ALGEBRAICITY OF HARMONIC MAASS FORMS

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Abstract. In 1947 D. H. Lehmer conjectured that Ramanujan's tau-function never vanishes. In the 1980s, B. Gross and D. Zagier proved a deep formula expressing the central derivative of suitable Hasse-Weil L-functions in terms of the Neron-Tate height of a Heegner point. This expository article describes recent work (with J. H. Bruinier and R. Rhoades) which reformulates both topics in terms of the algebraicity of harmonic Maass forms.

1. Introduction

This expository article describes recent work by the author, J. H. Bruinier and R. Rhoades [5, 6] on the unexpected algebraicity of certain coefficients of suitable harmonic Maass forms. We begin by recalling the notion of a harmonic Maass form (see [22] for more on harmonic Maass forms and their roles in number theory). Throughout, let $z = x + iy \in \mathbb{H}$, the upper-half of the complex plane, with $x, y \in \mathbb{R}$. For $k \in \frac{1}{2}\mathbb{Z}$, we define the weight k hyperbolic Laplacian by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \tag{1.1}$$

A *harmonic Maass form of weight k on* $\Gamma_0(N)$ (where $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$) is any smooth function on $\mathbb{H}$ satisfying:

(i) $f|_k\gamma = f$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$;

(ii) $\Delta_k f = 0$;

(iii) There is a polynomial $P_f = \sum_{n \leq 0} c_f^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) - P_f(z) = O(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

The polynomial $P_f \in \mathbb{C}[q^{-1}]$ is called the *principal part* of f at the corresponding cusp. We denote the vector space of these harmonic Maass forms by $H_k(\Gamma_0(N))$.

Remark 1. Harmonic Maass forms were introduced by Bruinier and Funke [4] in their work on theta lifts. Harmonic Maass forms have recently appeared prominently in many areas of mathematics: arithmetic geometry, combinatorics, gauge theory, Lie theory, physics, and topology (for example, see [22]).

The differential operator $\xi_w := 2iy^w \cdot \overline{\frac{\partial}{\partial \overline{z}}}$, plays a central role in the theory, and the key fact (see Proposition 3.2 of [4]) is that

$$\xi_{2-k} : H_{2-k}(\Gamma_0(N)) \longrightarrow S_k(\Gamma_0(N)), \tag{1.2}$$

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where $S_w(\Gamma_0(N))$ is the subspace of cusp forms. It is not difficult to make this more precise using Fourier expansions. Every weight $2-k$ harmonic Maass form $f(z)$ has a Fourier expansion of the form

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n, \tag{1.3}$$

where $\Gamma(a,x)$ is the incomplete Gamma-function and $q := e^{2\pi i z}$. A straightforward calculation shows that $\xi_{2-k}(f)$ has the Fourier expansion

$$\xi_{2-k}(f) = -(4\pi)^{k-1} \sum_{n=1}^{\infty} \overline{c_f^-(-n)} n^{k-1} q^n. \tag{1.4}$$

As (1.3) reveals, $f(z)$ naturally decomposes into two summands

$$f^+(z) := \textstyle\sum_{n \gg -\infty} c_f^+(n) q^n, \tag{1.5}$$

$$f^-(z) := \textstyle\sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n. \tag{1.6}$$

Therefore, $\xi_{2-k}(f)$ is given simply in terms of $f^-(z)$, the *non-holomorphic part* of f. The complementary part, the *holomorphic part* $f^+(z)$, is the object of this paper.

We are concerned with the algebraicity of the coefficients $c_f^+(n)$ for certain harmonic Maass forms f. For the f we consider, we expect the coefficients $c_f^+(n)$ to "generically" be transcendental. However, as we shall see, certain arithmetic conditions force the algebraicity of such coefficients. The conditions we have discovered are related to:

- Vanishing of Hecke eigenvalues
- Vanishing of derivatives of modular L-functions.

