A MOCK THETA FUNCTION FOR THE DELTA-FUNCTION

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For George E. Andrews in celebration of his 70th birthday.

ABSTRACT. We define a "mock theta" function

$$M_\Delta(z) = \sum_{n=-1}^{\infty} a_\Delta(n) q^n = 11! \cdot q^{-1} - \frac{2615348736000}{691} - \cdots$$

for Ramanujan's Delta-function. Although it is not a modular form, we are able to determine its images under the Hecke operators. These images are in terms of Ramanujan's tau-function and the modular polynomials $j_m(z)$ which encode the denominator formula for the infinite dimensional Monster Lie algebra. Using these results, we obtain a criterion for Lehmer's Conjecture on the nonvanishing of $\tau(n)$. We show that $\tau(p) = 0$ if and only if $M_\Delta(z) \mid T_{-10}(p)$ is a modular form. We also relate Lehmer's Conjecture to integrality properties of the $a_\Delta(n)$.

1. Introduction and Statement of Results

As usual, let

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - \cdots \tag{1.1}$$

(note. $q := e^{2\pi i z}$ throughout) be the unique normalized weight 12 cusp form on $\mathrm{SL}_2(\mathbb{Z})$. Its coefficients, the so-called values of Ramanujan's tau-function, provide examples of some of the deepest phenomena in the theory of holomorphic modular forms. Here we show how its values also arise in the theory of non-holomorphic modular forms, the so-called *Maass forms.*

To make this precise, we first recall some classical modular forms. As usual, let

$$E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_6(z) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

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denote the normalized weight 4 and 6 Eisenstein series for the modular group $\mathrm{SL}_2(\mathbb{Z})$. Throughout, we let $\sigma_\nu(n) := \sum_{d|n} d^\nu$. Let $j(z)$ be the classical modular function

$$j(z) := \frac{E_4(z)^3}{\Delta(z)} = \sum_{n=-1}^{\infty} c(n)q^n = q^{-1} + 744 + 196884q + \cdots . \tag{1.2}$$

We now recall an important sequence of modular functions related to the Monster. Let $j_0(z) := 1$, and for every positive integer m, let $j_m(z)$ be the unique modular function which is holomorphic on $\mathbb{H}$, the upper-half of the complex plane, whose q-expansion is of the form

$$j_m(z) = q^{-m} + \sum_{n=1}^{\infty} c_m(n)q^n. \tag{1.3}$$

It is a standard fact that each $j_m(z)$ is a monic degree m polynomial in $j(z)$ with integer coefficients. The first few j_m are:

$$\begin{aligned}
j_0(z) &= 1,\\
j_1(z) &= j(z) - 744 = q^{-1} + 196884q + \cdots ,\\
j_2(z) &= j(z)^2 - 1488j(z) + 159768 = q^{-2} + 42987520q + \cdots ,\\
j_3(z) &= j(z)^3 - 2232j(z)^2 + 1069956j(z) - 36866976 = q^{-3} + 2592899910q + \cdots .
\end{aligned}$$

These polynomials are easily described using Hecke operators. For primes p and integral weights k, the Hecke operator $T_k(p)$ is defined by

$$\left(\sum a(n)q^n\right) \mid T_k(p) := \sum \left(a(np) + p^{k-1}a(n/p)\right) q^n. \tag{1.4}$$

These operators generate all of the Hecke operators, and it turns out that

$$j_m(z) = m(j_1(z) \mid T_0(m)).$$

Remark. The $j_m(z)$ have an elegant generating function (see [2, 11, 12]). If polynomials $J_m(x)$ are defined by

$$\sum_{m=0}^{\infty} J_m(x)q^m := \frac{E_4(z)^2E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z) - x} = 1 + (x - 744)q + \cdots , \tag{1.5}$$

then we have that $j_m(z) = J_m(j(z))$.

