EULERIAN SERIES AS MODULAR FORMS

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Abstract. In 1988, Hickerson proved the celebrated "mock theta conjectures", a collection of ten identities from Ramanujan's "lost notebook" which express certain modular forms as linear combinations of mock theta functions. In the context of Maass forms, these identities arise from the peculiar phenomenon that two different harmonic Maass forms may have the same non-holomorphic parts. Using this perspective, we construct several infinite families of modular forms which are differences of mock theta functions.

1. Introduction and Statement of Results

Eulerian series are combinatorial formal power series which are constructed from basic hypergeometric series. There are famous examples of Eulerian series which essentially are modular forms. For example, with $q := e^{2\pi i z}$, the celebrated Rogers-Ramanujan identities

$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)},$$
$$\prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

provide series expansions of infinite products which correspond to weight 0 modular forms. As another example, the partition number generating function satisfies

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}.$$

Since this series is essentially the reciprocal of Dedekind's weight 1/2 modular form, this provides another example of an Eulerian series which is a modular form.

The literature on such identities is extensive, and the pursuit of further identities and their interpretations remains an active area of research largely due to applications in combinatorics, Lie theory, number theory and physics (for example, see [1], [14], and [16] to name a few). Among these identities, the "mock theta conjectures" of Ramanujan proved to be some of the most difficult to confirm.

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To state the conjectures, we first fix notation. For non-negative integers n, let $(x)_n := (x;q)_n := \prod_{j=0}^{n-1}(1-xq^j)$, and let $(x)_\infty := (x;q)_\infty := \prod_{j=0}^{\infty}(1-xq^j)$ (note that empty products equal 1). As usual, let $f_0(q), f_1(q), \Phi(q)$, and $\Psi(q)$ denote the mock theta functions (see [2] for background and historical references)

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n}, \qquad \Phi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}$$

$$f_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q)_n}, \qquad \Psi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}.$$

The mock theta conjectures of Ramanujan are a list of ten identities involving these functions. Andrews and Garvan [3] proved that these ten conjectures are equivalent to the truth of the following pair of identities that essentially express two weight 1/2 modular forms as linear combinations of Eulerian series (see [3] $(3.5)_R$ and $(3.10)_R$):

$$\frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = f_0(q) + 2\Phi(q^2), \tag{1.1}$$

$$\frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = f_1(q) + 2q^{-1}\Psi(q^2). \tag{1.2}$$

These identities defied proof until Hickerson's important paper of 1988 [12]. In a follow-up to [12], Hickerson proved several more identities of this type [13], and, more recently, Choi and Yesilyurt have obtained even further such identities (for example, see [8, 9, 21]) using methods similar to those of Hickerson.

Recently, much light has been shed on the nature of Ramanujan's mock theta functions. By work of Zwegers and the first two authors [6, 7, 22], it is now known that Ramanujan's mock theta functions are the "holomorphic parts" (see Section 2) of weight 1/2 harmonic weak Maass forms. From this perspective, the mock theta conjectures, as well as the related works of Choi, Hickerson, and Yesilyurt [8, 9, 13, 21], arise naturally in the theory of Maass forms; such identities result from the presence of linear relations between the non-holomorphic parts of Maass forms. Armed with this deeper point of view, one can now construct infinite families of modular forms by forcing such arithmetic relations, thereby reducing the search for explicit identities to standard calculations involving modular forms. Here we present such families using well known q-series which have been previously investigated by many authors.

First we consider identities which arise by the annihilation of non-holomorphic parts of Maass forms. To this end, let $R(w;q)$ be the Eulerian series defined by

$$R(w;q) := 1 + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}. \tag{1.3}$$

The integers $N(m,n)$ count the number of partitions of an integer n with rank m, where the *rank* of a partition is defined to be its largest part minus the number of its parts. Refining this notion, for integers $0 \leq r < t$, let $N(r,t;n)$ be the number of partitions of n whose rank is congruent to $r \pmod t$. Dyson conjectured [10], and Atkin and Swinnerton-Dyer proved [4], for every $n \geq 0$ and each r, that

$$N(r,5;5n+4) = \frac{p(5n+4)}{5},$$
$$N(r,7;7n+5) = \frac{p(7n+5)}{7},$$

equalities which combinatorially "explain" the Ramanujan congruences

$$p(5n+4) \equiv 0 \pmod 5 \quad \text{and} \quad p(7n+5) \equiv 0 \pmod 7.$$

One readily sees, using conjugates of partitions, that

$$N(r,t;n) = N(t-r,t;n). \tag{1.4}$$

These functions also satisfy [4] some non-trivial sporadic identities such as

$$N(1,7;7n+1) = N(2,7;7n+1) = N(3,7;7n+1). \tag{1.5}$$

Atkin and Swinnerton-Dyer [4] proved some surprising identities such as (see also (5.19) of [13])

$$-\frac{(q;q^7)_\infty^2(q^6;q^7)_\infty^2(q^7;q^7)_\infty^2}{(q;q)_\infty} = \sum_{n=0}^{\infty}\left(N(0,7;7n+6) - N(1,7;7n+6)\right)q^n. \tag{1.6}$$

As in (1.1), (1.2), and the work of Choi, Hickerson, and Yesilyurt [8, 9, 13], this identity expresses a weight 1/2 modular form as a linear combination of Eulerian series. This example is a special case of a much more general phenomenon.

Generalizing these examples, we have the following theorem.

Theorem 1.1. *Suppose that $t \geq 5$ is prime, $0 \leq r_1, r_2 < t$ and $0 \leq d < t$. Then the following are true:*

(1) *If $\left(\frac{1-24d}{t}\right) = -1$, then*

$$\sum_{n=0}^{\infty}\left(N(r_1,t;tn+d) - N(r_2,t;tn+d)\right)q^{24(tn+d)-1}$$

is a weight 1/2 weakly holomorphic modular form on $\Gamma_1(576t^6)$.

(2) *Suppose that $\left(\frac{1-24d}{t}\right) = 1$. If $r_1, r_2 \not\equiv \frac{1}{2}(\pm 1 \pm \alpha) \pmod t$, where α is any integer for which $0 \leq \alpha < 2t$ and $1 - 24d \equiv \alpha^2 \pmod{2t}$, then*

$$\sum_{n=0}^{\infty}\left(N(r_1,t;tn+d) - N(r_2,t;tn+d)\right)q^{24(tn+d)-1}$$

is a weight 1/2 weakly holomorphic modular form on $\Gamma_1(576t^6)$.

Two remarks.
1) Example (1.6) is the $t = 7$ and $d = 6$ case of Theorem 1.1 (2). In this case, the only choices of r_1 and r_2 satisfying the hypotheses are $0, 1$, and 6. Since $N(1, 7; n) = N(6, 7; n)$, (1.6) is the only nontrivial example of Theorem 1.1 (2) in this case. The proof of the theorem will show, for all other pairs of r_1 and r_2 (apart from the trivial examples such as those arising from (1.4)), that

$$\sum_{n=0}^{\infty} \left(N(r_1, t; tn + d) - N(r_2, t; tn + d)\right) q^{24(tn+d)-1}$$

is not a weakly holomorphic modular form. The corresponding Maass forms turn out to have non-trivial non-holomorphic parts.

2) Andrews and Garvan reformulated (1.1) and (1.2) in terms of partition ranks and the mock theta functions $\chi_0(q)$ and $\chi_1(q)$ (see (4.9) and (4.10) of [3]). Using the results in Zwegers' thesis (see Section 4.5 of [22]), or by arguing as in [7], it is not difficult to realize these two mock theta functions as the holomorphic parts of weight $1/2$ harmonic Maass forms. Using the proof of Theorem 1.1, one finds that these reformulations are implied by identities between harmonic Maass forms whose non-holomorphic parts cancel. This reduces the proof of the mock theta conjectures to the verification of two simple identities for classical weakly holomorphic modular forms. Elementary facts about modular forms easily implies these identities, thereby giving a new proof of the mock theta conjectures.

Theorem 1.1 gives modular forms as differences of the generating functions for the functions $N(r, t; tn + d)$. There are similar theorems where the rank modulus t is independent of the modulus of the arithmetic progression of the sizes of the partitions. To make this precise, for integers $0 < a < t$, let $\mathfrak{f}_t := \frac{2t}{\gcd(t,6)}$, $\mathfrak{l}_t := \mathrm{lcm}(2t^2, 24)$, and let $\widetilde{\mathfrak{l}}_t := \mathfrak{l}_t/24$. We then have the following theorem.

Theorem 1.2. *Suppose that $t > 1$ is an odd integer. If $0 \leq r_1, r_2 < t$ are integers, and $\mathcal{P} \nmid 6t$ is prime, then*

$$\sum_{\substack{n \geq 1 \\ \left(\frac{24\mathfrak{l}_t n - \mathfrak{l}_t}{\mathcal{P}}\right) = -\left(\frac{-24\widetilde{\mathfrak{l}}_t}{\mathcal{P}}\right)}} \left(N(r_1, t; n) - N(r_2, t; n)\right) q^{\mathfrak{l}_t n - \frac{\mathfrak{l}_t}{24}}$$

is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1\left(144\mathfrak{f}_t^2\widetilde{\mathfrak{l}}_t\mathcal{P}^4\right)$.

