HOOK LENGTHS AND 3-CORES

GUO-NIU HAN AND KEN ONO

Abstract. Recently, the first author generalized a formula of Nekrasov and Okounkov which gives a combinatorial formula, in terms of hook lengths of partitions, for the coefficients of certain power series. In the course of this investigation, he conjectured that $a(n) = 0$ if and only if $b(n) = 0$, where integers $a(n)$ and $b(n)$ are defined by

$$\sum_{n=0}^{\infty} a(n)x^n := \prod_{n=1}^{\infty}(1-x^n)^8,$$
$$\sum_{n=0}^{\infty} b(n)x^n := \prod_{n=1}^{\infty}\frac{(1-x^{3n})^3}{1-x^n}.$$

The numbers $a(n)$ are given in terms of hook lengths of partitions, while $b(n)$ equals the number of 3-core partitions of n. Here we prove this conjecture.

1. Introduction and statement of results

In their work on random partitions and Seiberg-Witten theory, Nekrasov and Okounkov [8] proved the following striking formula:

$$F_z(x) := \sum_{\lambda} x^{|\lambda|} \prod_{h\in\mathcal{H}(\lambda)} \left(1-\frac{z}{h^2}\right) = \prod_{n=1}^{\infty}(1-x^n)^{z-1}. \tag{1.1}$$

Here the sum is over integer partitions λ, $|\lambda|$ denotes the integer partitioned by λ, and $\mathcal{H}(\lambda)$ denotes the multiset of classical hooklengths associated to a partition λ. In a recent preprint, the first author [3] has obtained an extension of (1.1), one which has a specialization which gives the classical generating function

$$C_t(x) := \sum_{n=0}^{\infty} c_t(n)x^n = \prod_{n=1}^{\infty}\frac{(1-x^{tn})^t}{1-x^n} \tag{1.2}$$

for the number of t-core partitions of n. Recall that a partition is a *t-core* if none of its hook lengths are multiples of t.

In the course of his work, the first author [4] formulated a number of conjectures concerning hook lengths of partitions. One of these conjectures is related to classical identities of Jacobi. For positive integers t, he compared the functions $F_{t^2}(x)$ and $C_t(x)$. If $t = 1$, we obviously have that

$$F_1(x) = C_1(x) = 1.$$

The second author thanks the support of the NSF, and he thanks the Manasse family.

For $t = 2$, by two famous identities of Jacobi, we have

$$F_4(x) = \prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)x^{(k^2+k)/2},$$
$$C_2(x) = \prod_{n=1}^{\infty}\frac{(1-x^{2n})^2}{1-x^n} = \sum_{k=0}^{\infty}x^{(k^2+k)/2}.$$

In both pairs of power series one sees that the non-zero coefficients are supported on the same terms. For $t = 3$, we then have

$$\begin{aligned}(1.3)\qquad F_9(x) = \sum_{n=0}^{\infty}a(n)x^n &:= \prod_{n=1}^{\infty}(1-x^n)^8 \\ &= 1 - 8x + 20x^2 - 70x^4 + \cdots - 520x^{14} + 57x^{16} + 560x^{17} + 182x^{20} + \cdots\end{aligned}$$

and

$$\begin{aligned}(1.4)\qquad C_3(x) = \sum_{n=0}^{\infty}b(n)x^n &:= \prod_{n=1}^{\infty}\frac{(1-x^{3n})^3}{1-x^n} \\ &= 1 + x + 2x^2 + 2x^4 + \cdots + 2x^{14} + 3x^{16} + 2x^{17} + 2x^{20} + \cdots .\end{aligned}$$

Remark. It is clear that $b(n) = c_3(n)$.

In accordance with the elementary observations when $t = 1$ and 2, one notices that the non-zero coefficients of $F_9(x)$ and $C_3(x)$ appear to be supported on the same terms. Based on substantial numerical evidence, the first author made the following conjecture.

Conjecture 4.6. (Conjecture 4.6 of [4])
Assuming the notation above, we have that $a(n) = 0$ if and only if $b(n) = 0$.