In Section 2 we discuss algebraicity of coefficients of harmonic Maass forms and the vanishing of Hecke eigenvalues, and in Section 3 we discuss algebraicity and the vanishing of central derivatives of modular L-functions.

2. Vanishing of Hecke eigenvalues

As usual, let

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n := q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q^2 + 252q^3 - \dots \tag{2.7}$$

be the unique normalized weight 12 cusp form on $\mathrm{SL}_2(\mathbb{Z})$. Its coefficients, the so-called values of Ramanujan's tau-function, provide tantalizing examples of some of the deepest phenomena in the theory of modular forms. It served as a testing ground for Hecke's theory of Hecke operators, and Serre's theory of modular ℓ-adic Galois representations [26]. It also served as the prototype for the celebrated Ramanujan-Petersson Conjectures, which was proved by Deligne [9] for integer weight. Despite these advances, the following seemingly simple conjecture has remained open.

Conjecture. (D. H. Lehmer [19], 1947)
If n is a positive integer, then $\tau(n)$ is non-zero.

The strongest results on this conjecture are based on congruences. For example, we have the well known Ramanujan congruences (for example, see [1, 25, 26, 30])

$$\begin{aligned} \tau(n) &\equiv n^{70}\sigma_{71}(n) \pmod{5^3} && \text{if } 5 \nmid n,\\ \tau(n) &\equiv n\sigma_3(n) \pmod{7} && \text{if } 7 \nmid n,\\ \tau(n) &\equiv \sigma_{11}(n) \pmod{691}, \end{aligned}$$

where $\sigma_\nu(n) := \sum_{d|n} d^\nu$. These congruences place restrictions on those n for which $\tau(n)$ might vanish. For example, the mod 691 congruence alone implies that $\tau(p)$ can only vanish for those primes $p \equiv -1 \pmod{691}$. The difficulty in approaching Lehmer's Conjecture using explicit congruence arises from the fact that for every positive integer M there are infinitely many primes p (in fact, a proportion) for which $\tau(p) \equiv 0 \pmod{M}$. This line of reasoning alone does not preclude the possibility that

$$\lim_{X\to+\infty} \frac{\#\{1 \le n \le X \ : \ \tau(n) = 0\}}{X} = 1.$$

In an important paper [28], Serre amplified this approach using ℓ-adic Galois representations and the Chebotarev Density Theorem, and he confirmed that it is not the case that "almost all" of the values of $\tau(n)$ vanish.

Here we shed new light on Lehmer's Conjecture. Bruinier, Rhoades and the author [6] have obtained a theoretical criterion which implies the nonvanishing of all of the coefficients of a newform.

Suppose that $g \in S_k(\Gamma_0(N))$ is a normalized even weight newform, and let F_g be the number field obtained by adjoining the coefficients of g to $\mathbb{Q}$. We say that a harmonic Maass form $f \in H_{2-k}(\Gamma_0(N))$ is *good for g* if it satisfies the following properties:

(i) The principal part of f at the cusp ∞ belongs to $F_g[q^{-1}]$.
(ii) The principal parts of f at the other cusps of $\Gamma_0(N)$ are constant.
(iii) We have that $\xi_{2-k}(f) = \|g\|^{-2} g$, where $\|g\|$ denotes the usual Petersson norm.

Remark 2. For every newform g, there is a harmonic Maass form f which is good for g.

The next theorem follows from the main results in [6].

Theorem 2.1. *Suppose that $g = \sum_{n=1}^\infty c_g(n)q^n \in S_k(\Gamma_0(N))$ is a normalized even weight newform, and suppose that $f \in H_{2-k}(\Gamma_0(N))$ is good for g. Then the following are true:*

(1) *If g has complex multiplication, then all of the coefficients $c_f^+(n)$ are algebraic.*
(2) *If $c_g(p) = 0$ for a prime $p \nmid N$, then $c_f^+(n)$ is algebraic when $\mathrm{ord}_p(n)$ is odd.*

Remark 3. The proof of Theorem 2.1 shows that these algebraic coefficients of f^+ are in an explicit abelian extension of F_g. Moreover, the proof shows that the coefficients $c_f^+(n)$ are always in the field $F_g(c_f^+(1))$.