Now we present further infinite families of Eulerian series which are weakly holomorphic modular forms of weight $1/2$. Unlike Theorems 1.1 and 1.2, which arise from specializations of the single Eulerian series $R(w; q)$, these modular forms will be presented as linear combinations of different pairs of Eulerian series. More precisely, we present the next theorem to illustrate that two seemingly unrelated Eulerian series can be the holomorphic parts of distinct Maass forms with equal non-holomorphic parts.

Define the series $H'(a,c,w;z)$, $K'(w;z)$, $K''(w;z)$, with $0<a<c$, by

$$H'(a,c,w;z) := \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}(-q)_n}{(wq^{\frac{a}{c}})_{n+1}(wq^{1-\frac{a}{c}})_{n+1}}, \tag{1.7}$$

$$K'(w;z) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(wq^2;q^2)_n(w^{-1}q^2;q^2)_n}, \tag{1.8}$$

$$K''(w;z) := \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_{n-1}}{(wq;q^2)_n(w^{-1}q;q^2)_n}. \tag{1.9}$$

These series are already of independent interest. In particular, it turns out, thanks to work of Berkovich and Garvan [5] and Lovejoy [17], that $K'(w;z)$ is the generating function for the M_2-rank partition statistic (see Section 6 of [5]).

Let $\zeta_c := e^{2\pi i/c}$ and $f_c := 2c/\gcd(c,4)$. For $0<a<c$, we let

$$\widetilde{H}(a,c;z) := q^{\frac{a}{c}(1-\frac{a}{c})}\left(H'(a,c,1;z)+H'(a,c,-1;z)\right), \tag{1.10}$$

$$\widetilde{K}(a,c;z) := \frac{1}{4}\csc\left(\pi\frac{a}{c}\right) q^{-\frac{1}{8}} K'(\zeta_c^a;z) + \sin\left(\pi\frac{a}{c}\right) q^{-\frac{1}{8}} K''(\zeta_c^a;z). \tag{1.11}$$

Theorem 1.3. *Let $0<a<c$. In the notation above, $\widetilde{H}(a,c;4f_c^2z)$ is a weight $1/2$ weakly holomorphic modular form on $\Gamma_1(64f_c^4)$ and $\widetilde{K}(a,c;2f_c^2z)$ is a weight 1/2 weakly holomorphic modular form on $\Gamma_1(64f_c^4)$.*

Example. By letting $a=1$ and $c=2$ in Theorem 1.3, it turns out that one may obtain

$$\sum_{n\geq 0}\frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} + \sum_{n\geq 0}\frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}^2} = 2\frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^4}.$$

Three remarks.

1) Throughout, we essentially adopt Shimura's [19] notion of a half-integral weight modular form. The only departure from Shimura's notion is that a weakly holomorphic modular form is permitted to have poles at cusps.

2) We stress that these theorems, as stated, were chosen for their aesthetics. For example, there are versions of Theorems 1.1 and 1.2 which hold for all integers $t>1$. Moreover, even these theorems are special cases of general results for the Maass forms $D\left(\frac{a}{t};z\right)$ described in Section 3.

3) In principle one may explicitly compute the weakly holomorphic modular forms in all of the theorems above in terms of expressions in modular units and Dedekind's eta-function. In general such expressions will be quite complicated.

In Section 2 we recall facts about harmonic weak Maass forms. In Section 3, we recall work of the first two authors concerning Maass forms related to Dyson's ranks, and we prove Theorems 1.1 and 1.2. In Section 4, we prove Theorem 1.3 after explicitly constructing more infinite families of harmonic weak Maass forms.

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2. Preliminaries on harmonic Maass forms

Here we recall the notion of a harmonic weak Maass form of weight $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. If $z = x + iy$ with $x, y \in \mathbb{R}$, then the weight k hyperbolic Laplacian is given by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right). \tag{2.1}$$

If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{2.2}$$

A *harmonic weak Maass form of weight k on a subgroup* $\Gamma \subset \Gamma_0(4)$ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \mathbb{H}$, we have
$$f(Az) = \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz + d)^k \ f(z).$$
Here $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol.
(2) We have that $\Delta_k f = 0$.
(3) The function $f(z)$ has at most linear exponential growth at all the cusps of Γ.

Remark. The transformation law above agrees with Shimura's notion of a half-integral weight modular form [19]. This theory technically involves automorphic forms on the metaplectic group. Throughout this paper we work within this framework. As Shimura shows, one can faithfully work on subgroups of $\Gamma_0(4)$ due to natural identifications with subgroups of the metaplectic group. In particular, these identifications allow one to deduce modularity on congruence subgroups from suitable transformation laws with respect to the generators of these groups. This principle is fundamental to the proofs of the results in Section 4.

All harmonic weak Maass forms have Fourier expansions of the form

$$f(z) = \sum_{n=n_0}^{\infty} \gamma(f, n; y) q^{-n} + \sum_{n=n_1}^{\infty} a(f, n) q^n, \tag{2.3}$$

with $n_0, n_1 \in \mathbb{Z}$. The $\gamma(f, n; y)$ are functions in y arising from the incomplete gamma function, while the $a(f, n)$ are complex numbers. To see this, one can check that the second condition in the definition of a harmonic Maass form implies that the Fourier expansion of all such $f(z)$ may be given in terms of classical Whittaker functions.

Specializing these functions leads to both the incomplete gamma function as well as the customary exponential functions. We refer to $\sum_{n=n_0}^{\infty}\gamma(f,n;y)q^{-n}$ as the "non-holomorphic part" of $f(z)$, and we refer to $\sum_{n=n_1}^{\infty}a(f,n)q^n$ as its "holomorphic part."

3. Dyson's ranks and Theorems 1.1 and 1.2

Here we prove Theorem 1.1 and 1.2 using earlier work of the first two authors.

3.1. **Dyson's ranks and Maass forms.** Here we briefly recall some results from [7]. As in the introduction, let

$$R(w;q) := 1+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}N(m,n)w^mq^n = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}.$$

Suppose that $0<a<t$ are integers. If $\mathfrak{f}_t=\frac{2t}{\gcd(t,6)}$, then define the weight $3/2$ theta function $\Theta\left(\frac{a}{t};\tau\right)$ by

$$\Theta\left(\frac{a}{t};\tau\right) := \sum_{m \pmod{\mathfrak{f}_t}}(-1)^m\sin\left(\frac{a\pi(6m+1)}{t}\right)\cdot\theta\left(6m+1,6\mathfrak{f}_t;\frac{\tau}{24}\right), \tag{3.1}$$

where $\theta(\alpha,\beta;\tau) := \sum_{n\equiv\alpha \pmod{\beta}} ne^{2\pi i n^2\tau}$.

As in the introduction, let $\mathfrak{l}_t := \operatorname{lcm}(2t^2,24)$, and let $\widetilde{\mathfrak{l}_t} := \mathfrak{l}_t/24$, and define the function $S\left(\frac{a}{t};z\right)$ by the period integral

$$S\left(\frac{a}{t};z\right) := -\frac{i\sin\left(\frac{\pi a}{t}\right)\mathfrak{l}_t^{\frac{1}{2}}}{\sqrt{3}}\int_{-\bar{z}}^{i\infty}\frac{\Theta\left(\frac{a}{t};\mathfrak{l}_t\tau\right)}{\sqrt{-i(\tau+z)}}\,d\tau. \tag{3.2}$$

Using this notation, define $D\left(\frac{a}{t};z\right)$ by

$$D\left(\frac{a}{t};z\right) := -S\left(\frac{a}{t};z\right)+q^{-\frac{\mathfrak{l}_t}{24}}R(\zeta_t^a;q^{\mathfrak{l}_t}). \tag{3.3}$$

The first two authors proved (Theorem 1.1 of [7]) that the $D\left(\frac{a}{t};z\right)$ are weight $1/2$ harmonic weak Maass forms. This fact, combined with the combinatorial description of $R(w;q)$, gives a description of the generating functions for the $N(r,t;n)$ in terms of Maass forms. To state it, let $\Gamma(\alpha;x)$ denote the usual incomplete gamma function

$$\Gamma(\alpha;x) := \int_x^{\infty}e^{-t}t^{\alpha-1}\,dt.$$

Theorem 1.3 and Proposition 4.1 of [7], combined with the formulas in Section 3.3 of [7], imply the following theorem.