The $j_m(z)$ encode the identity

$$j(\tau) - j(z) = p^{-1} \exp\left(-\sum_{m=1}^{\infty} j_m(z) \cdot \frac{p^m}{m}\right),$$

where $p = e^{2\pi i \tau}$. It is equivalent, by a straightforward calculation, to the famous denominator formula for the Monster Lie algebra

$$j(\tau) - j(z) = p^{-1} \prod_{m>0 \text{ and } n\in\mathbb{Z}} (1 - p^m q^n)^{c(mn)}.$$

We show that these polynomials also arise in the Hecke theory of certain harmonic weak Maass forms (For more on such Maass forms and their applications, see [3, 4, 5, 6, 7, 8, 9]). We shall recall the definition of these forms in Section 2. The mock theta functions of Ramanujan provide non-trivial examples of such Maass forms (for example, see [3, 4, 17]). As an example, it turns out that

$$q^{-1} f(q^{24}) + 2i\sqrt{3} \cdot N_f(z) \tag{1.6}$$

is a weight 1/2 harmonic weak Maass form, where

$$N_f(z) := \int_{-24\overline{z}}^{i\infty} \frac{\sum_{n=-\infty}^{\infty} \left(n + \frac{1}{6}\right) e^{3\pi i \left(n+\frac{1}{6}\right)^2 \tau}}{\sqrt{-i(\tau + 24z)}} \, d\tau$$

is a period integral of a theta function, and $f(q)$ is Ramanujan's mock theta function

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

All of Ramanujan's mock theta functions are parts of such Maass forms in this way.

Since such period integrals of modular forms are non-holomorphic, we shall refer to the holomorphic projection of a harmonic weak Maass form as a *mock theta function*. Using Poincaré series, in Section 2.1 we define such a mock theta function $M_\Delta(z)$

(1.7)

$$\begin{aligned} M_\Delta(z) = \sum_{n=-1}^{\infty} a_\Delta(n) q^n &= 39916800 q^{-1} - \frac{2615348736000}{691} - \cdots \\ &= 11! \cdot q^{-1} + \frac{24 \cdot 11!}{B_{12}} - 73562460235.68364\ldots q - 929026615019.11308\ldots q^2 \\ &\quad - 8982427958440.32917\ldots q^3 - 71877619168847.70781\ldots q^4 - \cdots. \end{aligned}$$

Here $B_{12} = -691/2730$ is the 12th Bernoulli number. This function enjoys the property that

$$M_\Delta(z) + N_\Delta(z)$$

is a weight -10 harmonic weak Maass form on $\mathrm{SL}_2(\mathbb{Z})$, where $N_\Delta(z)$ is the period integral of $\Delta(z)$

$$N_\Delta(z) = (2\pi)^{11} \cdot 11i \cdot \beta_\Delta \int_{-\overline{z}}^{i\infty} \frac{\overline{\Delta(-\overline{\tau})}}{(-i(\tau + z))^{-10}} \, d\tau. \tag{1.8}$$

We shall see that $\beta_\Delta \sim 2.840287\ldots$ is the first coefficient of a certain Poincaré series.

For positive integers n, the coefficients $a_\Delta(n)$ appear to be real numbers without nice arithmetic properties. Indeed, we do not have a simple description for any of these coefficients. However, we shall show that these coefficients satisfy deep arithmetic properties thanks to the rationality of $a_\Delta(-1)$ and $a_\Delta(0)$. In particular, we shall describe the behavior of the Hecke operators on $M_\Delta(z)$ in terms of power series with rational coefficients, and we shall also be able to speak of congruences.

Although $M_\Delta(z)$ is not a modular form, for every prime p we shall show that

$$\begin{aligned} &p^{11}\cdot M_\Delta(z)\mid T_{-10}(p)-\tau(p)\cdot M_\Delta(z)\\ &\qquad\qquad=\sum_{n=-p}^{\infty}\left(p^{11}a_\Delta(pn)-\tau(p)a_\Delta(n)+a_\Delta(n/p)\right)q^n \end{aligned} \tag{1.9}$$

is a weight -10 weakly holomorphic modular form. It will turn out to be a modular form with integer coefficients. We explicitly describe these forms in terms of Ramanujan's tau-function and the modular polynomials $j_m(z)$. For convenience, if p is prime, then define the modular functions $A_p(z)$ and $B_p(z)$ by

$$A_p(z):=\frac{24}{B_{12}}(1+p^{11})+j_p(z)-264\sum_{m=1}^{p}\sigma_9(m)j_{p-m}(z), \tag{1.10}$$

$$B_p(z):=-\tau(p)\left(-264+\frac{24}{B_{12}}+j_1(z)\right). \tag{1.11}$$

Theorem 1.1. *If p is prime, then*

$$\sum_{n=-p}^{\infty}\left(p^{11}a_\Delta(pn)-\tau(p)a_\Delta(n)+a_\Delta(n/p)\right)q^n=\frac{11!}{E_4(z)E_6(z)}\cdot\left(A_p(z)+B_p(z)\right).$$

Theorem 1.1 gives many congruences for the mock theta function $M_\Delta(z)$ such as those given by the following corollary.

