

LIFTING ELLIPTIC CUSP FORMS TO MAASS FORMS WITH AN APPLICATION TO PARTITIONS

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ABSTRACT. For $2 < k \in \frac{1}{2}\mathbb{Z}$, we define lifts of cuspidal Poincaré series in $S_k(\Gamma_0(N))$ to weight $2 - k$ harmonic weak Maass forms. This construction answers a question of Dyson by providing the general framework “explaining” Ramanujan’s mock theta functions. As an application, we show that the number of partitions of a positive integer n is the “trace” of singular moduli of a Maass form arising from the lift of a weight 4 cusp form corresponding to a Calabi-Yau threefold.

1. INTRODUCTION AND STATEMENT OF RESULTS

Ramanujan proved that the partition function $p(n)$ satisfies the congruences

$$\begin{aligned} p(5n + 4) &\equiv 0 \pmod{5}, \\ p(7n + 5) &\equiv 0 \pmod{7}, \\ p(11n + 6) &\equiv 0 \pmod{11}. \end{aligned}$$

Although these congruences are not difficult to prove, the generic theory [1, 2, 21] of partition congruences is quite complicated, and depends critically on the interplay between deeper structures in the theory of modular forms. Congruences such as

$$p(48037937n + 1122838) \equiv 0 \pmod{17}$$

depend on the Deligne-Serre theory of ℓ -adic Galois representations, and Shimura’s theory of half-integral weight modular forms. Shimura’s theory is built around lifts which map half-integral weight cusp forms to integer weight cusp forms.

We describe another lift, one which maps cuspidal Poincaré series to harmonic weak Maass forms. Using these maps, we obtain an arithmetic formula exhibiting $p(n)$ as the “trace” of singular moduli of a Maass form arising from a Calabi-Yau threefold.

First we describe these lifts. Suppose that $2 < k \in \frac{1}{2}\mathbb{Z}$, and that N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$). Let $\text{Maass}_{2-k}(\Gamma_0(N), 0)$ be the space of weight $2 - k$ harmonic weak Maass forms on $\Gamma_0(N)$ (see Section 2), and let $\text{Weak}_k(\Gamma_0(N))$ be the space of weight k weakly holomorphic modular forms on $\Gamma_0(N)$, where a *weakly holomorphic modular form* is any meromorphic modular form whose poles (if any) are

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supported at cusps. The differential operator $\xi_w := 2iy^w \frac{\partial}{\partial \bar{z}}$ defines a map

$$\xi_{2-k} : \text{Maass}_{2-k}(\Gamma_0(N), 0) \longrightarrow \text{Weak}_k(\Gamma_0(N)).$$

Let $\text{Maass}_{2-k}^*(\Gamma_0(N))$ be the subspace of those $f(z) \in \text{Maass}_{2-k}(\Gamma_0(N), 0)$ for which $\xi_{2-k}(f(z)) \in S_k(\Gamma_0(N))$, the weight k elliptic cusp forms on $\Gamma_0(N)$. It turns out that $\ker(\xi_{2-k}) = \text{Weak}_{2-k}(\Gamma_0(N))$.

For every positive integer m , let $H(m, k, N; z) \in S_k(\Gamma_0(N))$ be the classical holomorphic Poincaré series (see Section 3.2). These forms generate $S_k(\Gamma_0(N))$. Similarly, for every positive integer m we construct (see Section 3.3) Maass-Poincaré series

$$F(m, 2 - k, N; z) \in \text{Maass}_{2-k}^*(\Gamma_0(N)).$$

Using these series, we let

$$(1.1) \quad \mathcal{L}_{k,N}(H(m, k, N; z)) := F(m, 2 - k, N; z).$$

This defines the lifting of cuspidal Poincaré series in $S_k(\Gamma_0(N))$ to $\text{Maass}_{2-k}^*(\Gamma_0(N))$ which is dual to the differential operator ξ_{2-k} .