Remark 4. Guerzhoy, Kent, and the author [23] have obtained results which relate Theorem 2.1 to Eichler cohomology for those g with complex multiplication. This leads to beautiful results concerning the p-adic properties of holomorphic parts of certain harmonic Maass forms.

2.1. **Dedekind eta-product newforms.** We now discuss some numerics related to Theorem 2.1 in the case of Ramanujan's tau-function, and all of the even weight Dedekind-eta product newforms. We recall the complete list (for example, see [10, 20]) of such newforms which are products of Dedekind's eta-function

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.8}$$

Each of these eta-products is in a one dimensional space of cusp forms. Moreover for each level N, there is at most one weight, namely $k := 2\lceil \frac{12}{N+1} \rceil$, for which there is such a newform. For convenience we shall denote these newforms using the notation

$$f_N(z) = \sum_{n=1}^{\infty} a_N(n) q^n \in S_k(\Gamma_0(N)). \tag{2.9}$$

The table below lists these newforms, and it indicates whether any vanishing coefficients are known.

(N,k)	Newform	CM	Vanishing Coefficients
$(1,12)$	$\eta(z)^{24}$	No	None known
$(2,8)$	$\eta(z)^8\eta(2z)^8$	No	None known
$(3,6)$	$\eta(z)^6\eta(3z)^6$	No	None known
$(4,6)$	$\eta(2z)^{12}$	No	Yes
$(5,4)$	$\eta(z)^4\eta(5z)^4$	No	Yes
$(6,4)$	$\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2$	No	None known
$(8,4)$	$\eta(2z)^4\eta(4z)^4$	No	Yes
$(9,4)$	$\eta(3z)^8$	Yes	Yes
$(11,2)$	$\eta(z)^2\eta(11z)^2$	No	Yes
$(14,2)$	$\eta(z)\eta(2z)\eta(7z)\eta(14z)$	No	Yes
$(15,2)$	$\eta(z)\eta(3z)\eta(5z)\eta(15z)$	No	Yes
$(20,2)$	$\eta(2z)^2\eta(10z)^2$	No	Yes
$(24,2)$	$\eta(2z)\eta(4z)\eta(6z)\eta(12z)$	No	Yes
$(27,2)$	$\eta(3z)^2\eta(9z)^2$	Yes	Yes
$(32,2)$	$\eta(3z)^2\eta(9z)^2$	Yes	Yes
$(36,2)$	$\eta(6z)^4$	Yes	Yes

Table 1

Remark 5. In the case where $(N,k) = (1,12)$, we have that $f_1(z) = \Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$. Using the congruence properties of $\tau(n)$, it is simple to check that $\tau(n) \neq 0$ for all $n \leq 10^{15}$. Similar remarks apply for each newform above with "None known".

Remark 6. Thanks to the modularity of elliptic curves, a famous theorem of Elkies [11] then implies that every weight 2 newform

$$f(z)=\sum_{n=1}^{\infty}a(n)q^n\in S_2(\Gamma_0(N))$$

with integer coefficients has vanishing Fourier coefficients. Indeed, there are infinitely many primes $p\nmid N$, the supersingular primes, for which $a(p)=0$. In this direction, we also note that every weight 1 newform has the property that "almost all" of its coefficients vanish.

Since the newforms we consider are in one dimensionsal spaces of cusp forms, we may represent them easily in terms of Poincaré series. Moreover, these Poincaré series are images under explicit harmonic Maass forms which may be described in terms of Maass-Poincaré series.