Theorem 3.1. *If $0\le r<t$ are integers, where t is odd, then*

$$\sum_{n=0}^{\infty}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{\mathfrak{l}_t n-\widetilde{\mathfrak{l}_t}}$$

is the holomorphic part of a weight 1/2 harmonic weak Maass form on $\Gamma_1\left(144\mathfrak{f}_t^2\widetilde{\mathfrak{l}_t}\right)$. *Moreover, the non-holomorphic part of this weak Maass form is*

$$\frac{i\mathfrak{l}_t^{\frac{1}{2}}}{t\sqrt{3}}\sum_{m \pmod{\mathfrak{f}_t}}\ \sum_{n\equiv 6m+1 \pmod{6\mathfrak{f}_t}} A(r,t,m)\cdot\gamma(t,y;n)q^{-\widetilde{\mathfrak{l}_t}n^2}.$$

Here we have that

$$A(r,t,m) := (-1)^m\sum_{j=1}^{t-1}\zeta_t^{-rj}\sin\left(\frac{\pi j}{t}\right)\sin\left(\frac{\pi j(6m+1)}{t}\right), \tag{3.4}$$

and we have that

$$\gamma(t,y;n) := \frac{i}{\sqrt{2\pi\widetilde{\mathfrak{l}_t}}}\cdot\Gamma\left(\frac{1}{2},4\pi\widetilde{\mathfrak{l}_t}n^2y\right).$$

Using quadratic twists, one can directly obtain weakly holomorphic modular forms. Theorem 1.4 of [7] asserts the following result.

Theorem 3.2. *If* $0\le r<t$ *are integers, where* t *is odd, and* $\mathcal{P}\nmid 6t$ *is prime, then*

$$\sum_{\substack{n\ge 1\\ \left(\frac{24\mathfrak{l}_tn-\mathfrak{l}_t}{\mathcal{P}}\right)=-\left(\frac{-24\widetilde{\mathfrak{l}_t}}{\mathcal{P}}\right)}}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{\mathfrak{l}_tn-\frac{\mathfrak{l}_t}{24}}$$

is a weight $1/2$ *weakly holomorphic modular form on* $\Gamma_1\left(144\mathfrak{f}_t^2\widetilde{\mathfrak{l}_t}\mathcal{P}^4\right)$.

3.2. **Proofs of Theorems 1.1 and 1.2.** Here we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.2. This result follows from Theorem 3.2 since the $p(n)/t$ summands cancel when taking the difference of the relevant generating functions. □

Proof of Theorem 1.1. By Theorem 3.1, for any $0\le r<t$ we have

$$\sum_{n=0}^{\infty}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{24t^2n-t^2}+\sum_{n\in\mathbb{Z}}\widetilde{A}(r,t,n)\cdot\gamma(t,y;n)q^{-t^2n^2}$$

is a weight 1/2 harmonic weak Maass form on $\Gamma_1(576t^4)$. Here $\widetilde{A}(r,t,n)$ is a complex number given by

$$\widetilde{A}(r,t,n) = i\sqrt{8}\sum_{\substack{m \pmod{2t}\\ 6m+1\equiv n \pmod{12t}}} A(r,t,m), \tag{3.5}$$

where $\gamma(t,y;n)$ and $A(r,t;m)$ are defined in Theorem 3.1. Applying the Atkin $U(t^2)$ operator, we have, by a straightforward generalization of Proposition 1.5 of [19], that

$$\mathfrak{R}(r,t;z) := \sum_{n=0}^{\infty}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{24n-1}+\sum_{n\in\mathbb{Z}}\widetilde{A}(r,t,n)\cdot\gamma(t,y;n)q^{-n^2} \tag{3.6}$$

is a weight 1/2 harmonic weak Maass form on $\Gamma_1(576t^4)$.

Now we prove Theorem 1.1 (1). By a straightforward generalization of the classical argument on twists of modular forms (for example, see Proposition 22 of [15]), the quadratic twist of $\mathfrak{R}(r,t;z)$ by $\left(\frac{\cdot}{t}\right)$, say $\mathfrak{R}(r,t;z)_t$, is a weight 1/2 harmonic weak Maass form on $\Gamma_1(576t^6)$. In particular, $\left(\frac{-1}{t}\right)\mathfrak{R}(r,t;z)_t$ has an expansion of the form

$$\left(\frac{-1}{t}\right)\mathfrak{R}(r,t;z)_t = \sum_{n=0}^{\infty}\left(\frac{1-24n}{t}\right)\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{24n-1}+\sum_{\substack{n\in\mathbb{Z}\\ t\nmid n}}\widetilde{A}(r,t,n)\cdot\gamma(t,y;n)q^{-n^2}$$

We find that $\mathfrak{R}(r,t;z)-\left(\frac{-1}{t}\right)\mathfrak{R}(r,t;z)_t$ is on $\Gamma_1(576t^6)$, and its non-holomorphic part is supported on terms of the form $q^{-t^2n^2}$. By taking the quadratic twist of this form again by $\left(\frac{\cdot}{t}\right)$, to annihilate these non-holomorphic terms, one then finds that

$$\sum_{\substack{n\geq 0,\\ \left(\frac{1-24n}{t}\right)=-1}}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{24n-1}$$

is weight 1/2 weakly holormorphic modular form. Using the orthogonality of Dirichlet characters modulo t, and facts about twists again, it follows that

$$\sum_{n=0}^{\infty}\left(N(r,t;tn+d)-\frac{p(tn+d)}{t}\right)q^{24(tn+d)-1}$$

is a weight 1/2 weakly holomorphic modular form. Theorem 1.1 (1) follows by taking the difference of these forms when $r=r_1$ and r_2. Since taking twists of twists can be viewed as a single twist by the trivial character, we find that the resulting form is on $\Gamma_1(576t^6)$.

Now we turn to the proof of Theorem 1.1 (2). Here we argue directly with (3.4) and (3.6). Using the theory of twists of weak Maass forms again, we see that the restriction of $\mathfrak{R}(r_1,t;z)-\mathfrak{R}(r_2,t;z)$ to forms whose holomorphic parts are supported on exponents of the form $24(tn+d)-1$, is a weight 1/2 harmonic weak Maass form on $\Gamma_1(576t^6)$.

It suffices to show that the non-holomorphic part of this form is zero under the given hypotheses on r_1 and r_2. By (3.6), one sees that the non-holomorphic part is supported on terms of the form q^{-n^2}. By construction, these n satisfy $n\equiv\alpha\pmod{2t}$, for some $0\leq\alpha<2t$ with $1-24d\equiv\alpha^2\pmod{2t}$. Therefore, by (3.4) and (3.6), it suffices to show that

$$A(r_1,t,m)-A(r_2,t,m)=0,$$

when $6m+1 \equiv \alpha \pmod{12t}$. By (3.4), we have

$$A(r_1,t,m) - A(r_2,t,m) = (-1)^m \sum_{j=1}^{t-1} \left(\zeta_t^{-r_1 j} - \zeta_t^{-r_2 j}\right) \sin\left(\frac{\pi j}{t}\right) \sin\left(\frac{\pi j \alpha}{t}\right).$$

Using $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$, we have to show that

$$\sum_{j=1}^{t-1} \Big(e^{\frac{\pi i j}{t}(1-2r_1+\alpha)} - e^{\frac{\pi i j}{t}(1-2r_1-\alpha)} - e^{\frac{\pi i j}{t}(-1-2r_1+\alpha)} + e^{\frac{\pi i j}{t}(-1-2r_1-\alpha)}$$
$$- e^{\frac{\pi i j}{t}(1-2r_2+\alpha)} + e^{\frac{\pi i j}{t}(1-2r_2-\alpha)} + e^{\frac{\pi i j}{t}(-1-2r_2+\alpha)} - e^{\frac{\pi i j}{t}(-1-2r_2-\alpha)} \Big) = 0.$$

This follows since $\pm 1 - 2r_i \pm \alpha$, for $i = 1$ and 2, are even and coprime to t. $\square$

4. Further Explicit Examples of Maass Forms

Here we construct two new classes of weight $1/2$ harmonic Maass forms (see Theorems 4.1 and 4.2). Then in Section 4.3 we prove Theorem 1.3.

Since this section is quite technical, we begin by giving general comments on the main ideas behind the construction. For positive integers c, we construct sets of q-series $\mathcal{H}(a,b,c;z)$ and $\mathcal{K}(a,b,c;z)$, where a and b range over suitable residue classes modulo c, which are related to each other under the modular transformations which generate $\Gamma_0(4)$ (see Theorem 4.4). Under the map $z \to z+1$, the transformation laws follow from an inspection of Fourier expansions. These transformation laws are complicated by the fact that the individual functions are essentially permuted. Under the map $z \to -1/4z$, the situation is more complicated. It turns out that $\mathcal{K}(a,b,c;-1/4z)$ equals $\sqrt{-4iz} \cdot \mathcal{H}(a,b,c;z)$ up to an explicit non-holomorphic function which is a "period integral" of a certain weight $3/2$ cuspidal theta function. Theorem 4.4 (2) is obtained by a lengthy series of calculations involving the Residue Theorem and the partial fraction decompositions of special power series which are related to such period integrals. Therefore, we obtain a complete description of the modular transformation properties for these sets of functions under $\Gamma_0(4)$. Obtaining this conclusion requires the modular transformation properties of the period integrals which arise above (see Lemmas 4.5 and 4.6). These results imply Theorems 4.1 and 4.2 which concern the functions $D_1(a,b,c;z)$ and $D_2(a,b,c;z)$ which are certain linear combinations of the q-series and period integrals described above. To obtain these two theorems, one then needs to show that they transform suitably under the indicated congruence subgroups. Thankfully, one observes that the transformation laws of the q-series are precisely compatible with the transformation laws of the period integrals of the theta functions under the generators of $\Gamma_0(4)$. The conclusion then follows by either working directly with generators and relations, or by using the known transformation laws for present theta functions which imply the transformation laws for the relevant period integrals under the desired congruence subgroups.