Theorem 1.1. *Assume the notation and hypotheses above. The following are true:*

(1) *We have that*

$$f_{H(m,k,N;z)}(z) := \mathcal{L}_{k,N}(H(m, k, N; z)) - i(k-1)(2\pi m)^{k-1} \int_{-\bar{z}}^{i\infty} \frac{\overline{H(m, k, N; -\bar{\tau})}}{(-i(\tau+z))^{2-k}} d\tau$$

is a holomorphic function on the complex upper half-plane \mathbb{H} .

(2) *We have that*

$$\xi_{2-k}(\mathcal{L}_{k,N}(H(m, k, N; z))) = (k-1) \cdot (4\pi m)^{k-1} \cdot H(m, k, N; z).$$

Remark. Since Poincaré series in $S_k(\Gamma_0(N))$ are dependent, we stress that these lifting maps are defined on Poincaré series, not the space $S_k(\Gamma_0(N))$.

Remark. Theorem 1.1 (1) is typical of results in the theory of automorphic integrals (for example, see works by Knopp and Niebur [17, 20]), where automorphic forms arise from period integrals of cusp forms. Thanks to explicit formulas for our Poincaré series, Theorem 1.1 (1) follows from an elementary integral identity (see Proposition 4.1).

Now we turn to the motivating problem of providing an arithmetic formula for $p(n)$. To this end, let $g_C(z) \in S_4(\Gamma_0(6))$ be the eta-product

$$(1.2) \quad g_C(z) := \eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z) = \sum_{n=1}^{\infty} a(n)q^n = q - 2q^2 - 3q^3 + 4q^4 + \dots,$$

where $q := e^{2\pi iz}$, and where $\eta(z)$ is Dedekind's eta-function. This form corresponds (see Mortenson's thesis [18] and Verrill's paper [24]) to the Calabi-Yau threefold

$$Y : x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + xy + \frac{1}{xy} + yz + \frac{1}{yz} + xyz + \frac{1}{xyz} = t + \frac{1}{t} - 2.$$

For odd primes p , this implies that $a(p) = p^3 - p^2 - 13 - N(p)$, where $N(p) := \#Y(\mathbb{F}_p)$. If $\|\cdot\|$ denotes the usual Petersson norm, then define $\mathcal{C}(z) \in \text{Maass}_0(\Gamma_0(6), -2)$ by

$$(1.3) \quad \mathcal{C}(z) := -\frac{1}{96\pi^3 \|\mathcal{g}_C\|^2} \cdot \left(\frac{1}{2\pi i} \frac{\partial}{\partial z} + \frac{1}{2\pi \text{Im}z} \right) (\mathcal{L}_{4,6}(\mathcal{g}_C(z))).$$

Due to the fact that the lifting maps are defined on Poincaré series, $\mathcal{C}(z)$ is not well defined as given above. Later in the paper we resolve this issue by describing $\mathcal{g}_C(z)$ in terms of the first Poincaré series in $S_4(\Gamma_0(6))$.

Remark. In terms of symmetric-square L -functions, (1.3) implies

$$\mathcal{C}(z) = -\frac{64}{27\pi \cdot L(\text{Sym}^2(\mathcal{g}_C), 1)} \cdot \left(\frac{1}{2\pi i} \frac{\partial}{\partial z} + \frac{1}{2\pi \text{Im}z} \right) (\mathcal{L}_{4,6}(\mathcal{g}_C(z))).$$