Corollary 1.2. *The following congruences are true:*

(1) *If p is prime, then*

$$\sum_{n=-p}^{\infty}\left(p^{11}a_\Delta(pn)-\tau(p)a_\Delta(n)+a_\Delta(n/p)\right)q^n\equiv 0 \pmod{11!}.$$

(2) *If p is prime, then*

$$\frac{1}{11!}\sum_{n=-p}^{\infty}\left(p^{11}a_\Delta(pn)-\tau(p)a_\Delta(n)+a_\Delta(n/p)\right)q^n\equiv j_p(z)-\tau(p)j_1(z) \pmod{24}.$$

Remark. Since $\frac{24}{B_{12}} = -\frac{65520}{691}$ and since the coefficients of each $j_m(z)$ are integers, it follows that the coefficients in Corollary 1.2 (1) are in $\frac{11!}{691}\mathbb{Z}$. The absence of 691 in the denominators follows from the famous Ramanujan congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

It is interesting to note that this congruence and the appearance of 691 in the formula for $a_\Delta(0)$ both arise from the fact that the numerator of B_{12} is 691.

Remark. One may simplify the right hand side of Corollary 1.2 (2) for primes $p \geq 5$. For such primes p, classical congruences of Ramanujan imply that

$$\tau(p) \equiv 1 + p \pmod{24}.$$

A famous conjecture of Lehmer asserts that $\tau(n) \neq 0$ for every positive integer n. It is well known that the truth of this conjecture would follow from the nonvanishing of $\tau(p)$ for all primes p. With this in mind, we obtain the following corollary which reinterpretes this conjecture in terms of the mock theta function $M_\Delta(z)$.

Corollary 1.3. *The following are true for all primes p.*

(1) *We have that $\tau(p) = 0$ if and only if*

$$\sum_{n=-p}^{\infty} \left(p^{11} a_\Delta(pn) + a_\Delta(n/p)\right) q^n$$

is a weight -10 modular form on $\mathrm{SL}_2(\mathbb{Z})$.

(2) *We have that $\tau(p) = 0$ if and only if*

$$\sum_{n=-p}^{\infty} \left(p^{11} a_\Delta(pn) + a_\Delta(n/p)\right) q^n = \frac{11!}{E_4(z)E_6(z)} \cdot A_p(z).$$

(3) *If $\tau(p) = 0$, then for every positive integer n coprime to p we have that $p^{11} a_\Delta(pn)$ is an integer for which*

$$p^{11} a_\Delta(pn) \equiv 0 \pmod{11!}.$$

Based on numerical experiments, we make the following conjecture which implies Lehmer's Conjecture.

Conjecture. *The coefficients $a_\Delta(n)$ are irrational for every positive integer n.*

Remark. After this paper was written, Bruinier, Rhoades and the author [10] proved general theorems relating the algebraicity of coefficients of certain integer weight harmonic weak Maass forms to the vanishing Hecke eigenvalues.

Remark. This conjecture is closely related to results obtained by the author and Bruinier on Heegner divisors and derivatives of L-functions (see [9]). In that work results were obtained on the transcendence on the coefficients of harmonic weak Maass forms. These results, combined with a well known conjecture of Goldfeld on modular L-functions, implies that "almost all" coefficients of certain weight $1/2$ harmonic weak Maass forms are transcendental, and hence irrational. This work provides further evidence supporting the plausibility of our conjecture.

