zagier ([**12**], p. 66) show that if $t \in \mathfrak{h}$, $n \geq 1$ and $\psi(t) \in Z_\ell^{(p)} \cap Z_n^{(p)}$, then

$$a\ell t^2 + \frac{\ell\lambda - \ell\lambda'}{\sqrt{p}} t + b = 0$$

for $(a,b,\lambda) \in \mathbb{Z}\oplus\mathbb{Z}\oplus\mathfrak{p}^{-1}$ with $\ell\lambda\lambda'+abp=n$. This follows as a result of considering the inverse image $\phi^{-1}(Z_\ell^{(p)}) \subseteq (\mathfrak{h}\times\mathfrak{h})/\mathrm{SL}_2(\mathcal{O}_K,\mathfrak{p})$.

Write $\ell\lambda = c + d\frac{1+\sqrt{p}}{2}$, for $c,d \in \mathbb{Z}$. We have that the discriminant of the equation above is $d^2-4ab\ell$. However, this implies that

$$\frac{(2c+d)^2-4n\ell}{p} = d^2-4ab\ell.$$

Thus, the discriminant is of the form $(x^2-4n\ell)/p$. From Hirzebruch and Zagier's Theorem 3 ([**12**], p. 77), computing the number of transverse intersections of $Z_\ell^{(p)}$ and $Z_n^{(p)}$, we see that each $z \in \mathfrak{h}$ with discriminant of the form $(x^2-4n\ell)/p$ occurs with the appropriate multiplicity.

3.2. Traces of singular moduli on modular curves aprés Bruinier and Funke. Throughout, we let ℓ be 1 or an odd prime. Recently, Bruinier and Funke [**7**] have generalized Zagier's results on the modularity of generating functions for traces of singular moduli, and they have obtained results for groups which do not necessarily possess a Hauptmodul. A particularly elegant example of their work applies to modular functions on $\Gamma_0^*(\ell)$. Suppose that $f(z) = \sum_{n\gg-\infty} a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(\ell))$ has constant term $a(0)=0$. The discriminant $-D$ trace is given by

$$t_f^*(D) := \sum_{Q\in\mathcal{Q}_{D,\ell}/\Gamma_0^*(\ell)} \frac{1}{\#\Gamma_0^*(\ell)_Q} \cdot f(\tau_Q). \tag{3.2}$$

Here $\Gamma_0^*(\ell)_Q$ is the stabilizer of Q in $\Gamma_0^*(\ell)$. Following Kohnen [**16**], we let, for $\epsilon \in \{\pm1\}$, $\mathcal{M}^{+,\epsilon}_{k+\frac{1}{2}}(\Gamma_0(4\ell))$ be the space of those weight $k+\frac{1}{2}$ weakly holomorphic modular forms $f(z) = \sum_{n\gg-\infty} a(n)q^n$ on $\Gamma_0(4\ell)$ whose Fourier coefficients satisfy

$$a(n) = 0 \quad \text{whenever} \quad (-1)^k n \equiv 2,3 \pmod 4 \quad \text{or} \quad \left(\frac{(-1)^k n}{\ell}\right) = -\epsilon. \tag{3.3}$$

Theorem 3.1. (Bruinier and Funke; Theorem 1.1 of [**7**])
If $\ell=1$ or is an odd prime and $f(z) = \sum_{n\gg-\infty} a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(\ell))$, with $a(0)=0$, then

$$G_\ell(f,z) := -\sum_{m,n\geq1} ma(-mn)q^{-m^2} + \sum_{n\geq1} \left(\sigma_1(n)+\ell\sigma_1(n/\ell)\right) a(-n) + \sum_{D>0} t_f^*(D)q^D$$

is an element of $\mathcal{M}^{+,+}_{\frac{3}{2}}(\Gamma_0(4\ell))$.