Remark. The obvious generalization of Conjecture 4.6 and the examples above is not true for $t = 4$. In particular, one easily finds that

$$F_{16}(x) = 1 - 15x + 90x^2 - \cdots + 641445x^{52} + 1537330x^{54} + \cdots ,$$
$$C_4(x) = 1 + x + 2x^2 + 3x^3 + \cdots + 5x^{52} + 8x^{53} + 10x^{54} + \cdots .$$

The coefficient of x^{53} vanishes in $F_{16}(x)$ and is non-zero in $C_4(x)$.

Here we prove that Conjecture 4.6 is true. We have the following theorem.

Theorem 1.1. *Assuming the notation above, we have that $a(n) = 0$ if and only if $b(n) = 0$. Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative n for which $\mathrm{ord}_p(3n+1)$ is odd for some prime $p \equiv 2 \pmod 3$.*

Remark. As usual, $\mathrm{ord}_p(N)$ denotes the largest power of a prime p dividing an integer N.

Remark. Theorem 1.1 shows that $a(n) = b(n) = 0$ in a systematic way. The vanishing coefficients are associated to primes $p \equiv 2 \pmod 3$. If $n \equiv 1 \pmod 3$ has the property that $\mathrm{ord}_p(n)$ is odd, then we have

$$a\left(\frac{n-1}{3}\right) = b\left(\frac{n-1}{3}\right) = 0.$$

For example, since $\mathrm{ord}_5(10) = 1 \equiv 1 \pmod 2$, we have that $a(3) = b(3) = 0$.

As an immediate corollary, we have the following.

Corollary 1.2. *For positive integers N, we have that*

$$\sum_{\lambda \vdash N} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{9}{h^2}\right) = 0$$

if and only if there are no 3-core partitions of N.

Theorem 1.1 implies that "almost all" of the $a(n)$ and $b(n)$ are 0. More precisely, we have the following.

Corollary 1.3. *Assuming the notation above, we have that*

$$\lim_{X \to +\infty} \frac{\#\{0 \le n \le X \ : \ a(n) = b(n) = 0\}}{X} = 1.$$

Acknowledgements

The authors thank Mihai Cipu for insightful comments related to Conjecture 4.6.

2. Proofs

It is convenient to renormalize the functions $a(n)$ and $b(n)$ using the series

$$\begin{aligned} \mathcal{A}(z) &= \sum_{n=1}^{\infty} a^*(n)q^n := \sum_{n=0}^{\infty} a(n)q^{3n+1} \\ &= q - 8q^4 + 20q^7 - 70q^{13} + 64q^{16} + 56q^{19} - 125q^{25} - 160q^{28} + \cdots . \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \mathcal{B}(z) &= \sum_{n=1}^{\infty} b^*(n)q^n := \sum_{n=0}^{\infty} b(n)q^{3n+1} \\ &= q + q^4 + 2q^7 + 2q^{13} + q^{16} + 2q^{19} + q^{25} + 2q^{28} + \cdots . \end{aligned} \tag{2.2}$$

Here we have that $z \in \mathbb{H}$, the upper-half of the complex plane, and we let $q := e^{2\pi i z}$. We make these changes since $\mathcal{A}(z)$ and $\mathcal{B}(z)$ are examples of two special types of modular forms (for background on modular forms, see [1, 6, 7, 9]). The modularity of these two series follows easily from the properties of Dedekind's eta-function

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.3}$$

The proofs of Theorem 1.1 and Corollary 1.3 shall rely on exact formulas we derive for the numbers $a^*(n)$ and $b^*(n)$.

2.1. **Exact formulas for $a^*(n)$.** The modular form $\mathcal{A}(z)$ given by

$$\mathcal{A}(z) = \eta(3z)^8 = \sum_{n=1}^{\infty} a^*(n) q^n$$

is in $S_4(\Gamma_0(9))$, the space of weight 4 cusp forms on $\Gamma_0(9)$. This space is one dimensional (see Section 1.2.3 in [9]). Therefore, every cusp form in the space is a multiple of $\mathcal{A}(z)$. It turns out that $\mathcal{A}(z)$ is a form with *complex multiplication*.