For positive $d \equiv 0, 3 \pmod{4}$, let $\mathcal{Q}_d^{(p)}$ be the set of positive definite integral binary quadratic forms (including imprimitive forms) $Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$ of discriminant $-d = b^2 - 4ac$, where $6 \mid a$. The group $\Gamma_0(6)$ acts on $\mathcal{Q}_d^{(p)}$ in the usual way. For each Q , let τ_Q be the unique root of $Q(x, 1) = 0$ in \mathbb{H} , and let $\Gamma_{\tau_Q} \subseteq \Gamma_0(6)$ be its isotropy subgroup. As in the theory of complex multiplication, we refer to each $\mathcal{C}(\tau_Q)$ as a *singular modulus*. For positive integers n , let

$$(1.4) \quad \text{Tr}^{(p)}(n) := \sum_{Q \in \mathcal{Q}_{24n-1}^{(p)}/\Gamma_0(6)} \frac{\chi_{12}(Q)\mathcal{C}(\tau_Q)}{\#\Gamma_{\tau_Q}},$$

where $\chi_{12}(Q) = \chi_{12}([a, b, c]) := \left(\frac{12}{b}\right)$. The following gives the formula for $p(n)$.

Theorem 1.2. *If n is a positive integer, then*

$$p(n) = \frac{\text{Tr}^{(p)}(n)}{24n - 1}.$$

Remark. This phenomenon, where coefficients of half-integral weight forms are “traces” of singular moduli was observed by Zagier [25]. Recent papers by the authors [6] and Bruinier and Funke [12] give generalizations.

In Section 2 we recall facts about weak Maass forms. In Section 3 we construct the Poincaré series $H(m, k, N; z)$ and $F(m, 2 - k, N; z)$, and we compute their Fourier expansions. In Sections 4 and 5 we prove Theorems 1.1 and 1.2. In Section 6 we explain how Theorem 1.1 is related to Ramanujan’s mock theta functions.

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2. WEAK MAASS FORMS

We recall the notion of a weak Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$. If $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$, then the weight k hyperbolic Laplacian is given by

$$(2.1) \quad \Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) +iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

For odd integers d , define ϵ_d by

$$(2.2) \quad \epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}, \\ i & \text{if } d \equiv 3 \pmod{4}. \end{cases}$$

If N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), then a *weak Maass form of weight k on $\Gamma_0(N)$* is any smooth function $M : \mathbb{H} \rightarrow \mathbb{C}$ satisfying the following:

- (1) For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and all $z \in \mathbb{H}$, we have

$$M(Az) = \begin{cases} (cz + d)^k M(z) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz + d)^k M(z) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

Here $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol, and \sqrt{z} is the principal branch of the holomorphic square root.

- (2) There is a complex number λ for which $\Delta_k M = \lambda M$.
(3) The function $M(z)$ has at most linear exponential growth at cusps.

Remark. These transformation laws occur in Shimura's theory of half-integral weight modular forms [23].

Let $\text{Maass}_k(\Gamma_0(N), \lambda)$ denote the space of weight k weak Maass forms on $\Gamma_0(N)$ with eigenvalue λ with respect to Δ_k . Those forms with $\lambda = 0$ are called *harmonic*, and they are relevant for Theorem 1.1. Here we recall some facts due to Bruinier and Funke (see Proposition 3.2 of [11]).

Lemma 2.1. *The differential operator $\xi_k := 2iy^k \frac{\partial}{\partial \bar{z}}$ maps*

$$\xi_k : \text{Maass}_k(\Gamma_0(N), 0) \longrightarrow \text{Weak}_{2-k}(\Gamma_0(N)),$$

and $\ker(\xi_k) = \text{Weak}_k(\Gamma_0(N))$.

3. POINCARÉ SERIES

To prove Theorem 1.1, we rely on explicit Fourier expansions (one could also argue directly with the defining series). Throughout, we rely on classical special functions whose properties and definitions may be found in [3]. We give them since they are useful in applications (for example, see [5]). In Section 3.1 we first recall the classical construction of Poincaré series (see [10, 15, 16]). In Sections 3.2 and 3.3 we give explicit Fourier expansions in terms of classical special functions (see [3] for more on these special functions).