In Section 2.1 we construct the mock theta function $M_\Delta(z)$ using results in earlier joint work with Bringmann [5]. In particular, we give complicated exact formulas for the coefficients of $M_\Delta(z)$, and we give a description of the real number β_Δ in (1.8). In Section 3 we prove Theorem 1.1 and Corollaries 1.2 and 1.3. In the last section we conclude with a detailed discussion of Theorem 1.1 and Corollary 1.2 when $p = 2$.

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2. Harmonic weak Maass forms and $M_\Delta(z)$

We recall the notion of a harmonic weak Maass form of integer weight k on $\mathrm{SL}_2(\mathbb{Z})$ due to Bruinier and Funke [8]. If $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$, then the weight k hyperbolic Laplacian is given by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \tag{2.1}$$

A *weight k harmonic weak Maass form on* $\mathrm{SL}_2(\mathbb{Z})$ is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and all $z \in \mathbb{H}$, we have
$$M(Az) = (cz+d)^k M(z).$$
(2) We have that $\Delta_k M = 0$.
(3) The function $M(z)$ has at most linear exponential growth at the cusp infinity.

2.1. Poincaré series and the definition of $M_\Delta(z)$.

Here we recall two relevant Poincaré series. The Poincaré series $H(z)$ will turn out to equal $\beta_\Delta \Delta(z)$, and the second Poincaré series $R(z)$ shall be of the form

$$R(z) = M_\Delta(z) + N_\Delta(z).$$

Here we recall the constructions of these Poincaré series following [5]. We rely on classical special functions whose properties and definitions may be found in [1]. Suppose that $k \in \mathbb{Z}$. For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, define $j(A, z)$ by

$$j(A, z) := (cz + d). \tag{2.2}$$

As usual, for such A and functions $f : \mathbb{H} \to \mathbb{C}$, we let

$$(f\,|_k\, A)(z) := j(A, z)^{-k} f(Az). \tag{2.3}$$

Let m be an integer, and let $\varphi_m : \mathbb{R}^+ \to \mathbb{C}$ be a function which satisfies $\varphi_m(y) = O(y^\alpha)$, as $y \to 0$, for some $\alpha \in \mathbb{R}$. If $e(\alpha) := e^{2\pi i \alpha}$ as usual, then let

$$\varphi_m^*(z) := \varphi_m(y)e(mx). \tag{2.4}$$

Such functions are fixed by the translations $\Gamma_\infty := \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \ : \ n \in \mathbb{Z}\}$. Given this data, we define the Poincaré series

$$P(m, k, \varphi_m; z) := \sum_{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (\varphi_m^* \mid_k A)(z). \tag{2.5}$$

Now define the function $H(z)$ by

$$H(z) := P(1, 12, e(iy); z) = \sum_{n=1}^{\infty} b(n)q^n. \tag{2.6}$$

It is well known that $H(z)$ is a weight 12 cusp form on $\mathrm{SL}_2(\mathbb{Z})$, and so it follows that

$$H(z) = \beta_\Delta \Delta(z),$$

where $\beta_\Delta = b(1) \sim 2.840287\dots$. It is well known that (for example, see Chapter 3 of [15]) that β_Δ may be described in terms of the Petersson norm of $H(z)$.

Now we construct the relevant weight -10 harmonic weak Maass form. Let $M_{\nu,\,\mu}(z)$ be the usual M-Whittaker function. For complex s, let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|),$$

and for positive m let $\varphi_{-m}(z) := \mathcal{M}_{1-\frac{k}{2}}(-4\pi m y)$. We now let

$$R(z) := P(-1, -10, \varphi_{-1}; z). \tag{2.7}$$

Now recall the period integral $N_\Delta(z)$ from (1.8), and define $M_\Delta(z)$ by

$$M_\Delta(z) := R(z) - N_\Delta(z). \tag{2.8}$$

We shall give an exact formula for the Fourier expansion of $M_\Delta(z)$ in terms of the classical Kloosterman sums

$$K(m, n, c) := \sum_{v(c)^\times} e\left(\frac{m\overline{v} + nv}{c}\right). \tag{2.9}$$

In the sums above, v runs through the primitive residue classes modulo c, and $\overline{v}$ denotes the multiplicative inverse of v modulo c. Theorem 1.1 of [5] (also see [13, 14] for related earlier works) then implies the following result relating $H(z), R(z)$ and $M_\Delta(z)$.