4.1. **New Classes of Harmonic Maass Forms.** Two classes of harmonic Maass forms will be derived from series related to the functions $\widetilde{H}(a,c;z)$ and $\widetilde{K}(a,c;z)$ defined in (1.10) and (1.11). It is convenient to work with the series $H(a,b,c;z)$ and $K(a,b,c;z)$, with $0 \leq a,b < c$ and a and b not both zero defined by

$$H\left(a,b,c;z\right) := \frac{(-q)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+\frac{a}{c}} q^{n(n+1)}}{1-\zeta_c^b q^{n+\frac{a}{c}}}, \tag{4.1}$$

and

$$K(a,b,c;z) := \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \left(\frac{ie^{-\pi i \frac{a}{c}} q^{\frac{b}{c}}}{2\left(1-e^{-2\pi i \frac{a}{c}} q^{2\frac{b}{c}}\right)} + \sum_{m=1}^{\infty} S(a,b,c,m;z) q^{m(2m+1)} \right), \tag{4.2}$$

with

$$S(a,b,c,m;z) := \frac{\sin\left(\frac{\pi a}{c} - \left(\frac{b}{c} + 2k(b,c)m\right)2\pi z\right) + \sin\left(\frac{\pi a}{c} - \left(\frac{b}{c} - 2k(b,c)m\right)2\pi z\right) q^{2m}}{1 - 2\cos\left(\frac{2\pi a}{c} - \frac{4\pi b z}{c}\right) q^{2m} + q^{4m}},$$

and

$$k(b,c) := \begin{cases} 0 & \text{if } 0 \leq \frac{b}{c} < \frac{1}{4}, \\ 1 & \text{if } \frac{1}{4} < \frac{b}{c} < \frac{3}{4}, \\ 2 & \text{if } \frac{3}{4} < \frac{b}{c} < 1. \end{cases} \tag{4.3}$$

Although a must be non-zero for the $\widetilde{K}(a,c;z)$ from the introduction, the reader should note that $a=0$ is permitted for $K(a,b,c;z)$, provided that $b \neq 0$

From the definitions of $\widetilde{H}$ and $\widetilde{K}$, see (1.10) and (1.11), and some calculations which follow, we have that

$$\widetilde{K}\left(a,c;z\right) = q^{-1/8} K(a,0,c;z) - q^{3/8} K(a,c/2,c;z), \tag{4.4}$$

and

$$\widetilde{H}\left(a,c;z\right) = q^{\frac{a}{c}\left(1-\frac{a}{c}\right)} \left(H(a,0,c;z) - H(a,c/2,c;z)\right). \tag{4.5}$$

To see this we note that for $b=0$ or $b=c/2$ we have

$$H(a,b,c;z) = \frac{(-q)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+\frac{a}{c}} q^{n(n+1)}}{1-\zeta_c^b q^{n+\frac{a}{c}}} = \zeta_c^b \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}(-q)_n}{(\zeta_c^b q^{\frac{a}{c}})_{n+1} (\zeta_c^b q^{1-\frac{a}{c}})_{n+1}}.$$

The first equality is by definition, the second equality follows from the Watson-Whipple transformation (see [18]). More precisely, one may use III.17 on page 242 of [11] with $a=q$, $b=\pm q^{a/c}$, c $= \pm q^{1-a/c}$, $d=-q$, and $e,f \to \infty$.

Additionally, we have

$$K'(\zeta_c^a;z)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(\zeta_c^a q^2;q^2)_n(\zeta_c^{-a}q^2;q^2)_n}$$
$$=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1+q^{2n})\left(2-2\cos\left(2\pi\frac{a}{c}\right)\right)}{1-2q^{2n}\cos(2\pi\frac{a}{c})+q^{4n}}q^{n(2n+1)}\right)=4\sin\left(\pi\frac{a}{c}\right)K(a,0,c;z),$$

where again the equality can be obtained from the Watson-Whipple transformation (again see [18]). Again this can be obtained by first simplifying III.17 on page 242 of [11] by letting $e,f\to\infty$ and then setting $a=1$, $d=q^{1/2}$, $b=\zeta_c^a$ and $c=\zeta_c^{-a}$ and finally $q\mapsto q^2$. We can use a similar type of transformation to rewrite the second sum appearing in the definition of $\widetilde{K}$. Namely, we obtain

$$-q^{1/2}K(a,c/2,c;z)=\sin\left(\pi\frac{a}{c}\right)K''(\zeta_c^a;z).$$

To describe the families of weak Maass forms, we begin by defining the functions

$$V_1\left(a,b,c;z\right):=\zeta_c^{-b}\int_{-\overline{z}}^{i\infty}\frac{T(a,b,c;2\tau)}{\sqrt{-i(\tau+z)}}d\tau, \tag{4.6}$$

$$V_2\left(a,b,c;z\right):=-\frac{1}{2}\zeta_c^{-b}\int_{-\overline{z}}^{i\infty}\frac{(-i\tau)^{-\frac{3}{2}}T\left(a,b,c;-\frac{1}{2\tau}\right)}{\sqrt{-i(\tau+z)}}d\tau, \tag{4.7}$$

where T is defined by

$$T(a,b,c;\tau):=ie^{2\pi i\frac{b}{c}\left(2\frac{a}{c}-1\right)}\sum_{n=-\infty}^{\infty}A(a,b,c,n;\tau)e^{2\pi i\tau\left(\left(n+\frac{1}{4}\right)^2+\left(\frac{b}{c}\right)^2\right)}, \tag{4.8}$$

and

$$A(a,b,c,n;\tau):=\left(n+\frac{1}{4}\right)\cosh\left(2\pi i\left(n+\frac{1}{4}\right)\left(\left(2\frac{a}{c}-1\right)+2\frac{b}{c}\tau\right)\right)$$
$$+\frac{b}{c}\sinh\left(2\pi i\left(n+\frac{1}{4}\right)\left(\left(2\frac{a}{c}-1\right)+2\frac{b}{c}\tau\right)\right).$$

Now define

$$D_1\left(a,b,c;z\right):=q^{4f_c^2\frac{a}{c}\left(1-\frac{a}{c}\right)}H(a,b,c;4f_c^2z)+V_2(a,b,c;2f_c^2z) \tag{4.9}$$

$$D_2(a,b,c;z):=$$

$$e^{-2k(b,c)\frac{a}{c}\pi i+4\pi i\left(\frac{a}{c}-1\right)\frac{b}{c}}q^{4f_c^2\left(k(b,c)\frac{b}{c}-\left(\frac{b}{c}\right)^2-\frac{1}{16}\right)}K(a,b,c;2f_c^2z)+V_1(a,b,c;2f_c^2z). \tag{4.10}$$

We now have the following theorems.

Theorem 4.1. *If $0<a<c$ and $b=0$ or $b=c/2$, then $D_1\left(a,b,c;z\right)$ is a weight 1/2 harmonic weak Maass form on $\Gamma_1\left(64f_c^4\right)$.*

We have an analogous theorem for the family of functions $D_2(a,b,c;z)$.

Theorem 4.2. *If $0 < a < c$ and $b = 0$ or $b = c/2$, then $D_2(a, b, c; z)$ is a weight $1/2$ harmonic weak Maass form on $\Gamma_1(64f_c^4)$.*

Remark. Asymptotics and congruences for the coefficients of the holomorphic parts of these harmonic weak Maass forms may be obtained by arguing as in [6] and [7]. In special cases, one may presumably obtain exact formulas analogous to those in [6].

4.2. **Sketch of the Proofs of Theorems 4.1 and 4.2.** In this section we follow work of Watson [20] and previous work of the first two authors [7] to establish transformation properties of the $H(a, b, c; z)$ and $K(a, b, c; z)$ functions.

To describe the transformation laws of these functions we define

$$L(a,b,c;\alpha) := \int_{-\infty}^{\infty} \frac{e^{-\alpha x - \alpha x^2 + 2\alpha x \frac{a}{c}}}{\cosh\left(\alpha x - 2\pi i \frac{b}{c}\right)}\, dx. \tag{4.11}$$

We have the following theorem which relates the three functions.