We now briefly recall the notion of a newform with complex multiplication (for example, see Chapter 12 of [6] or Section 1.2 of [9], [10]). Let $D < 0$ be the fundamental discriminant of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$. Let O_K be the ring of integers of K, and let $\chi_K := \left(\frac{D}{\bullet}\right)$ be the usual Kronecker character associated to K. Let $k \geq 2$, and let c be a Hecke character of K with exponent $k-1$ and conductor $\mathfrak{f}_c$, a non-zero ideal of O_K. By definition, this means that

$$c : I(\mathfrak{f}_c) \longrightarrow \mathbb{C}^{\times}$$

is a homomorphism, where $I(\mathfrak{f}_c)$ denotes the group of fractional ideals of K prime to $\mathfrak{f}_c$. In particular, this means that

$$c(\alpha O_K) = \alpha^{k-1}$$

for $\alpha \in K^{\times}$ for which $\alpha \equiv 1 \bmod^{\times} \mathfrak{f}_c$. To c we naturally associate a Dirichlet character ω_c defined, for every integer n coprime to $\mathfrak{f}_c$, by

$$\omega_c(n) := \frac{c(nO_K)}{n^{k-1}}.$$

Given this data, we let

$$\Phi_{K,c}(z) := \sum_{\mathfrak{a}} c(\mathfrak{a}) q^{N(a)}, \tag{2.4}$$

where $\mathfrak{a}$ varies over the ideals of O_K prime to $\mathfrak{f}_c$, and where $N(\mathfrak{a})$ is the usual ideal norm. It is known that $\Phi_{K,c}(z) \in S_k(\Gamma_0(|D| \cdot N(\mathfrak{f}_c)), \chi_K \cdot \omega_c)$ is a normalized newform.

Using this theory, we obtain the following theorem.

Theorem 2.1. *Assume the notation above. Then the following are true:*

(1) *If $p = 3$ or $p \equiv 2 \pmod 3$ is prime, then $a^*(p) = 0$.*
(2) *If $p \equiv 1 \pmod 3$ is prime, then*

$$a^*(p) = 2x^3 - 18xy^2,$$

where x and y are integers for which $p = x^2 + 3y^2$ with $x \equiv 1 \pmod 3$.

Remark. It is a classical fact that every prime $p \equiv 1 \pmod 3$ is of the form $x^2 + 3y^2$. Moreover, there is a unique pair of positive integers x and y for which $x^2 + 3y^2 = p$. Therefore, the formula for $a^*(p)$ is well defined.

Proof. There is a form with complex multiplication in $S_4(\Gamma_0(9))$. Following the recipe above, it is obtained by letting $k = 4$, $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\sqrt{-3})$ and $\mathfrak{f}_c := (\sqrt{-3})$. For primes p, the coefficients of q^p in this form agree with the claimed formulas. Since $S_4(\Gamma_0(9))$ is one dimensional, this form must be $\mathcal{A}(z)$. □

Using this theorem, we obtain the following immediate corollary.

Corollary 2.2. *The following are true about* $a^*(n)$*.*

(1) *If m and n are coprime positive integers, then*
$$a^*(mn) = a^*(m)a^*(n).$$
(2) *For every positive integer s, we have that $a^*(3^s) = 0$.*
(3) *If $p \equiv 2 \pmod 3$ is prime and s is a positive integer, then*
$$a^*(p^s) = \begin{cases} 0 & \text{if } s \text{ is odd,} \\ (-1)^{s/2}p^{3s/2} & \text{if } s \text{ is even.} \end{cases}$$
(4) *If $p \equiv 1 \pmod 3$ is prime and s is a positive integer, then $a^*(p^s) \neq 0$. Moreover, we have that*
$$a^*(p^s) \equiv (8x^3)^s \pmod p,$$
where $p = x^2 + 3y^2$ with $x \equiv 1 \pmod 3$.

