Ramanujan's congruences and Dyson's crank

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The achievement of Karl Mahlburg in this issue of PNAS (1) adds a lustrous chapter to a unique mathematical object: the crank.

In 1944, the crank was first hinted at by Freeman Dyson (2), then an undergraduate at Cambridge University. He had written an article, titled *Some Guesses in the Theory of Partitions*, for *Eureka*, the undergraduate mathematics journal of Cambridge. Dyson discovered the many conjectures in this article by attempting to find a combinatorial explanation of Ramanujan's famous congruences for $p(n)$, the number of partitions of n.

The three simplest of Ramanujan's congruences assert that:

$$5 \text{ divides } p(5n+4),$$

$$7 \text{ divides } p(7n+5),$$

$$11 \text{ divides } p(11n+6).$$

In Dyson's article, he defines the rank of a partition to be the largest part of the partition minus the number of parts. For example, the rank of the partition (of 22) $5 + 5 + 3 + 3 + 3 + 2 + 1$ is $5 - 7 = -2$. Dyson conjectured (and later Atkin and Swinnerton-Dyer proved, see ref. 3) that if $N(m, t, n)$ denotes the number of partitions of n with rank congruent to m modulo t, then

$$N(m, 5, 5n+4) = \frac{1}{5}p(5n+4), \quad 0 \le m \le 4$$

and

$$N(m, 7, 7n+5) = \frac{1}{7}p(7n+5), \quad 0 \le m \le 6.$$

Surprisingly, the obvious conjecture for the congruence at 11 is false, which led to the following concluding remarks in Dyson's article:

> I hold in fact: That there exists an arithmetical coefficient similar to, but more recondite than, the rank of a partition; I shall call this hypothetical coefficient the "crank" of the partition, and denote by $M(m, q, n)$ the number of partitions of n whose crank is congruent to m modulo q: that $M(m, q, n) = M(q - m, q, n)$, that $M(0, 11, 11n + 6) = M(1, 11, 11n + 6) = M(2, 11, 11n + 6) = M(3, 11, 11n + 6) = M(4, 11, 11n + 6)$; that numerous other relations exist. . . .
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> Whether these guesses are warranted by the evidence, I leave to the reader to decide. Whatever the final verdict of posterity may be, I believe the "crank" is unique among arithmetical functions in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan!

Unknown to Dyson and everyone else was the fact that in a somewhat chaotic handwritten collection of formulas (4) prepared by Srinivasa Ramanujan in 1919 lay the mathematical foundation of Dyson's mysterious crank.

Indeed, Ramanujan's formulas lay unread until 1976 when they were found in the Trinity College Library of Cambridge University among papers from the estate of the late G. N. Watson. In the early 1980s, Frank Garvan (5) wrote his Pennsylvania State Ph.D. thesis precisely on the formulas of Ramanujan relative to the soon-to-be-unearthed crank. In 1987, Garvan and Andrews (6) were able to find and describe the crank that had been hiding in Ramanujan's work for 68 years.

A few years ago, Ono (7) revisited the theory of Ramanujan's congruences. Thanks to many developments in the theory of modular forms, he was able to show that there are Ramanujan-type congruences for every prime >3. In general, such congruences are incredibly complicated. Indeed, the simplest congruence involving the prime 17 reads

$$p(48037937n + 1122838) \equiv 0 \bmod 17.$$

Subsequent work of Scott Ahlgren and Ono (8) has provided a comprehensive theoretical framework describing all such congruences. In particular, it is now known that there are congruences where the primes 5, 7, 11, and 17 above may be replaced by any positive integer without 2 or 3 as a factor.

In view of this theoretical description, it is then natural to ask what role the crank plays. Do the crank functions $M(n,q,m)$ reveal deeper insight into all of the partition congruences, or are they only relevant in the case of the primes 5, 7, and 11? Mahlburg's thesis research, under the direction of Ono, provides an elegant answer to these questions. Mahlburg's groundbreaking work (1) shows that the crank functions are intimately connected to all partition congruences. In short, Mahlburg shows that the crank functions themselves obey Ramanujan-type congruences, a surprising fact that clearly cements the central role of the crank in the theory of partitions.

The story of the crank is a long romantic tale, one that starts with Ramanujan, is inspired by the conjectures of Dyson, was fertilized by clues from the lost notebook, and has now reached a satisfying and unexpected conclusion with the work of Mahlburg. As Dyson said:

> Each step in the story is a work of art, and the story as a whole is a sequence of episodes of rare beauty, a drama built out of nothing but numbers and imagination.

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