We recall the Fourier expansions of these forms which are described in terms of the I and J-Bessel functions, and Kloosterman sums. Throughout, we let $e(\alpha):=e^{2\pi i\alpha}$, and we let $K(m,n,c)$ denote the classical Kloosterman sum

$$K(m,n,c):=\sum_{v(c)^\times}e\left(\frac{m\overline{v}+nv}{c}\right). \tag{2.10}$$

In the sum above, the index v ranges over the primitive residue classes modulo c, and $\overline{v}$ denotes the multiplicative inverse of v modulo c.

We require the classical family of modular Poincaré series (for example, see [15, 24])

$$P(m,k,N;z)=q^m+\sum_{n=1}^{\infty}a(m,k,N;n)q^n. \tag{2.11}$$

Theorem 2.2. *If $k\in 2\mathbb{N}$, and $m,N\geq 1$, then the following are true.*
1) We have that $P(m,k,N;z)\in S_k(\Gamma_0(N))$, and for positive integers n we have

$$a(m,k,N;n)=2\pi(-1)^{\frac{k}{2}}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\cdot\sum_{\substack{c>0\\ c\equiv 0 \pmod{N}}}\frac{K(m,n,c)}{c}\cdot J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

2) We have that $P(-m,k,N;z)\in M_k^!(\Gamma_0(N))$, the space of weight k weakly holomorphic modular forms on $\Gamma_0(N)$, and for positive integers n we have

$$a(-m,k,N;n)=2\pi(-1)^{\frac{k}{2}}\left(\frac{n}{m}\right)^{\frac{k-1}{2}}\cdot\sum_{\substack{c>0\\ c\equiv 0 \pmod{N}}}\frac{K(-m,n,c)}{c}\cdot I_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

Remark 7. For the cuspidal Poincaré series $P(m,k,N;z)$, we note that the coefficient of q^m is $1+a(m,k,N;m)$.

Remark 8. These series appear in many earlier works (for example, see [2, 3, 14, 21]).

The modular forms in Theorem 2.2 are related to the harmonic Maass forms in the following theorem.

Theorem 2.3. *If $k \in 2\mathbb{N}$ and $m, N \geq 1$, then*

$$Q(-m,k,N;z) := (1-k)\left(\Gamma(k-1,4\pi my) - \Gamma(k-1)\right) q^{-m} + \sum_{n\in\mathbb{Z}} c(m,k,N;n,y)\, q^n,$$

where $\Gamma(\alpha, x)$ is the incomplete Gamma-function, is in $H_{2-k}(\Gamma_0(N))$.
1) If $n < 0$, then

$$c(m,k,N;n,y) = 2\pi i^k (1-k)\,\Gamma(k-1,4\pi|n|y)\left|\frac{n}{m}\right|^{\frac{1-k}{2}} \times \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K(-m,n,c)}{c}\cdot J_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

2) If $n > 0$, then

$$c(m,k,N;n,y) = -2\pi i^k\Gamma(k)\left(\frac{n}{m}\right)^{\frac{1-k}{2}} \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K(-m,n,c)}{c}\cdot I_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

3) If $n = 0$, then

$$c(m,k,N;0,y) = -2^k\pi^k i^k m^{k-1} \sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K(-m,0,c)}{c^k}.$$

Remark 9. If n is positive, then notice that $c(m,k,N;n,y)$ does not depend on $y = \mathrm{Im}(z)$ in any way.

The next theorem (see §6.2 of [6]) gives the explicit relationships between the $P(m,k,N;z)$, the weakly holomorphic modular forms $P(-m,k,N;z)$, and the harmonic Maass forms $Q(-m,k,N;z)$. They are related by the ξ_{2-k} and $D := \frac{1}{2\pi i}\cdot\frac{d}{dz} = q\cdot\frac{d}{dq}$ differential operators.

Theorem 2.4. *Suppose that $k \in 2\mathbb{N}$, and that $m, N \geq 1$. Then the following are true.*

(1) *Under the operator ξ_{2-k}, we have that*
$$\xi_{2-k}(Q(-m,k,N;z)) = (4\pi)^{k-1}m^{k-1}(k-1)\cdot P(m,k,N;z) \in S_k(\Gamma_0(N)).$$
(2) *Under the operator D^{k-1}, if k is an integer, then we have that*
$$D^{k-1}Q(-m,k,N;z) = -m^{k-1}\Gamma(k)\cdot P(-m,k,N;z) \in M_k^!(\Gamma_0(N)).$$

Theorem 2.4, combined with the strong form of Theorem 2.1, implies the following result for the eta-products $f_N(z)$.