Theorem 2.1. *The following are true:*

(1) *The function $R(z)$ is a weight -10 harmonic weak Maass form on* $\mathrm{SL}_2(\mathbb{Z})$.

(2) *The function $M_{\Delta}(z)$ is holomorphic on $\mathbb{H}$, and it has a Fourier expansion of the form*

$$M_{\Delta}(z) = \sum_{n=-1}^{\infty} a_{\Delta}(n) q^n = \Gamma(12) q^{-1} - \frac{2^{12}\pi^{12}}{\zeta(12)} + \cdots,$$

where for positive integers n we have

$$a_{\Delta}(n) = -2\pi\Gamma(12) n^{-\frac{11}{2}} \cdot \sum_{c=1}^{\infty} \frac{K(-1,n,c)}{c} \cdot I_{11}\left(\frac{4\pi\sqrt{n}}{c}\right).$$

Here $I_{11}(x)$ is the usual I_{11}-Bessel function.

Remark. We note that $M_{\Delta}(z)$ begins with the terms

$$M_{\Delta}(z) = 39916800 q^{-1} - \frac{2615348736000}{691} + \cdots.$$

Obviously, we have that $a_{\Delta}(-1) = \Gamma(12) = 11!$. Using classical formulas for Riemann's zeta function at positive even integers, we find that $a_{\Delta}(0) = \frac{24 \cdot 11!}{B_{12}}$. Strictly speaking, the constant term for $M_{\Delta}(z)$ in [5] is given in terms of the infinite sum

$$\sum_{c=1}^{\infty} \frac{K(-1,0,c)}{c^{12}}.$$

A straightforward calculation relates these Kloosterman sums to $\mu(c)$, the classical Möbius function, and consequently provides the connection to the reciprocal of $\zeta(12)$.

Remark. It would be very interesting to have a simpler description of the coefficients $a_{\Delta}(n)$ for positive n.

Remark. It turns out that there are infinitely many choices for the mock theta function $M_{\Delta}(z)$. This ambiguity arises for two reasons. First of all, one may choose other descriptions of $\Delta(z)$ in terms of Poincaré series to obtain other mock theta functions. We simply selected the simplest choice. Secondly, we point out that if $F(z)$ is a weakly holomorphic modular form of weight -10 on $\mathrm{SL}_2(\mathbb{Z})$ with algebraic coefficients, then the sum

$$M_{\Delta}(z) + N_{\Delta}(z) + F(z)$$

is also a weight -10 harmonic weak Maass form, and so $M_{\Delta}(z) + F(z)$ is also a mock theta function for $\Delta(z)$. In both situations one can easily modify the arguments here to obtain the corresponding versions of Theorem 1.1 and Corollary 1.2.

3. Proofs

Here we prove Theorem 1.1 and Corollary 1.2. The proof of Theorem 1.1 relies on facts about harmonic weak Maass forms and their behavior under Hecke operators. The proofs of Corollaries 1.2 and 1.3 are elementary.

3.1. **Proof of Theorem 1.1.** The Hecke operators $T_{-10}(p)$ act independently on the holomorphic and non-holomorphic parts of $R(z)$. Since $\Delta(z)$ is a Hecke eigenform, it follows that $N_\Delta(z)$ is an eigenform of $T_{-10}(p)$ (for example, see Section 7 of [9]). In particular, since

$$\Delta(z) \mid T_{12}(p) = \tau(p)\Delta(z),$$

it then follows that

$$L_p(z) := R(z) \mid T_{-10}(p) - \tau(p)p^{-11} \cdot R(z)$$

is holomorphic on $\mathbb{H}$, and so it is a weight -10 weakly holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$. Recall that a weakly holomorphic modular form is a meromorphic modular form whose poles (if any) are supported at cusps. Obviously, we then have that

$$L_p(z) = M_\Delta(z) \mid T_{-10}(p) - \tau(p)p^{-11} \cdot M_\Delta(z). \tag{3.1}$$

Consequently, it follows that $E_4(z)E_6(z)L_p(z)$ is a weakly holomorphic modular form of weight 0. All such forms are polynomials in $j(z)$.