3.1. The fundamental Poincaré series. Suppose that $k \in \frac{1}{2}\mathbb{Z}$, and that N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$). For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, define $j(A, z)$ by

$$(3.1) \quad j(A, z) := \begin{cases} \sqrt{cz + d} & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right) \varepsilon_d^{-1} \sqrt{cz + d} & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases}$$

where ε_d is defined by (2.2), and where \sqrt{z} is the principal branch of the holomorphic square root as before. As usual, for $A \in \Gamma_0(N)$ and functions $f : \mathbb{H} \rightarrow \mathbb{C}$, we let

$$(3.2) \quad (f |_k A)(z) := j(A, z)^{-2k} f(Az).$$

Let m be an integer, and let $\varphi_m : \mathbb{R}^+ \rightarrow \mathbb{C}$ be a function which satisfies $\varphi_m(y) = O(y^\alpha)$, as $y \rightarrow 0$, for some $\alpha \in \mathbb{R}$. If $e(\alpha) := e^{2\pi i \alpha}$ as usual, then

$$(3.3) \quad \varphi_m^*(z) := \varphi_m(y) e(mx)$$

is fixed by the group of translations $\Gamma_\infty := \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$. Given this data we let

$$(3.4) \quad P(m, k, N, \varphi_m; z) := \sum_{A \in \Gamma_\infty \backslash \Gamma_0(N)} (\varphi_m^* |_k A)(z).$$

The explicit Fourier expansions are given in terms of the Kloosterman sums

$$(3.5) \quad K_k(m, n, c) := \begin{cases} \sum_{v(c) \times} e\left(\frac{m\bar{v} + nv}{c}\right) & \text{if } k \in \mathbb{Z}, \\ \sum_{v(c) \times} \left(\frac{c}{v}\right)^{2k} \varepsilon_v^{2k} e\left(\frac{m\bar{v} + nv}{c}\right) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

In the sums above, v runs through the primitive residue classes modulo c , and \bar{v} denotes the multiplicative inverse of v modulo c . The following lemma gives the fundamental properties of such Poincaré series (for example, see Proposition 3.1 of [13] where $N = 4$).

Lemma 3.1. *If $k > 2 - 2\alpha$, then the following are true.*

- (1) *Each Poincaré series $P(m, k, N, \varphi_m; z)$ is a weight k $\Gamma_0(N)$ -invariant function.*
- (2) *Near the cusp at ∞ , the function $P(m, k, N, \varphi_m; z) - \varphi_m^*(z)$ has moderate growth. Near the other cusps, $P(m, k, N, \varphi_m; z)$ has moderate growth.*
- (3) *If $P(m, k, N, \varphi_m; z)$ is twice continuously differentiable, then it has the locally uniformly absolutely convergent Fourier expansion*

$$P(m, k, N, \varphi_m; z) = \varphi_m^*(z) + \sum_{n \in \mathbb{Z}} a(n, y) e(nx),$$

where

$$a(n, y) := \sum_{\substack{c > 0 \\ c \equiv 0 \pmod{N}}}^{\infty} c^{-k} K_k(m, n, c) \int_{-\infty}^{\infty} z^{-k} \varphi_m\left(\frac{y}{c^2 |z|^2}\right) e\left(-\frac{mx}{c^2 |z|^2} - nx\right) dx.$$

3.2. The holomorphic Poincaré series $H(m, k, N; z)$. Suppose that $2 < k \in \frac{1}{2}\mathbb{Z}$, and that N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$). For positive integers m , let

$$(3.6) \quad H(m, k, N; z) := P(m, k, N, 1; z).$$

Lemma 3.1, combined with facts about Petersson norms, implies the following well known proposition (for example, see Chapter 3 of [16]).