Proof. Since $S_4(\Gamma_0(9))$ is one dimensional and since $a^*(1) = 1$, it follows that $\mathcal{A}(z)$ is a normalized Hecke eigenform. Claim (1) is well known to hold for all normalized Hecke eigenforms.

Claim (2) follows by inspection since $a^*(n) = 0$ if $n \equiv 0, 2 \pmod 3$.

To prove claims (3) and (4), we note that since $\mathcal{A}(z)$ is a normalized Hecke eigenform on $\Gamma_0(9)$, it follows, for every prime $p \neq 3$, that

$$a^*(p^s) = a^*(p)a^*(p^{s-1}) - p^3a^*(p^{s-2}). \tag{2.5}$$

If $p \equiv 2 \pmod 3$ is prime, then Theorem 2.1 implies that
$$a^*(p^s) = -p^3a(p^{s-2}).$$
Claim (3) now follows by induction since $a^*(1) = 1$ and $a^*(p) = 0$.

Suppose that $p \equiv 1 \pmod 3$ is prime. By Theorem 2.1, we know that $a^*(p) \neq 0$. More importantly, we have that
$$a^*(p) \equiv 8x^3 \pmod p,$$
where $p = x^2 + 3y^2$ with $x \equiv 1 \pmod 3$. To see this, one merely observes that
$$2x^3 - 18xy^2 = 2x(x^2 - 9y^2) = 2x(x^2 - 3(p - x^2)) \equiv 8x^3 \pmod p.$$
Since $|x| \leq \sqrt{p}$ and is non-zero, it follows that $a^*(p) \equiv 8x^3 \not\equiv 0 \pmod p$. By (2.5), we then have that
$$a^*(p^s) \equiv a^*(p)a^*(p^{s-1}) \equiv 8x^3a^*(p^{s-1}) \pmod p.$$
By induction, it follows that $a^*(p^s) \equiv (8x^3)^s \pmod p$, which is non-zero modulo p. This proves claim (4). □

Example 2.3. Here we give some numerical examples of the formulas for $a^*(n)$.
1) One easily finds that $a^*(13) = -70$. The prime $p = 13$ is of the form $x^2 + 3y^2$ where $x = 1$ and $y = 2$. Obviously, $x = 1 \equiv 1 \pmod 3$, and so Theorem 2.1 asserts that $a^*(13) = 2 \cdot 1^3 - 18 \cdot 1 \cdot 2^2 = -70$.

2) We have that $a^*(13) = -70$ and $a^*(16) = 64$. One easily checks that $a^*(13 \cdot 16) = a^*(208) = -70 \cdot 64 = -4480$. This is an example of Corollary 2.2 (1).

3) If $p = 5$ and $s = 3$, then Corollary 2.2 (3) asserts that $a^*(5^3) = 0$. If $p = 5$ and $s = 4$, then it asserts that $a^*(5^4) = 5^6 = 15625$. One easily checks both evaluations numerically.

4) Now we consider the prime $p = 13 \equiv 1 \pmod 3$. Since $x = 1$ and $y = 2$ for $p = 13$, Corollary 2.2 (4) asserts that $a^*(13^s) \equiv 8^s \pmod{13}$. One easily checks that

$$\begin{aligned} a^*(13) &= -70 \equiv 8 \pmod{13}, \\ a^*(13^2) &= 2703 \equiv 8^2 \pmod{13}, \\ a^*(13^3) &= -35420 \equiv 8^3 \pmod{13}. \end{aligned}$$

2.2. **Proof of Theorem 1.1 and Corollary 1.3.** Before we prove Theorem 1.1, we recall a known formula for $b(n)$ (also see Section 3 of [2]), the number of 3-core partitions of n.