Theorem 2.5. *Suppose that $f_N(z) = \sum_{n=1}^{\infty} a_N(n)q^n \in S_k(\Gamma_0(N))$ is one of the eta-product newforms in Table 1. Then the following are true:*

(1) *If $f_N(z)$ has complex multiplication, then then there is a positive integer $D(N)$ such that $D(N)a(-1,k,N;n)$ is an integer for every $n \geq 1$.*
(2) *If $a_N(p) = 0$ for a prime $p \nmid N$, then $a(-1,k,N;n)$ is rational when $\mathrm{ord}_p(n)$ is odd.*

Proof. Each $f_N(z)$ is in a one dimensional space of cusp forms. Therefore, it follows that $\mathbb{C}f_N = \mathbb{C}P(1,k,N;z) = S_k(\Gamma_0(N))$. Consequently, the theory of Petersson norms (for example, see Theorem 3.3 of [15]), combined with Theorem 2.4 (1), then implies that $Q(-1,k,N;z)$ is good for $f_N(z)$.

Since the $K(-1,n,c)$ are real, the strong form of Theorem 2.1 then implies that the relevant $c(-1,k,N;n,y)$ are rational, which in turn implies, by Theorem 2.4 (1), that the relevant coefficients of $P(-1,k,N;z)$ are rational. In the case of CM, $P(-1,k,N;z)$ is a weakly holomorphic modular form with rational coefficients. Since all such modular forms have bounded denominators, this guarantees the existence of the integer $D(N)$. □

Example. For each eta-product newform $f_N(z)$ with complex multiplication, we numerically computed the first few terms of the weakly holomorphic modular form

$$-(k-1)!\cdot P(-1,k,N;z) = -(k-1)!q^{-1} - (k-1)!\sum_{n=1}^{\infty} a(-1,k,N;n)q^n.$$

By Theorem 2.5, there is an integer $D(N)$ for which $-D(N)(k-1)!a(-1,k,N;n)$ is an integer for every positive n. For $(N,k)=(9,4)$, we numerically found that

$$\begin{aligned}&-6P(-1,4,9;z)\\ &\quad\sim -6q^{-1} - 11.9999q^2 + 294.0000q^5 - 287.9999q^8 - 4625.9999q^{11} + 16464.0000q^{14} - \dots .\end{aligned}$$

Here the detector appears to be $D(9)=1$.

Using the convergents of the simple continued fractions of the first few Fourier coefficients, we have obvious candidates for all the $D(N)$, which we refer to as a *CM detector*. As the table below indicates, we have "observed" that all of these detectors appear to equal 1.

(N,k)	CM Newform	CM Detector$^?$
$(9,4)$	$f_9(z)$	1
$(27,2)$	$f_{27}(z)$	1
$(32,2)$	$f_{32}(z)$	1
$(36,2)$	$f_{36}(z)$	1

Table 2

Example. The more interesting question concerns the problem of whether there are transcendental coefficients whenever the newform g in Theorem 2.1 does not have complex multiplication. For those forms in Table 1 without complex multiplication, we numerically computed $-a(-1,k,N;1)$, and we computed the convergents of their simple continued fraction expansions. Here we are content with describing these results because the proof of Theorem 2.1 shows that $c_f^+(n)$ is rational for all n if and only if $-a(1,k,N;1)$ is rational.