Proposition 3.2. *The set of Poincaré series $\{H(m, k, N; z) : m \geq 1\}$ spans $S_k(\Gamma_0(N))$. Moreover, if $\delta(m, n)$ is the Kronecker delta-function and J_{k-1} is the usual J -Bessel function, then $H(m, k, N; z) = \sum_{n=1}^{\infty} b_m(n)q^n$, where*

$$b_m(n) = \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \left(\delta(m, n) + 2\pi i^{-k} \sum_{\substack{c>0 \\ c \equiv 0 \pmod{N}}} \frac{K_k(m, n, c)}{c} \cdot J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) \right).$$

3.3. Maass-Poincaré series $F(m, 2-k, N; z)$. Although the series $F(m, 2-k, N; z)$ are less well known, they have appeared in earlier works [6, 10, 13, 15, 19]. To define them, again suppose that $2 < k \in \frac{1}{2}\mathbb{Z}$, and that N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$). To employ Lemma 3.1, we first select an appropriate function φ .

Let $M_{\nu, \mu}(z)$ be the usual M -Whittaker function. For complex s , let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2} \operatorname{sgn}(y), s-\frac{1}{2}}(|y|),$$

and for positive m let $\varphi_{-m}(z) := \mathcal{M}_{1-\frac{k}{2}}(-4\pi my)$. We now let

$$(3.7) \quad F(m, 2-k, N; z) := P(-m, 2-k, N, \varphi_{-m}; z).$$

Lemma 3.1 leads to the following proposition (for a proof in the $N = 4$ case see [6]).

Proposition 3.3. *Each $F(m, 2-k, N; z)$ is in $\text{Maass}_{2-k}(\Gamma_0(N), 0)$. Moreover, if I_{k-1} is the usual I -Bessel function, and $\Gamma(a, x)$ is the incomplete Γ -function, then*

$$F(m, 2-k, N; z) = (1-k) (\Gamma(k-1, 4\pi my) - \Gamma(k-1)) q^{-m} + \sum_{n \in \mathbb{Z}} c(n, y) q^n.$$

1) If $n < 0$, then

$$c(n, y) = 2\pi i^k (1-k) \Gamma(k-1, 4\pi|n|y) \left| \frac{n}{m} \right|^{\frac{1-k}{2}} \times \sum_{\substack{c>0 \\ c \equiv 0 \pmod{N}}} \frac{K_{2-k}(-m, n, c)}{c} \cdot J_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

2) If $n > 0$, then

$$c(n, y) = -2\pi i^k \Gamma(k) \left(\frac{n}{m}\right)^{\frac{1-k}{2}} \sum_{\substack{c>0 \\ c \equiv 0 \pmod{N}}} \frac{K_{2-k}(-m, n, c)}{c} \cdot I_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

3) If $n = 0$, then

$$c(0, y) = -2^k \pi^k i^k m^{k-1} \sum_{\substack{c > 0 \\ c \equiv 0 \pmod{N}}} \frac{K_{2-k}(-m, 0, c)}{c^k}.$$

4. PROOF OF THEOREM 1.1

Throughout, suppose that $2 < k \in \frac{1}{2}\mathbb{Z}$, and that N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$).

We begin with an elementary integral identity.

Proposition 4.1. *If n is a positive integer, then*

$$\int_{-\bar{z}}^{i\infty} \frac{e^{2\pi i n \tau}}{(-i(\tau + z))^{2-k}} d\tau = i(2\pi n)^{1-k} \cdot \Gamma(k-1, 4\pi n y) q^{-n}.$$

Proof. This identity follows by the direct calculation

$$\int_{-\bar{z}}^{i\infty} \frac{e^{2\pi i n \tau}}{(-i(\tau + z))^{2-k}} d\tau = \int_{2iy}^{i\infty} \frac{e^{2\pi i n(\tau-z)}}{(-i\tau)^{2-k}} d\tau = i(2\pi n)^{1-k} \cdot \Gamma(k-1, 4\pi n y) q^{-n}.$$

□

Proof of Theorem 1.1. We first prove Theorem 1.1 (2). For convenience, let

$$F(m, 2-k, N; z) = (k-1)\Gamma(k-1)q^{-m} + \sum_{n=0}^{\infty} a(n)q^n + \sum_{n=1}^{\infty} b(n)\Gamma(k-1, 4\pi n y)q^{-n}.$$