Theorem 4.3. *Suppose c is a positive integer and a and b are integers for which $0 \leq a, b < c$ with a and b not both zero and $\frac{b}{c} \notin \{\frac{1}{4}, \frac{3}{4}\}$. Furthermore, if $\mathrm{Re}(\alpha) > 0$, and $q := e^{-\alpha}$, and $q_1 := e^{-\pi^2/\alpha}$, we have that*

$$\begin{aligned} q^{\frac{a}{c}\left(1-\frac{a}{c}\right)} H\left(a,b,c;-\frac{\alpha}{2\pi i}\right) =& \sqrt{\frac{4\pi}{\alpha}} e^{-2k(b,c)\frac{a}{c}\pi i + 4\pi i\left(\frac{a}{c}-1\right)\frac{b}{c}} q_1^{4k(b,c)\frac{b}{c} - 4\left(\frac{b}{c}\right)^2 - \frac{1}{4}} K\left(a,b,c;\frac{\pi i}{\alpha}\right) \\ &- \sqrt{\frac{\alpha}{4\pi}} \zeta_c^{-b} L(a,b,c;\alpha). \end{aligned}$$

Proof. Since $b = 0$ is treated by McIntosh in [18], we shall assume that $0 < b < c$. We proceed as in [7] and [20]. By analytic continuation, it suffices to prove the identity for real α. Define the integral I by

$$\begin{aligned} I := I_1 + I_2 =& \frac{1}{2\pi i} \int_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} \frac{\pi}{\sin(\pi\tau)} \cdot \frac{e^{-\alpha(\tau + \frac{a}{c})}}{1 - \zeta_c^b e^{-\alpha(\tau + \frac{a}{c})}} \cdot e^{-\alpha\tau(\tau+1)}\, d\tau \\ &- \frac{1}{2\pi i} \int_{-\infty + i\varepsilon}^{+\infty + i\varepsilon} \frac{\pi}{\sin(\pi\tau)} \cdot \frac{e^{-\alpha(\tau + \frac{a}{c})}}{1 - \zeta_c^b e^{-\alpha(\tau + \frac{a}{c})}} \cdot e^{-\alpha\tau(\tau+1)}\, d\tau. \end{aligned} \tag{4.12}$$

The zeros of $1 - \zeta_c^b e^{-\alpha(\tau + \frac{a}{c})}$ are precisely at the values

$$\tau_n := -\frac{a}{c} + \frac{2\pi i(\frac{b}{c} + n)}{\alpha}.$$

We wish to choose $\varepsilon > 0$ small enough so that there are no zeros of $1 - \zeta_c^b e^{-\alpha(\tau + \frac{a}{c})}$ in the region of integration, so we pick ε so that

$$\frac{2\pi\left(\frac{b}{c} + n\right)}{\alpha} > \varepsilon$$

for all n. This is possible because $b \neq 0$. So the poles of the integrand are due only to $\sin(\pi\tau)$. But $\sin(\pi\tau)$ has zeros at all $n \in \mathbb{Z}$, and furthermore, the residue at n is

$$\frac{(-1)^n q^{n+\frac{a}{c}}}{1-\zeta_c^b q^{n+\frac{a}{c}}} \cdot q^{n(n+1)}.$$

For $\mathrm{Re}(\tau) \to \infty$, the integrand rapidly decays, so the Residue Theorem implies that

$$I = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+\frac{a}{c}}}{1-\zeta_c^b q^{n+\frac{a}{c}}} \cdot q^{n(n+1)} = \frac{(q)_\infty}{(-q)_\infty} H\left(a,b,c;-\frac{\alpha}{2\pi i}\right).$$

We now compute I_1 and I_2 separately by moving each contour and using the Residue Theorem. We first compute I_2. For τ in the upper-half plane

$$\frac{1}{\sin(\pi\tau)} = -2i \sum_{n=0}^{\infty} e^{(2n+1)\pi i\tau}.$$

Using this in I_2 we obtain

$$I_2 = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{-\infty+i\varepsilon}^{+\infty-i\varepsilon} 2\pi i \frac{e^{(2n+1)\pi i\tau-\alpha\left(\tau+\frac{a}{c}\right)-\alpha\tau(\tau+1)}}{1-\zeta_c^b e^{-\alpha\left(\tau+\frac{a}{c}\right)}}\, d\tau.$$

Define J_n by

$$J_n := \int_{-\infty+i\varepsilon}^{+\infty-i\varepsilon} \frac{e^{(2n+1)\pi i\tau-\alpha\left(\tau+\frac{a}{c}\right)-\alpha\tau(\tau+1)}}{1-\zeta_c^b e^{-\alpha\left(\tau+\frac{a}{c}\right)}}\, d\tau.$$

We now shift the path of integration through the points $w_n := \frac{(2n+1)\pi i}{2\alpha}$ which are the saddle points of $e^{(2n+1)\pi i\tau-\alpha\tau^2}$. This introduces more poles in the region of integration. By the Residue Theorem, we have

$$J_n = J_n' + \text{ sum of residues of poles of integrand between the two contours,}$$

where J_n' is the integral along the contour through w_n defined by

$$J_n' := \int_{-\infty+w_n}^{+\infty+w_n} \frac{e^{(2n+1)\pi i\tau-\alpha\left(\tau+\frac{a}{c}\right)-\alpha\tau(\tau+1)}}{1-\zeta_c^b e^{-\alpha\left(\tau+\frac{a}{c}\right)}}\, d\tau.$$

Hence we will take into account those poles τ_m for which $m \geq 0$ and $\mathrm{Im}(\tau_m) < \mathrm{Im}(w_n)$, or equivalently $m \geq 0$ and $2\left(\frac{b}{c}+m\right) - \frac{1}{2} < n$. Notice that we must have $\mathrm{Im}(\tau_m) \neq \mathrm{Im}(w_n)$. To guarantee this we take $\frac{b}{c} \notin \{\frac{1}{4}, \frac{3}{4}\}$. By the definition of $k(b,c)$ the condition is equivalent to

$$n \geq 2m + k(b,c).$$

At τ_m, the integrand has the residue

$$\begin{aligned}\lambda_{n,m} :=& \frac{2\pi i}{\alpha} e^{(2n+1)\pi i\tau_m-\alpha\left(\tau_m+\frac{a}{c}\right)-\alpha\tau_m(\tau_m+1)} \\ =& \frac{2\pi i}{\alpha} e^{-(2n+1)\pi i\frac{a}{c}+4\pi i\left(\frac{b}{c}+m\right)\left(\frac{a}{c}-1\right)} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} q_1^{2(2n+1)\left(\frac{b}{c}+m\right)-4\left(\frac{b}{c}+m\right)^2}.\end{aligned}$$

Hence, the Residue Theorem combined with reordering summation, gives

$$I_2 = \sum_{m\geq 0} \sum_{n\geq 2m+k(b,c)} \lambda_{n,m} + \sum_{n\geq 0} J'_n. \tag{4.13}$$

Using $\lambda_{n+1,m} = e^{-2\pi i\frac{a}{c}} q_1^{4\left(\frac{b}{c}+m\right)} \lambda_{n,m}$ we find that

$$\begin{aligned}\sum_{m=0}^{\infty} \sum_{n\geq 2m+k(b,c)} \lambda_{n,m} &= \sum_{m=0}^{\infty} \frac{\lambda_{2m+k(b,c),m}}{1-e^{-2\pi i\frac{a}{c}} q_1^{4\left(\frac{b}{c}+m\right)}}\\ &= \frac{2\pi i}{\alpha} e^{-(2k(b,c)+1)\frac{a}{c}\pi i+4\pi i\left(\frac{a}{c}-1\right)\frac{b}{c}} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} q_1^{2(2k(b,c)+1)\frac{b}{c}-4\left(\frac{b}{c}\right)^2} \sum_{m=0}^{\infty} \frac{q_1^{4m^2+2(2k(b,c)+1)m}}{1-e^{-2\pi i\frac{a}{c}} q_1^{4(m+\frac{b}{c})}}.\end{aligned}$$

Arguing in the same way, using the fact that for all τ in the lower half-plane we have

$$\frac{1}{\sin(\pi\tau)} = 2i \sum_{n=0}^{\infty} e^{-(2n+1)\pi i\tau},$$

we find that

$$I_1 = -\sum_{m=1}^{\infty} \sum_{n\geq 2m-k(b,c)} \mu_{n,m} + \sum_{n=0}^{\infty} K'_n,$$

where

$$\begin{aligned}\mu_{n,m} :=& \frac{2\pi i}{\alpha} e^{-(2n+1)\pi i\tau_{-m}-\alpha\left(\tau_{-m}+\frac{a}{c}\right)-\alpha\tau_{-m}(\tau_{-m}+1)}\\ =& \frac{2\pi i}{\alpha} e^{(2n+1)\pi i\frac{a}{c}+4\pi i\left(\frac{b}{c}-m\right)\left(\frac{a}{c}-1\right)} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} q_1^{2(2n+1)\left(-\frac{b}{c}+m\right)-4\left(\frac{b}{c}-m\right)^2},\end{aligned}$$

and where

$$K'_n := \int_{-\infty+\widetilde{w}_n}^{+\infty+\widetilde{w}_n} \frac{e^{-(2n+1)\pi i\tau-\alpha(\tau+\frac{a}{c})-\alpha\tau(\tau+1)}}{1-\zeta_c^b e^{-\alpha(\tau+\frac{a}{c})}}\, d\tau.$$

Here we have that $\widetilde{w}_n := -\frac{(2n+1)\pi i}{2\alpha}$. As before, we have

$$\mu_{n+1,m} = e^{2\pi i\frac{a}{c}} q_1^{4\left(-\frac{b}{c}+m\right)} \mu_{n,m}.$$