Due to the slow convergence of the formulas for $-a(1,k,N;1)$ when $k=2$, we report on our findings for the higher weight eta-product newforms. For $(N,k)=(1,12)$, we find that

$$-a(1,12,1;1) \sim -1842.89472\dots,$$

and its first few convergents are:

$$-1843, -\frac{16586}{9}, -\frac{35015}{19}, -\frac{9750756}{5291}, -\frac{9785771}{5310}, -\frac{48893840}{26531}, \dots.$$

We have more convergents, and it seems highly unlikely that $-a(1, 12, 1; 1)$ is rational. This phenomenon is shared by all of the eta-product newforms without CM.

We conclude this section with some natural questions.

Problem 1. *Assuming the notation and hypotheses in Theorem 2.1, is it true that all of the* $c_f^+(n)$ *are algebraic if and only if* g *has CM? In particular, is it true that* $c_f^+(1)$ *is algebraic if and only if* g *has CM?*

Problem 2. *Determine methods for deducing the irrationality of sums of the form*

$$\sum_{\substack{c>0 \\ c\equiv 0 \pmod N}} \frac{K(-m,n,c)}{c} \cdot I_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

3. Vanishing of Derivatives

The algebraicity of coefficients of harmonic Maass forms also sheds light on the vanishing of central derivatives of modular L-functions. To explain the context of these new results, we recall famous theorems of Waldspurger, Kohnen and Zagier.

In the 1980s, Waldspurger [32], and Kohnen and Zagier [16, 17, 18] proved that half-integral weight modular forms serve as generating functions for central values of quadratic twists of even weight modular L-functions. Using the Shimura correspondence [29], they proved that certain coefficients of half-integral weight cusp forms are essentially square-roots of such values.

These results appear prominently in works related to rational elliptic curves. For example, Tunnell [31] made great use of these results in his work on the ancient "congruent number problem": the determination of those positive integers which are areas of right triangles with rational sidelengths. More generally, these results of Kohnen, Waldspurger and Zagier play important roles in the deep works of Gross, Zagier and Kohnen [12, 13] on the Birch and Swinnerton-Dyer Conjecture.

The author and Bruinier [5] have generalized this theorem of Waldspurger and Kohnen to prove that the Fourier coefficients of weight 1/2 harmonic Maass forms encode the vanishing and nonvanishing of both the central values and derivatives of quadratic twists of weight 2 modular L-functions.

We explain a special case of these results. Suppose that

$$G(z) = \sum_{n=1}^{\infty} B_G(n) q^n \in S_2(\Gamma_0(p)) \tag{3.12}$$

is a weight 2 newform with prime level p. As usual, we let

$$L(G,s) = \sum_{n=1}^{\infty} \frac{B_G(n)}{n^s} \tag{3.13}$$

denote its Hecke L-function. If Δ is a fundamental discriminant of a quadratic field coprime to p, then we let $L(G, \chi_\Delta, s)$ be the *quadratic twist* Hecke L-function

$$L(G, \chi_\Delta, s) = \sum_{n=1}^{\infty} \frac{B_G(n)\chi_\Delta(n)}{n^s}, \tag{3.14}$$

where $\chi_\Delta(\bullet) := \left(\frac{\Delta}{\bullet}\right)$ denotes the Kronecker character for $\mathbb{Q}(\sqrt{\Delta})$.

Here is a special case of the main result in [5].

Theorem 3.1. *Assume the hypotheses and notation above. In addition, suppose that the sign of the functional equation of $L(G, s)$ is $\epsilon(G) = -1$. Then there is a weight 1/2 harmonic Maass form $f(z)$ on $\Gamma_0(4p)$ with Fourier expansion*

$$f(z) = \sum_{n \gg -\infty} c_g^+(n) q^n + \sum_{n<0} c_g^-(n)\Gamma(1/2; 4\pi|n|y)q^n,$$

where $y = \mathrm{Im}(z)$, satisfying the following:

(1) *If $\Delta < 0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right) = 1$, then*

$$L(G, \chi_\Delta, 1) = \alpha_G \cdot \sqrt{|\Delta|} \cdot c_g^-(\Delta)^2,$$

where α_G is an explicit non-zero constant.