Now we turn to the problem of determining the exact formula for $E_4(z)E_6(z)L_p(z)$. We first note that

$$M_\Delta(z) \mid T_{-10}(p) = 11! \cdot p^{-11} \cdot q^{-p} + a_\Delta(0)(1 + p^{-11}) + O(q),$$

where $a_\Delta(0) = \frac{24 \cdot 11!}{B_{12}}$ as before. Therefore, since

$$E_4(z)E_6(z) = E_{10}(z) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n,$$

it follows that

$$\begin{aligned} &E_4(z)E_6(z) \cdot (M_\Delta(z) \mid T_{-10}(p)) \\ &\quad = 11! \cdot p^{-11} \cdot q^{-p} - 264 \cdot 11! \cdot p^{-11} \sum_{m=1}^{p-1} \sigma_9(m)q^{-p+m} \\ &\qquad\qquad - 264 \cdot 11! \cdot p^{-11}(1 + p^9) + a_\Delta(0)(1 + p^{-11}) + O(q). \end{aligned} \tag{3.2}$$

Similarly, we have that

$$-\tau(p)p^{-11} \cdot E_4(z)E_6(z)M_\Delta(z) = -\tau(p)p^{-11} \cdot \big(11! \cdot q^{-1} - 264 \cdot 11! + a_\Delta(0)\big) + O(q).$$

Combining these observations, we find that

$$\begin{aligned}
&E_4(z)E_6(z)L_p(z)\\
&\quad = 11!\cdot p^{-11}\cdot q^{-p} - 264\cdot 11!\cdot p^{-11}\sum_{m=1}^{p-1}\sigma_9(m)q^{-p+m}\\
&\qquad\qquad - 264\cdot 11!\cdot p^{-11}(1+p^9) + a_\Delta(0)(1+p^{-11})\\
&\qquad\qquad\quad - \tau(p)p^{-11}\cdot\big(11!\cdot q^{-1} - 264\cdot 11! + a_\Delta(0)\big) + O(q)\\
&\quad = 11!\cdot p^{-11}\left(q^{-p} - 264\sum_{m=1}^{p-1}\sigma_9(m)q^{-p+m} - 264(1+p^9) + \frac{a_\Delta(0)p^{11}}{11!}\cdot(1+p^{-11})\right)\\
&\qquad - \tau(p)\cdot 11!\cdot p^{-11}\cdot\left(q^{-1} - 264 + \frac{a_\Delta(0)}{11!}\right) + O(q).
\end{aligned}$$

Since every polynomial in $j(z)$ is uniquely determined by its "principal part" (i.e. the coefficients corresponding to non-positive exponents), it follows that

$$E_4(z)E_6(z)L_p(z) = \frac{11!}{p^{11}}\cdot\left(A_p(z) + B_p(z)\right).$$

Here we use the fact that each $j_m(z)$ satisfies

$$j_m(z) = q^{-m} + O(q).$$

This completes the proof.

3.2. **Proof of Corollary 1.2.** Here we prove the two claimed congruences.

(1) With exception of the constant terms, by (1.10) and (1.11) we have that all of the coefficients of $A_p(z)$ and $B_p(z)$ are integers. Since $\frac{24}{B_{12}} = -\frac{65520}{691}$, and since $E_4(z)E_6(z)$ has integer coefficients, the first claim follows immediately from Theorem 1.1 and the Ramanujan congruence

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

(2) By Theorem 1.1, we have that

$$\frac{1}{11!}\sum_{n=-p}^{\infty}\left(p^{11}a_\Delta(pn) - \tau(p)a_\Delta(n) + a_\Delta(n/p)\right)q^n = \frac{1}{E_4(z)E_6(z)}\cdot(A_p(z) + B_p(z)).$$

Moreover, its coefficients are in $\mathbb{Z}$ by part (1). Therefore, we may reduce this modulo 24. Since $E_4(z)E_6(z) \equiv 1 \pmod{24}$, by (1.10) and (1.11) we then find that

$$\begin{aligned}
\frac{1}{11!}\sum_{n=-p}^{\infty}\left(a_\Delta(pn) - \tau(p)a_\Delta(n) + a_\Delta(n/p)\right)q^n &\equiv A_p(z) + B_p(z) \pmod{24}\\
&\equiv j_p(z) - \tau(p)j_1(z) \pmod{24}.
\end{aligned}$$

This gives the second claim.