Hence we obtain

$$\begin{aligned}\sum_{m=1}^{\infty} \sum_{n\geq 2m-k(b,c)} \mu_{n,m} &= \sum_{m\geq 1} \frac{\mu_{2m-k(b,c),m}}{1-e^{2\pi i\frac{a}{c}} q_1^{4\left(-\frac{b}{c}+m\right)}}\\ &= \frac{2\pi i}{\alpha} e^{(-2k(b,c)+1)\frac{a}{c}\pi i+4\pi i\left(\frac{a}{c}-1\right)\frac{b}{c}} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} q_1^{2(2k(b,c)-1)\frac{b}{c}-4\left(\frac{b}{c}\right)^2} \sum_{m=1}^{\infty} \frac{q_1^{4m^2+2(-2k(b,c)+1)m}}{1-e^{2\pi i\frac{a}{c}} q_1^{4\left(m-\frac{b}{c}\right)}}.\end{aligned}$$

Therefore, we have

$$I_1+I_2-\sum_{n\geq 0}\left(J_n'+K_n'\right)=\frac{4\pi}{\alpha}e^{-2k(b,c)\frac{a}{c}\pi i+4\pi i\left(\frac{a}{c}-1\right)\frac{b}{c}}q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)}q_1^{4k(b,c)\frac{b}{c}-4\left(\frac{b}{c}\right)^2}$$
$$\times\left(\frac{ie^{-\pi i\frac{a}{c}}q_1^{2\frac{b}{c}}}{2\left(1-e^{-2\pi i\frac{a}{c}}q_1^{4\frac{b}{c}}\right)}+\sum_{m=1}^{\infty}\overline{K}(a,b,c,m;\beta)q_1^{4m^2+2m}\right),$$

where

$$\overline{K}(a,b,c,m;\beta):=\frac{i}{2}\cdot\left(\frac{q_1^{2\frac{b}{c}+4mk(b,c)}e^{-\pi i\frac{a}{c}}}{1-e^{-2\pi i\frac{a}{c}}q_1^{4\left(m+\frac{b}{c}\right)}}-\frac{q_1^{-2\frac{b}{c}-4mk(b,c)}e^{\pi i\frac{a}{c}}}{1-e^{2\pi i\frac{a}{c}}q_1^{4\left(m-\frac{b}{c}\right)}}\right)$$
$$=\frac{\sin\left(\frac{\pi a}{c}-i\beta\left(2\frac{b}{c}+4k(b,c)m\right)\right)+\sin\left(\frac{\pi a}{c}-i\beta\left(2\frac{b}{c}-4k(b,c)m\right)\right)q_1^{4m}}{1-2\cos\left(2\pi\frac{a}{c}-4i\beta\frac{b}{c}\right)q_1^{4m}+q_1^{8m}},$$

and $\beta:=\pi^2/\alpha$. Now, from the definition of $S(a,b,c,m;z)$, we see that

$$I_1+I_2$$
$$=\frac{4\pi}{\alpha}e^{-2k(b,c)\frac{a}{c}\pi i+4\pi i\left(\frac{a}{c}-1\right)\frac{b}{c}}q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)}q_1^{4k(b,c)\frac{b}{c}-4\left(\frac{b}{c}\right)^2}\frac{(q_1^4;q_1^4)_\infty}{(q_1^2;q_1^4)_\infty}K\left(a,b,c;\frac{\pi i}{\alpha}\right)+\sum_{n\geq 0}(J_n'+K_n').$$

It remains to calculate $\sum_{n\geq 0}(J_n'+K_n')$. We will first compute J_n'. For this we use the identity $\frac{t}{1-t}=\frac{1+t}{t^{-1}-t}$, with $t=\zeta_c^be^{-\alpha\left(\tau+\frac{a}{c}\right)}$, which gives that the integrand of J_n' equals

$$\zeta_c^{-b}\cdot e^{(2n+1)\pi i\tau-\alpha\tau^2}\left(e^{-\alpha\tau}\frac{1+\zeta_c^be^{-\alpha\left(\tau+\frac{a}{c}\right)}}{\zeta_c^{-b}e^{\alpha\left(\tau+\frac{a}{c}\right)}-\zeta_c^be^{-\alpha\left(\tau+\frac{a}{c}\right)}}\right)$$
$$=\zeta_c^{-2b}\cdot e^{(2n+1)\pi i\tau-\alpha\tau^2+\alpha\frac{a}{c}}\left(\frac{\zeta_c^be^{-\alpha\left(\tau+\frac{a}{c}\right)}+\zeta_c^{2b}e^{-2\alpha\left(\tau+\frac{a}{c}\right)}}{\zeta_c^{-b}e^{\alpha\left(\tau+\frac{a}{c}\right)}-\zeta_c^be^{-\alpha\left(\tau+\frac{a}{c}\right)}}\right).$$

We now make the substitution $\tau=-\frac{a}{c}+w_n+x$. Since $w_n=\frac{(2n+1)\pi i}{2\alpha}$, the integral J_n' becomes

$$J_n'=\frac{(-1)^{n+1}i}{2}\zeta_c^{-2b}q_1^{\frac{(2n+1)^2}{4}}q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)}$$
$$\times\int_{-\infty}^{\infty}\frac{\left(\zeta_c^be^{-\frac{(2n+1)\pi i}{2}}e^{-\alpha x}+\zeta_c^{2b}e^{(2n+1)\pi i}e^{-2\alpha x}\right)}{\cosh\left(\alpha x-2\pi i\frac{b}{c}\right)}e^{-\alpha x^2+2\alpha\frac{a}{c}x}\,dx$$
$$=\frac{(-1)^{n+1}i}{2}\zeta_c^{-2b}q_1^{\frac{(2n+1)^2}{4}}q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)}\int_{-\infty}^{\infty}\frac{\left(i(-1)^{n+1}\zeta_c^be^{-\alpha x}-\zeta_c^{2b}e^{-2\alpha x}\right)}{\cosh\left(\alpha x-2\pi i\frac{b}{c}\right)}e^{-\alpha x^2+2\alpha\frac{a}{c}x}\,dx.$$

In the same way, we obtain

$$K_n' = \frac{(-1)^n i}{2}\zeta_c^{-2b} q_1^{\frac{(2n+1)^2}{4}} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} \int_{-\infty}^{\infty} \frac{\left(i(-1)^n\zeta_c^b e^{-\alpha x} - \zeta_c^{2b}e^{-2\alpha x}\right)}{\cosh\left(\alpha x - 2\pi i\frac{b}{c}\right)} e^{-\alpha x^2+2\alpha\frac{a}{c}x}\, dx.$$

Therefore, we see that

$$K_n' + J_n' = -\,\zeta_c^{-b} q_1^{\frac{(2n+1)^2}{4}} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} \int_{-\infty}^{\infty} \frac{e^{-\alpha x-\alpha x^2+2\alpha\frac{a}{c}x}}{\cosh\left(\alpha x - 2\pi i\frac{b}{c}\right)}\, dx.$$

Using this we easily see that

$$\sum_{n\geq 0}(K_n' + J_n') = -\zeta_c^{-b} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} \int_{-\infty}^{\infty} \frac{e^{-\alpha x-\alpha x^2+2\alpha\frac{a}{c}x}}{\cosh\left(\alpha x - 2\pi i\frac{b}{c}\right)}\, dx \sum_{n\geq 0} q_1^{\frac{1}{4}(2n+1)^2}.$$

By applying the identity $\sum_{n\geq 0} q_1^{\frac{1}{4}(2n+1)^2} = q_1^{\frac{1}{4}}(q_1^4;q_1^4)_\infty/(q_1^2;q_1^4)_\infty$, we obtain

$$\sum_{n\geq 0}(K_n' + J_n') = -\,\zeta_c^{-b} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} q_1^{\frac{1}{4}} \frac{(q_1^4;q_1^4)_\infty}{(q_1^2;q_1^4)_\infty} L(a,b,c;\alpha). \tag{4.14}$$

Combining the two different evaluations for I we have

$$\begin{aligned}\tfrac{(q)_\infty}{(-q)_\infty} H\left(a,b,c;-\tfrac{\alpha}{2\pi i}\right) &= \tfrac{4\pi}{\alpha} e^{-2k(b,c)\frac{a}{c}\pi i + 4\pi i\left(\frac{a}{c}-1\right)\frac{b}{c}} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} q_1^{4k(b,c)\frac{b}{c}-4\left(\frac{b}{c}\right)^2} \\ &\times \tfrac{(q_1^4;q_1^4)_\infty}{(q_1^2;q_1^4)_\infty} K\left(a,b,c;\tfrac{\pi i}{\alpha}\right) - \zeta_c^{-b} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} q_1^{\frac{1}{4}} \tfrac{(q_1^4;q_1^4)_\infty}{(q_1^2;q_1^4)_\infty} L(a,b,c;\alpha).\end{aligned} \tag{4.15}$$

By the transformation law for Dedekind's eta-function,

$$(q)_\infty = \sqrt{\frac{2\pi}{\alpha}} q^{-\frac{1}{24}} q_1^{\frac{1}{6}} (q_1^4;q_1^4)_\infty,$$

we may deduce that

$$\frac{(q)_\infty}{(-q)_\infty} = 2\sqrt{\frac{\pi}{\alpha}} q_1^{\frac{1}{4}} \frac{(q_1^4;q_1^4)_\infty}{(q_1^2;q_1^4)_\infty}.$$

With (4.15) this gives

$$\begin{aligned}\sqrt{\frac{4\pi}{\alpha}} q_1^{\frac{1}{4}} H\left(a,b,c;-\frac{\alpha}{2\pi i}\right) =& \frac{4\pi}{\alpha} e^{-2k(b,c)\frac{a}{c}\pi i + 4\pi i\left(\frac{a}{c}-1\right)\frac{b}{c}} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} q_1^{4k(b,c)\frac{b}{c}-4\left(\frac{b}{c}\right)^2} K\left(a,b,c;\frac{\pi i}{\alpha}\right) \\ &- \zeta_c^{-b} q^{-\frac{a}{c}\left(1-\frac{a}{c}\right)} q_1^{\frac{1}{4}} L(a,b,c;\alpha),\end{aligned}$$

which gives the result. □

We use Theorem 4.3 to construct harmonic weak Maass forms on subgroups of $\Gamma_0(4) = \left\langle \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix} \right\rangle$. Since $\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ -4 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -\frac{1}{4}\\ 1 & 0\end{pmatrix}$, it will suffices to determine the images of our functions under $z \mapsto z+1$ and $z \mapsto -\frac{1}{4z}$, see the proofs of Theorems 4.1 and 4.2 for more details.