(2) *If $\Delta > 0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right) = 1$, then $L'(G, \chi_\Delta, 1) = 0$ if and only if $c_g^+(\Delta)$ is algebraic.*

Remark 10. Theorem 3.1 is a special case of the general result which holds for all levels, and any arbitrary sign.

Example. Here is an example which numerically illustrates the most general form of Theorem 3.1 for the weight 2 newform G which corresponds to the conductor 37 rank 1 elliptic curve

$$E: \quad y^2 = x^3 + 10x^2 - 20x + 8.$$

The table below includes some of the coefficients of a suitable f which were numerically computed by F. Strömberg (also see [7]), and the quadratic twist elliptic curves $E(-\Delta)$.

Δ	$c_g^+(-\Delta)$	$L'(E(\Delta),1) = L'(G,\chi_\Delta,1)$
-3	$1.0267149116\ldots$	$1.4792994920\ldots$
-4	$1.2205364009\ldots$	$1.8129978972\ldots$
-7	$1.6900297463\ldots$	$2.1107189801\ldots$
$\vdots$	$\vdots$	$\vdots$
-136	$-4.8392675993\ldots$	$5.7382407649\ldots$
-139	-6	0
-151	$-0.8313568817\ldots$	$6.6975085515\ldots$
$\vdots$	$\vdots$	$\vdots$
-815	$121.1944103120\ldots$	$4.7492583693\ldots$
-823	312	0

Strictly speaking, the cases where $\Delta = -139$ and -823 were not obtained numerically. We have that $L'(E(-139),1) = L'(E(-823),1) = 0$ by the Gross-Zagier formula (and correspond to vanishing coefficients of the modular form of weight 3/2 which arises in the context of the Shimura correspondence). The evaluations $c_g^+(139) = -6$ and $c_g^+(823) = 312$ arise from explicit generalized Borcherds products (for example, see Example 8.3 of [5]). The rank 3 elliptic curve $E(-139)$ is quite famous, for it was used as input data for Goldfeld's celebrated effective solution to Gauss's "Class Number Problem". For the other Δ in the table, the derivatives are non-vanishing and the coefficients $c_g^+(-\Delta)$ are transcendental.

Theorem 3.1 relates the algebraicity of coefficients of harmonic Maass forms to the vanishing of central derivatives of modular L-functions. It is natural to ask the following question.

Problem 3. *Assume the notation and hypotheses in Theorem 3.1. Is there an exact formula relating $c_g^+(\Delta)$ to $L'(G,\chi_\Delta,1)$?*

Along similar lines, Bruinier and T. Yang have recently obtained some very deep results. They are able to make use of their theory of Green's functions and theta lifts for harmonic Maass forms to obtain very general exact formulas for derivatives along the lines of the work of Gross and Zagier. In addition to formulating a deep conjecture about derivatives of L-functions and heights, they have proved the following striking theorem (see [8]).

Theorem 3.2. *If $G \in S_2(\Gamma_0(N))$ is a weight 2 newform with the property that the sign of the functional equation of $L(G,s)$ is $\epsilon(G) = -1$, then there is a weight 1/2 harmonic Maass form f, a weight 3/2 cusp form g, and a Heegner divisor $Z(f)$ whose Neron-Tate height pairing is given by*

$$\langle Z(f), Z(f)\rangle_{NT} = \frac{2\sqrt{N}}{\pi \|g\|^2} L'\big(G,1).$$

Remark 11. To ease notation in Theorems 3.1 and 3.2, we did not describe the relationship between G, g, f and $Z(f)$. Loosely speaking, they are related as follows:

(1) We have that g is a weight $3/2$ form whose image under the Shimura lift is G.
(2) The form f is selected so that its principal part is defined over the number field generated by the coefficients of G, and so that it also satisfies
$$\xi_{\frac{1}{2}}(f) = \|g\|^{-2} g.$$
(3) The Heegner divisor $Z(f)$ is defined using the principal part of f.

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