3.3. Traces on Hilbert modular surfaces. We are now in a position to sketch the proofs of Theorems 1.2, 1.3, and 1.4.

Sketch of the proof of Theorem 1.2. It is well known that the Jacobi theta function

$$\Theta(z) = \sum_{x\in\mathbb{Z}} q^{x^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots . \tag{3.4}$$

is a weight 1/2 holomorphic modular form on $\Gamma_0(4)$. Suppose that

$$f(z) = \sum_{n\gg-\infty} a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(\ell))$$

satisfies the hypotheses of Theorem 1.2. By (3.1) and Theorem 3.1, an easy calculation reveals that

$$\Phi^{(p)}_{\ell,f}(z) = \epsilon(\ell)\left(G_\ell(f,pz)\Theta(z)\right) \mid U(4) \mid \left(U(\ell)+\ell V(\ell)\right), \tag{3.5}$$

where for $d \geq 1$ the operators $U(d)$ and $V(d)$ are defined on formal power series by

$$\left(\sum a(n)q^n\right) \mid U(d) := \sum a(dn)q^n, \tag{3.6}$$

and

$$\left(\sum a(n)q^n\right) \mid V(d) := \sum a(n)q^{dn}. \tag{3.7}$$

The proof now follows from generalizations of classical facts about the U and V operators to spaces of weakly holomorphic modular forms. □

Sketch of the proof of Theorem 1.3. We work directly with (1.1). We recall the following classical theta function identities:

$$\Theta(z) = \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} = \sum_{x\in\mathbb{Z}} q^{x^2} = 1 + 2q + 2q^4 + \cdots , \tag{3.8}$$

$$\Theta_0(z) = \frac{\eta(z)^2}{\eta(2z)} = \sum_{x\in\mathbb{Z}} (-1)^x q^{x^2} = 1 - 2q + 2q^4 - 2q^9 + \cdots , \tag{3.9}$$

and

$$\Theta_{\mathrm{odd}}(z) = \frac{\eta(16z)^2}{\eta(8z)} = \sum_{x\geq 0} q^{(2x+1)^2} = q + q^9 + q^{25} + q^{49} + \cdots . \tag{3.10}$$

By (1.1), (3.5), and (3.9), we have that

$$\begin{aligned}\Phi^{(p)}_{1,J_1}(z) &= -\left(g_1(pz)\Theta(z)\right) \mid U(4)\\ &= -\left(\frac{\Theta_0(pz)E_4(4pz)}{\eta(4pz)^6}\cdot\Theta(z)\right) \mid U(4).\end{aligned}$$

For integers v, we have the identities $E_4(p(z+\nu)) = E_4(pz)$ and

$$\eta(p(z+\nu))^6 = i^\nu \eta(pz)^6,$$

which when inserted into the definition of $U(4)$ gives

$$\Phi^{(p)}_{1,J_1}(z) = -\frac{E_4(pz)}{4\eta(pz)^6}\sum_{\nu=0}^{3} i^{-\nu}\Theta_0(p(z+\nu)/4)\Theta((z+\nu)/4).$$

By (3.9) and (3.10), one finds that

$$\Phi^{(p)}_{1,J_1}(z) = -\frac{E_4(pz)}{4\eta(pz)^6} \cdot \sum_{x,y\in\mathbb{Z}} q^{(px^2+y^2)/4} \cdot (-1)^x \left(\sum_{\nu=0}^{3} i^{p\nu x^2+y^2\nu-\nu}\right).$$

Since we have that

$$\sum_{\nu=0}^{3} i^{p\nu x^2+y^2\nu-\nu} = \begin{cases} 0 & \text{if } x \equiv y \pmod 2, \\ 4 & \text{if } x \not\equiv y \pmod 2, \end{cases}$$

it follows that

$$\begin{aligned}\Phi^{(p)}_{1,J_1}(z) &= -\frac{E_4(pz)}{\eta(pz)^6} \cdot \left(\sum_{x,y\in\mathbb{Z}} q^{((2y+1)^2+4px^2)/4} - \sum_{x,y\in\mathbb{Z}} q^{(4y^2+(2x+1)^2p)/4}\right) \\ &= -\frac{2E_4(pz)}{\eta(pz)^6} \cdot \left(\Theta(pz)\Theta_{\mathrm{odd}}(z/4) - \Theta(z)\Theta_{\mathrm{odd}}(pz/4)\right).\end{aligned}$$

The claimed formula now follows easily from (1.20), (3.8), and (3.10). □

Sketch of the proof of Theorem 1.4. If $p \equiv 1 \pmod 4$ is prime, then a lengthy, but straightforward calculation, reveals that

$$N^*_p(z) = E_4(z)^{a(p)} \cdot \Delta(z)^{c(p)} \cdot F_p(j(z)), \tag{3.11}$$

where $F_p(x) \in \mathbb{Z}[x]$ is a monic polynomial with

$$\deg(F_p(x)) = \begin{cases} (5p-5)/12 & \text{if } p \equiv 1 \pmod{12}, \\ (5p-1)/12 & \text{if } p \equiv 5 \pmod{12}. \end{cases}$$

Hence it suffices to compute the factorization of $F_p(x)$ over $\mathbb{Z}[x]$.