We define the following functions which are modified versions of the functions we dealt with in the previous section. For $0 \leq a, b < c$ with a and b not both zero, $\frac{b}{c} \notin \{\frac{1}{4}, \frac{3}{4}\}$ and $q = e^{2\pi i z}$ define $\mathcal{H}(a,b,c;z)$ and $\mathcal{K}(a,b,c;z)$ by

$$\mathcal{H}(a,b,c;z) := q^{2\frac{a}{c}\left(1-\frac{a}{c}\right)} \cdot H(a,b,c;2z), \tag{4.16}$$

$$\mathcal{K}(a,b,c;z) := e^{-2k(b,c)\frac{a}{c}\pi i + 4\pi i\left(\frac{a}{c}-1\right)\frac{b}{c}} q^{2\left(k(b,c)\frac{b}{c} - \left(\frac{b}{c}\right)^2 - \frac{1}{16}\right)} K(a,b,c;z). \tag{4.17}$$

The following theorem describes how these functions transform under the two transformations of interest.

Theorem 4.4. *Suppose that $a, b,$ and c are integers for which $0 \leq a, b < c$, $\frac{b}{c} \notin \{\frac{1}{4}, \frac{3}{4}\}$, and $k(b,c)$ is defined as in (4.3).*

(1) *For all $z \in \mathbb{H}$ we have*

$$\begin{aligned} \mathcal{K}\left(a,b,c;z+1\right) =& \zeta_{c^2}^{-2b^2} \cdot \zeta_8^{-1} \cdot \zeta_c^{2b\,k(b,c)} \cdot \mathcal{K}(a-2b,b,c;z), \\ \mathcal{H}\left(a,b,c;z+1\right) =& \zeta_c^{4a} \cdot \zeta_{c^2}^{-2a^2} \cdot \mathcal{H}(a,2a+b,c;z). \end{aligned}$$

(2) *For all $z \in \mathbb{H}$ we have*

$$\begin{aligned} \frac{1}{\sqrt{-iz}}\mathcal{K}\left(a,b,c;-\frac{1}{4z}\right) =& \mathcal{H}(a,b,c;z) + \sqrt{-iz}\zeta_c^{-b} L(a,b,c;-4\pi i z), \\ \frac{1}{\sqrt{-4iz}}\mathcal{H}\left(a,b,c;-\frac{1}{4z}\right) =& \mathcal{K}(a,b,c;z) + \frac{1}{4iz}\zeta_c^{-b} L\left(a,b,c;\frac{\pi i}{z}\right) \end{aligned}.$$

Remark. Two remarks are in order. We need to be careful about the parameters in the functions above. For example, the parameters of $\mathcal{K}(a-2b,b,c;z)$ may lie outside of the defined ranges when $a-2b$ is not in the interval $[0,c)$. In such cases one defines the functions in the obvious way, and then observes that the functions equal, up to a precise root of unity, the corresponding functions where $a-2b$ is replaced by its reduced residue modulo c. The second remark is in regards to the constraints on our parameters $a, b,$ and c. Recall, $\mathcal{K}(a,b,c;z)$ and $\mathcal{H}(a,b,c;z)$ are only defined when a and b are not both zero. This could potentially be a problem in the transformations above. However, in the transformation $z \mapsto z+1$ for $\mathcal{K}$, we notice that $a-2b$ and b cannot both be congruent to 0 modulo c unless $a \equiv b \equiv 0 \pmod{c}$. Similarly we avoid any such problem with the transformation $z \mapsto z+1$ for the $\mathcal{H}$ family.

Proof of Theorem 4.4. The first set of claims follows from direct calculations using the definitions given in (4.1) and (4.2). More precisely, we use the explicit Fourier expansions, and carefully keep track of the parameters which change under $z \to z+1$. The second claim follows from Theorem 4.3 by letting $\alpha = -4\pi i z$ and $\alpha = \pi i/z$. □

At this point it is more useful to rewrite $L(a,b,c;\alpha)$ in terms of a period integral of the function $T(a,b,c;\tau)$, which was defined in (4.8).

Lemma 4.5. *In the notation above, we have that*

$$\frac{1}{4iz}L\left(a,b,c;\frac{\pi i}{z}\right)=\int_0^{i\infty}\frac{T(a,b,c;2\tau)}{\sqrt{-i(\tau+z)}}d\tau,$$
$$\sqrt{-iz}L(a,b,c;-4\pi iz)=-\frac{1}{2}\int_0^{i\infty}\frac{(-i\tau)^{-\frac{3}{2}}T(a,b,c;-\frac{1}{2\tau})}{\sqrt{-i(\tau+z)}}d\tau.$$

Proof. We will show the first of these claims. The second follows by a similar calculation. For more details see Lemma 3.2 of [7]. Setting $z=it/2$ and making a change of variables $y=x/t$, we have

$$L\left(a,b,c;\frac{2\pi}{t}\right)=t\int_{-\infty}^{\infty}e^{-2\pi ty^2}\frac{e^{2\pi y\left(2\frac{a}{c}-1\right)}}{\cosh\left(2\pi y-2\pi i\frac{b}{c}\right)}\,dy.$$

By the Mittag-Leffler partial fraction decomposition, we write

$$\frac{e^{2\pi\left(2\frac{a}{c}-1\right)y}}{\cosh\left(2\pi y-2\pi i\frac{b}{c}\right)}=\frac{1}{2\pi i}\sum_{n\in\mathbb{Z}}e^{2\pi\left(2\frac{a}{c}-1\right)y_n^+}\cdot\frac{1}{y-y_n^+}-\frac{1}{2\pi i}\sum_{n\in\mathbb{Z}}e^{2\pi\left(2\frac{a}{c}-1\right)y_n^-}\cdot\frac{1}{y-y_n^-},$$

where $y_n^\pm:=i\left(n\pm\frac{1}{4}+\frac{b}{c}\right)$ are the zeros of $\cosh\left(2\pi y-2\pi i\frac{b}{c}\right)$. We remark at this point that we will be rearranging the order of summation and integration during the course of the proof, while we do not justify it, it can be justified as in [22] or [7] Lemma 3.2.

We consider

$$\pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}e^{-2\pi ty^2}\sum_{n\in\mathbb{Z}}e^{2\pi i\left(2\frac{a}{c}-1\right)y_n^\pm}\cdot\frac{1}{y-y_n^\pm}.$$

We switch the order of summation and integration to obtain

$$\pm\frac{1}{2\pi i}\sum_n e^{2\pi i\left(2\frac{a}{c}-1\right)y_n^\pm}\int_{-\infty}^{\infty}\frac{e^{-2\pi ty^2}}{y-y_n^\pm}dy$$
$$=\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(n+\frac{1}{4}\pm\frac{b}{c}\right)e^{\pm2\pi i\left(2\frac{a}{c}-1\right)\left(n+\frac{1}{4}\pm\frac{b}{c}\right)}\int_0^{\infty}\frac{e^{-\pi\left(n+\frac{1}{4}\pm\frac{b}{c}\right)^2u}}{\sqrt{u+2t}}du,$$

where the equality follows from the integral identity (see [22])

$$\int_{-\infty}^{\infty}\frac{e^{-\pi tx^2}}{x-is}dx=\pi is\int_0^{\infty}\frac{e^{-\pi us^2}}{\sqrt{u+t}}du.$$

Combining the two sums, by interchanging summation and integration again, gives

$$L\left(a,b,c;\frac{2\pi}{t}\right)$$
$$=t\int_0^{\infty}\frac{e^{2\pi i\left(2\frac{a}{c}-1\right)\frac{b}{c}}}{\sqrt{u+2t}}\sum_{n\in\mathbb{Z}}\left(\left(n+\frac{1}{4}\right)\cosh(X)+\frac{b}{c}\sinh(X)\right)e^{-\pi u\left(\left(\frac{b}{c}\right)^2+\left(n+\frac{1}{4}\right)^2\right)}du,$$

where for convenience we write $X := 2\pi i(n+\frac{1}{4})\left((2\frac{a}{c}-1)+iu\frac{b}{c}\right)$. Now making the substitution $\tau = iu/4$ and replacing $t=-2iz$ we obtain the stated result. $\square$

Continuing the study of these period integrals, we have the following lemma.