3.3. **Proof of Corollary 1.3.** Theorem 1.1, combined with the fact that $M_\Delta(z)$ is not modular, gives the first claim. Theorem 1.1 and (1.11) immediately implies the second claim. The third claim now follows from Corollary 1.2 (1).

4. Examples

Here we give a detailed discussion of Theorem 1.1 and Corollary 1.2 when $p = 2$. In particular, we use the exact formulas for the coefficients of $M_\Delta(z)$ to numerically approximate the theoretical exact formulas for $M_\Delta(z) \mid T_{-10}(2)$ and compare them with the exact formulas we derive using just the first two coefficients of $M_\Delta(z)$.

We begin by giving the numerical approximations for the first few terms of $M_\Delta(z)$. Using Theorem 2.1, we find that

$$\begin{aligned} M_\Delta(z) = \sum_{n=-1}^{\infty} a_\Delta(n)q^n &= 39916800q^{-1} - \frac{2615348736000}{691} \\ &- 73562460235.68364\ldots q - 929026615019.11308\ldots q^2 \\ &- 8982427958440.32917\ldots q^3 - 71877619168847.70781\ldots q^4 \\ &- 497966668914961.54321\ldots q^5 - 3074946857439412.02739\ldots q^6 - \cdots \end{aligned}$$

Again we stress that $a_\Delta(-1)$ and $a_\Delta(0)$ are exact.

4.1. **Theorem 1.1 when $p = 2$.** .

Using Theorem 2.1, one numerically finds that

(4.1)
$$\begin{aligned} M_\Delta(z) \mid T_{-10}(2) - \tau(2)2^{-11} \cdot M_\Delta(z) = \frac{155925}{8} \cdot q^{-2} + 467775q^{-1} - 3831077250 \\ - 929888675100.00000\ldots q - 71888542118662.49999\ldots q^2 \\ - 3075052120267049.99999\ldots q^3 - \cdots . \end{aligned}$$

The first three terms are exact.

The modular functions $A_2(z)$ and $B_2(z)$ are:

$$\begin{aligned} A_2(z) &= j_2(z) - 264j_1(z) - \frac{227833992}{691}, \\ B_2(z) &= 24j_1(z) - \frac{5950656}{691}. \end{aligned}$$

Therefore, Theorem 1.1 implies that

$$\begin{aligned} M_\Delta(z) \mid T_{-10}(2) - \tau(2)2^{-11} \cdot M_\Delta(z) &= \frac{11!}{2^{11}E_4(z)E_6(z)} \cdot (A_2(z) + B_2(z)) \\ &= \frac{155925(j_2(z) - 240j_1(z) - 338328)}{8E_4(z)E_6(z)} \\ &= \frac{155925}{8} \cdot q^{-2} + 467775q^{-1} - 3831077250 - 929888675100q \\ &\quad - \frac{143777084237325}{2} \cdot q^2 - 3075052120267050q^3 - \cdots . \end{aligned}$$

This agrees with the numerics in (4.1).

4.2. **Corollary 1.2 when $p = 2$.** Using the calculations from the previous subsection, we have that

$$M_\Delta(z) \mid T_{-10}(2) - \tau(2)2^{-11}M_\Delta(z) = \frac{155925(j_2(z) - 240j_1(z) - 338328)}{8E_4(z)E_6(z)}.$$

The coefficients are easily seen to lie in $\frac{1}{8}\mathbb{Z}$. This illustrates Corollary 1.2 (1).

To illustrate Corollary 1.2 (2), simply multiply both sides of this expression by $2^{11}/11!$ to obtain

$$\begin{aligned} \frac{2^{11}\left(M_\Delta(z) \mid \ - \tau(2)2^{-11}M_\Delta(z)\right)}{11!} &= \frac{j_2(z) - 240j_1(z) - 338328}{E_4(z)E_6(z)} \\ &\equiv j_2(z) - \tau(2)j_1(z) \\ &\equiv j_2(z) \pmod{24}. \end{aligned}$$

Here we used $\tau(2) = -24$ and $E_4(z)E_6(z) \equiv 1 \pmod{24}$.

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