The operator ξ_{2-k} is anti-linear, and it has the property that $\xi_{2-k}(f) = 0$ for holomorphic functions f . We also have the identity

$$\xi_{2-k}(\Gamma(k-1, 4\pi n y)) = -(4\pi n)^{k-1} e^{-4\pi n y}.$$

These facts imply that

$$\xi_{2-k}(F(m, 2-k, N; z)) = -(4\pi)^{k-1} \sum_{n=1}^{\infty} n^{k-1} \overline{b(n)} q^n.$$

By the definition of $\mathcal{L}_{k,N}$, Theorem 1.1 (2) follows from the identity

$$\overline{K_{2-k}(-m, -n, c)} = K_k(m, n, c).$$

To prove Theorem 1.1 (1), it suffices to compare the Fourier expansion of

$$\int_{-\bar{z}}^{i\infty} \frac{\overline{H(m, k, N; -\bar{\tau})}}{(-i(\tau + z))^{2-k}} d\tau$$

with the non-holomorphic part of $F(m, 2 - k, N; z)$. If $H(m, k, N; z) = \sum_{n=1}^{\infty} b(n)q^n$, then by Proposition 4.1 we find that

$$\int_{-\bar{z}}^{i\infty} \frac{\overline{H(m, k, N; -\bar{\tau})}}{(-i(\tau + z))^{2-k}} d\tau = i(2\pi)^{1-k} \sum_{n=1}^{\infty} \frac{\overline{b(n)}}{n^{k-1}} \Gamma(k-1, 4\pi ny) q^{-n}.$$

The claim now follows from the formulas in Propositions 3.2 and 3.3. \square

5. PROOF OF THEOREM 1.2

In earlier work [7], the authors proved that if n is a positive integer, then

$$p(n) = \frac{1}{24n-1} \sum_{Q \in \mathcal{Q}_{24n-1}^{(p)}/\Gamma_0(6)} \frac{\chi_{12}(Q)P(\tau_Q)}{\#\Gamma_{\tau_Q}},$$

where

$$(5.1) \quad P(z) := 4\pi \sum_{A \in \Gamma_{\infty} \setminus \Gamma_0(6)} \operatorname{Im}(Az)^{\frac{1}{2}} I_{\frac{3}{2}}(2\pi \operatorname{Im}(Az)) e(-\operatorname{Re}(Az)).$$

This was obtained by reformulating Rademacher's exact formula using Salié sums.

Therefore, it suffices to show that

$$P(z) = -\frac{1}{96\pi^3 \|g_C\|^2} \cdot \left(\frac{1}{2\pi i} \frac{\partial}{\partial z} + \frac{1}{2\pi \operatorname{Im} z} \right) (\mathcal{L}_{4,6}(g_C(z))).$$

Since $\dim_{\mathbb{C}}(S_4(\Gamma_0(6))) = 1$, it follows (for example, see Chapter 3 of [16]) that

$$g_C(z) = 32\pi^3 \|g_C\|^2 \cdot H(1, 4, 6; z).$$

Therefore, it suffices to show that

$$P(z) = -\frac{1}{3} \left(\frac{1}{2\pi i} \frac{\partial}{\partial z} + \frac{1}{2\pi \operatorname{Im} z} \right) F(1, -2, 6; z).$$