Lemma 4.6. *With V_1 and V_2 as defined in (4.6) and (4.7) we have*

$$\begin{aligned}\frac{1}{\sqrt{-iz}}V_1\left(a,b,c;-\frac{1}{4z}\right) =& V_2(a,b,c;z) - \sqrt{-iz}\,\zeta_c^{-b}L(a,b,c;-4\pi iz)\\ \frac{1}{\sqrt{-4iz}}V_2\left(a,b,c;-\frac{1}{4z}\right) =& V_1(a,b,c;z) - \frac{1}{4iz}\,\zeta_c^{-b}L\left(a,b,c;\frac{\pi i}{z}\right),\end{aligned}$$

for all $z\in\mathbb{H}$.

Proof. This follows by a change of variables and Lemma 4.5 (see Lemma 3.3 of [7]). $\square$

To obtain the Maass forms and to prove Theorems 4.1 and 4.2, for $0\le a,b<c$ and a and b not both zero, we define

$$\mathcal{G}_1(a,b,c;z) := \mathcal{H}(a,b,c;z) + V_2(a,b,c;z), \tag{4.18}$$

$$\mathcal{G}_2(a,b,c;z) := \mathcal{K}(a,b,c;z) + V_1(a,b,c;z). \tag{4.19}$$

Proof of Theorems 4.1 and 4.2. We begin by considering the transformation properties of $D_1(a,b,c;z) = \mathcal{G}_1(a,b,c;2f_c^2z)$ and $D_2(a,b,c;z)=\mathcal{G}_2(a,b,c;2f_c^2z)$ under any element of $\Gamma_0(4)$. It follows from Theorem 4.4 (1) that $\mathcal{H}(a,b,c;2f_c^2(z+1)) = \mathcal{H}(a,b,c;2f_c^2z)$ and $\mathcal{K}(a,b,c;2f_c^2(z+1)) = \mathcal{K}(a,b,c;2f_c^2z)$. Hence, the $\mathcal{H}$ and $\mathcal{K}$ parts of the functions D_1 and D_2 remain fixed under the transformation given by $z\mapsto z+1$.

Using the Fourier expansion of the theta function $T(a,b,c;4f_c^2\tau)$ one easily deduces that $V_1(a,b,c;2f_c^2(z+1)) = V_1(a,b,c;2f_c^2z)$. Additionally, using classical facts about theta functions (for example, see equations (2.4) and (2.5) of [19]), one can also deduce that $V_2(a,b,c;2f_c^2(z+1)) = V_2(a,b,c;2f_c^2z)$.

For the transformation $z\mapsto -1/4z$, we begin by noting the similarities between the transformation laws appearing in Theorem 4.4 (2) and Lemma 4.6. Namely, the signs in front of the period integrals are opposite.

It is then clear that Theorem 4.4, Lemma 4.6, and the above calculation allow us to determine the images of $D_1(a,b,c;z)$ and $D_2(a,b,c;z)$ under any element of $\Gamma_0(4)$. In particular, one directly obtains (in a roundabout way) the modular transformation properties for the period integrals of the theta functions $T(a,b,c;4f_c^2z)$ under the generators of $\Gamma_0(4)$. On the other hand, these modular transformation properties are precisely dictated by the modular transformations of the theta functions themselves. Therefore it follows that $D_1(a,b,c;z)$ and $D_2(a,b,c;z)$ are modular on the same congruence subgroups of $\Gamma_0(4)$ as the corresponding theta functions. In other words, the modular transformation properties of $D_1(a,b,c;z)$ and $D_2(a,b,c;z)$ are inherited by the modularity properties of $T(a,b,c;-1/4f_c^2\tau)$ and $T(a,b,c;4f_c^2\tau)$. Rewriting $T(a,0,c;\tau)$, defined in (4.8), by

$$T(a,0,c;\tau) = \frac{i}{4} \sum_{m \pmod{f_c}} \sin\left(\frac{a}{c}\pi(4m+1)\right) \theta\left(4m+1, 4f_c, \frac{\tau}{16}\right),$$

with $\theta(\alpha,\beta,\tau) := \sum_{n\equiv\alpha \pmod{\beta}} ne^{2\pi i\tau n^2}$ we may apply Proposition 2.2 of [19] to see that $T(a,0,c;4f_c^2\tau)$ is a weight $3/2$ cusp form on $\Gamma_1(64f_c^4)$. Similarly, $T(a,c/2,c;4f_c^2\tau) = -T(a,0,c;4f_c^2\tau)$ determines the transformation properties of $D_2(a,c/2,c;z)$ and is on $\Gamma_1(64f_c^4)$.

To complete the proof for $D_2(a,b,c;z)$, it suffices to show that each component is annihilated by the weight $1/2$ Laplacian $\Delta_{\frac{1}{2}}$, and satisfies the necessary growth conditions at the cusps. This follows by a straightforward calculation just as in [7].

Similar calculations show that for $b=0$ or $c/2$, that $T(a,b,c;-1/4f_c^2\tau)$ is a weight $3/2$ cusp form on $\Gamma_1(64f_c^4)$. Thus $D_1(a,b,c;z)$ is a weight $1/2$ harmonic weak Maass form on $\Gamma_1(64f_c^4)$. □

4.3. **Proof of Theorem 1.3.** We can now establish Theorem 1.3.

Proof of Theorem 1.3. Combining Theorems 4.1 and 4.2 with (4.5) and (4.4) shows that the functions $\widetilde{H}(a,c;2f_c^2z)$ and $\widetilde{K}(a,c;2f_c^2z)$ are both weight $1/2$ harmonic weak Maass forms on $\Gamma_1(64f_c^4)$. To show that they are holomorphic we note that a direct computation shows that $T(a,0,c;\tau) = -T(a,c/2,c;\tau)$. Thus we may conclude that $\widetilde{H}(a,c;2f_c^2z)$ and $\widetilde{K}(a,c;2f_c^2z)$ are both holomorphic. □

References

[1] G. E. Andrews, *q-series: Their development and application in analysis, number theory, combinatorics physics, and computer algebra*, CBMS No. 66, Amer. Math. Soc., 1986, Providence, R.I.

[2] G. E. Andrews, *Mock theta functions,* Theta functions - Bowdoin 1987, Part 2 (Brunswick, ME., 1987), pages 283-297, Proc. Sympos. Pure Math. **49**, Part 2, Amer. Math. Soc., Providence, RI., 1989.

[3] G. E. Andrews and F. Garvan, *Ramanujan's "lost notebook". VI: The mock theta conjectures*, Adv. in Math. **73** (1989), pages 242-255.

[4] A. O. L. Atkin and H. P. F. Swinnerton-Dyer, *Some properties of partitions*, Proc. London Math. Soc. **66** No. 4 (1954), pages 84-106.

[5] A. Berkovich and F. G. Garvan, *Some Observations on Dyson's New Symmetries of Partitions*, J. Combin. Theory Ser. A **100** (2002), no. 1, pages 61-93.

[6] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. **165** (2006), pages 243-266.

[7] K. Bringmann and K. Ono, *Dyson's ranks and Maass forms*, Ann. of Math., in press.

[8] Y.-S. Choi, *Tenth order mock theta functions in Ramanujan's lost notebook,* Invent. Math. **136** (1999), pages 497-569.

[9] Y.-S. Choi, *Tenth order mock theta functions in Ramanujan's lost notebook, II*, Adv. in Math. **156** (2000), pages 180-285.

[10] F. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) **8** (1944), pages 10-15.

[11] G. Gasper and M. Rahman, *Basic Hypergeometric Series,* Cambridge University Press, Cambridge, 1990.

[12] D. Hickerson, *A proof of the mock theta conjectures*, Invent. Math. **94** (1988), pages 639-660.

[13] D. Hickerson, *On the seventh order mock theta functions*, Invent. Math. **94** (1988), pages 661-677.

[14] V. G.Kac, *Infinite dimensional Lie algebras*, Cambridge Univ. Press, Cambridge, 1990.

[15] N. Koblitz, *Introduction to elliptic curves and modular forms*, Springer-Verlag, Berlin, 1993.

[16] J. Lepowsky and R. L. Wilson, *A new family of algebras underlying the Rogers-Ramanujan identities and generalizations*, Proc. Natl. Acad. Sci., USA **78** (1981), pages 7254-7258.

[17] J. Lovejoy, *Rank and Conjugation for a Second Frobenius Representation of an Overpartition*, Ann. Comb. **9** (2005), no.3, pages 321-334..

[18] R. McIntosh, *H and K Family of Mock Theta Functions*, preprint.

[19] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. **97** (1973), pages 440-481.

[20] G. N. Watson, *The mock theta functions (2),* Proc. London Math. Soc. (2) **42** (1937), pages 274-304.

[21] H. Yesilyurt, *Four identities related to third order mock theta functions in Ramanujan's lost notebook*, Adv. in Math. **190** (2005), pages 278-299.

[22] S. P. Zwegers, *Mock theta functions*, Ph.D. Thesis, Universiteit Utrecht, 2002.

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