Loosely speaking, $F_p(x)$ captures the divisor of the modular form $N^*_p(z)$ in $\mathfrak{h}$. To compute the points in the divisor, we shall make use of Theorem 1.3. Since $\eta(z)$ is non-vanishing on $\mathfrak{h}$, the factors of $F_p(x)$ only arise from the zeros of the "norm" of $E_4(pz)$ and of

$$\mathfrak{f}_1(4z)^4\mathfrak{f}_2(z)^2 - \mathfrak{f}_1(4pz)^4\mathfrak{f}_2(pz)^2.$$

To determine these zeros and their corresponding multiplicities, we require classical facts about class numbers and the Eichler-Selberg trace formula. To begin, first observe that $E_4(\omega) = 0$, where $\omega := e^{2\pi/3} = \frac{-1+\sqrt{-3}}{2}$. Hence it follows that $E_4(pz)$ is zero for $z_p := \omega/p$. Since z_p has discriminant $-3p^2$, the irreducibility of $H_{3\cdot p^2}(x)$ implies that $H_{3\cdot p^2}(x) \mid F_p(x)$ in $\mathbb{Z}[x]$. Therefore, we may conclude that

$$F_p(x) = H_{3\cdot p^2}(x) \cdot I_p(x),$$

where $I_p(x) \in \mathbb{Z}[x]$ has

$$\deg(I_p(x)) = \begin{cases} (p-1)/12 & \text{if } p \equiv 1 \pmod{12}, \\ (p-5)/12 & \text{if } p \equiv 5 \pmod{12}. \end{cases}$$

To complete the proof, it suffices to determine the polynomial $I_p(x)$. To this end, observe that $I_p(x)$ is the polynomial which encodes the divisor of the norm of

$$\mathfrak{f}_1(4z)^4\mathfrak{f}_2(z)^2 - \mathfrak{f}_1(4pz)^4\mathfrak{f}_2(pz)^2.$$

To study this divisor, one notes that if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $b \equiv c \equiv 0 \pmod 4$ and $g(z) := \mathfrak{f}_1(4z)^4\mathfrak{f}_2(z)^2$, then $g\left(\frac{az+b}{cz+d}\right) = g(z)$. The proof is complete once we establish that

$$I_p(x) = H_3(x)^{a(p)} \cdot H_4(x)^{b(p)} \prod_{-D\in\mathcal{D}_p} H_D(x)^2.$$

To prove this assertion, we note that the modular transformation above implies that $z \in \mathfrak{h}$ is a root of $g(z) - g(pz)$ if $\frac{az+b}{cz+d} = pz$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $b \equiv c \equiv 0 \pmod 4$. This leads to the quadratic equation

$$\frac{pc}{4}z^2 + \frac{pd-a}{4}z - \frac{b}{4} = 0.$$

Using some class number relations, and the fact that Hilbert class polynomials are irreducible, we simply need to show that for a negative discriminant of the form $-D := \frac{x^2-4p}{16f^2}$ with $x, f \in \mathbb{Z}$ that there are two integral binary quadratic forms

$$\begin{aligned} Q_1 &:= \frac{pc_1}{4f}x^2 + \frac{pd_1 - a_1}{4f}xy - \frac{b_1}{4f}y^2 \\ Q_2 &:= \frac{pc_2}{4f}x^2 + \frac{pd_2 - a_2}{4f}xy - \frac{b_2}{4f}y^2, \end{aligned}$$

which are inequivalent under $\Gamma_0(p)$ with discriminants $-D$ such that $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $b_1 \equiv b_2 \equiv c_1 \equiv c_2 \equiv 0 \pmod 4$. This is an easy exercise. □

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