For this we need the Fourier expansion of $P(z)$ which was computed by Niebur [20]. Correcting some typographical errors in his paper, we find that

$$P(z) = 4\pi I_{\frac{3}{2}}(2\pi y) y^{\frac{1}{2}} e^{-2\pi i x} + \sum_{n \in \mathbb{Z}} b(n, y) q^n,$$

where

(5.2)

$$b(n, y) := \begin{cases} \frac{8\pi^3}{3y} \sum_{\substack{c>0 \\ c \equiv 0 \pmod{6}}} \frac{K_{-2}(-1, 0, c)}{c^4} & \text{if } n = 0, \\ 8\pi y^{1/2} e^{2\pi n y} K_{\frac{3}{2}}(2\pi n y) \sum_{\substack{c>0 \\ c \equiv 0 \pmod{6}}} \frac{K_{-2}(-1, n, c)}{c} \cdot I_3\left(\frac{4\pi\sqrt{n}}{c}\right) & \text{if } n > 0, \\ 8\pi y^{1/2} e^{2\pi n y} K_{\frac{3}{2}}(2\pi|n|y) \sum_{\substack{c>0 \\ c \equiv 0 \pmod{6}}} \frac{K_{-2}(-1, n, c)}{c} \cdot J_3\left(\frac{4\pi\sqrt{|n|}}{c}\right) & \text{if } n < 0. \end{cases}$$

Thanks to Proposition 3.3 and (5.2), the theorem is obtained (after some computation) using the identities:

$$\begin{aligned} \Gamma(3, y) &= e^{-y} (y^2 + 2y + 2), \\ \frac{\partial}{\partial z} \Gamma(3, 4\pi|n|y) &= 32\pi^3 i |n|^3 y^2 e^{-4\pi|n|y}, \\ I_{\frac{3}{2}}(z) &= \left(\frac{z}{2\pi}\right)^{1/2} \left(\frac{1}{z^2} (e^{-z} - e^z) + \frac{1}{z} (e^z + e^{-z})\right), \\ K_{\frac{3}{2}}(z) &= \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left(1 + \frac{1}{z}\right). \end{aligned}$$

6. RELATIONSHIP WITH RAMANUJAN'S MOCK THETA FUNCTIONS

Ramanujan's mock theta functions are a collection of 22 "strange" q -series such as

$$(6.1) \quad f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

They do not arise as the minor modification of a modular form. Nevertheless, a wealth of evidence, such as identities involving mock theta functions and modular forms, suggested a strong connection between these objects (for example, see [4] and the references therein). Determining their place in the theory of automorphic forms was a puzzle for many decades, a quandary nicely described [14] by Freeman Dyson in 1987:

"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future."

In his 2002 Ph.D. thesis [26, 27], Zagier made an important breakthrough. Loosely speaking, he "completed" the mock theta functions to obtain weight 1/2 weak Maass

forms. In the case of $f(q)$, it turns out that (see Corollary 2.3 of [5])

$$M_f(z) := q^{-1}f(q^{24}) - 2i\sqrt{3} \cdot N_f(z)$$

is a weight $1/2$ weak Maass form on $\Gamma_0(144)$ with Nebentypus $\chi_{12}(\cdot) := \left(\frac{12}{\cdot}\right)$, where

$$N_f(z) := - \int_{-24\bar{z}}^{i\infty} \frac{\sum_{n=-\infty}^{\infty} \left(n + \frac{1}{6}\right) e^{3\pi i \left(n + \frac{1}{6}\right)^2 \tau}}{\sqrt{-i(\tau + 24z)}} d\tau.$$

In the context of Theorem 1.1 (1), $N_f(z)$ plays the role of the period integral, and the mock theta function $f(q)$ plays the role of the holomorphic function $f_{H(m,k,N;z)}(z)$.

Zwegers' work, combined with recent work by the authors [5, 8, 9], establishes that the mock theta functions (resp. certain q -series arising from the Rogers-Fine basic hypergeometric series) are the holomorphic parts of weight $1/2$ (resp. $3/2$) harmonic weak Maass forms. In these cases, the non-holomorphic parts are indeed period integrals of weight $3/2$ (resp. $1/2$) theta functions. Theorem 1.1 illustrates this phenomenon for all other possible half-integral weights.

Despite this beautiful picture, many questions remain. For example, we ask:

Question. *Can any of the $f_{H(m,k,N;z)}(z)$ be represented as a combinatorial q -series such as those appearing in the theory of basic hypergeometric series?*

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