Lemma 2.4. *Assuming the notation above, we have that*

$$\mathcal{B}(z) = \sum_{n=1}^{\infty} b^*(n) q^n = \sum_{n=0}^{\infty} b(n) q^{3n+1} = \sum_{n=0}^{\infty} \sum_{d \mid 3n+1} \left(\frac{d}{3}\right) q^{3n+1},$$

where $\left(\frac{\bullet}{3}\right)$ denotes the usual Legendre symbol modulo 3.

Proof. We have that $\mathcal{B}(z) = \eta(9z)^3/\eta(3z)$ is in $M_1(\Gamma_0(9), \chi)$, where $\chi := \left(\frac{-3}{\bullet}\right)$. The lemma follows easily from this fact. One may implement the theory of weight 1 Eisenstein series to obtain the desired formulas.

Alternatively, one may use the weight 1 form

$$\Theta(z) = \sum_{n=0}^{\infty} c(n) q^n := \sum_{x,y \in \mathbb{Z}} q^{x^2+xy+y^2} = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + \cdots.$$

Using the theory of twists, we find that

$$\begin{aligned} \widetilde{\Theta}(z) = \sum_{n \equiv 1 \pmod 3} c(n) q^n &= 6q + 6q^4 + 12q^7 + 12q^{13} + 6q^{16} + 12q^{19} + 6q^{25} + \cdots \\ &= 6\left(q + q^4 + 2q^7 + 2q^{13} + q^{16} + 2q^{19} + q^{25} + \cdots\right). \end{aligned}$$

By dimensionality (see Section 1.2.3 of [9]) we have that $\mathcal{B}(z) = \frac{1}{6}\widetilde{\Theta}(z)$. The claimed formulas for the coefficients follows easily from the fact that $x^2 + xy + y^2$ corresponds to the norm form on the ring of integers of $\mathbb{Q}(\sqrt{-3})$.

□

Example 2.5. The only divisors of primes $p \equiv 1 \pmod 3$ are 1 and p, and so we have that $b^*(p) = 1 + \left(\frac{p}{3}\right) = 1 + \left(\frac{1}{3}\right) = 2$.

Proof of Theorem 1.1. The theorem follows immediately from Theorem 2.1, Corollary 2.2 and Lemma 2.4. One sees that the only $n \equiv 1 \pmod 3$ for which $a^*(n) = 0$ are those n for which $\mathrm{ord}_p(n)$ is odd for some prime $p \equiv 2 \pmod 3$. The same conclusion holds for $b^*(n)$. Using the fact that

$$a(n) = a^*(3n+1) \quad \text{and} \quad b(n) = b^*(3n+1),$$

the theorem follows. □

Proof of Corollary 1.3. In a famous paper [11], Serre proved that "almost all" of the coefficients of a modular form with complex multiplication are zero. This implies that almost all of the $a^*(n)$ are zero. The result now follows thanks to Theorem 1.1. □

References

[1] D. Bump, *Automorphic forms and representations*, Cambridge Univ. Press, Cambridge, 1998.

[2] A. Granville and K. Ono, *Defect zero p-blocks for finite simple groups*, Trans. Amer. Math. Soc. **348** (1996), no. 1, 331–347.

[3] G.-N. Han, *The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications*, arXiv:0805.1398 [math.CO]

[4] G.-N. Han, *Some conjectures and open problems on partition hook lengths*, Experimental Math., in press.

[5] E. Hecke, *Mathematische Werke*, Vandenhoeck & Ruprecht, Third edition, Göttingen, 1983.

[6] H. Iwaniec, *Topics in classical automorphic forms*, Grad. Studies in Math. **17**, Amer. Math. Soc., Providence, RI., 1997.

[7] T. Miyake, *Modular forms*, Springer-Verlag, Berlin, 2006.

[8] N. A. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, The unity of mathematics, Progr. Math. **244**, Birkhauser, Boston, 2006, pages 525-596.

[9] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, **102**, Amer. Math. Soc., Providence, RI, 2004.

[10] K. Ribet, *Galois representations attached to eigenforms with Nebentypus*, Springer Lect. Notes in Math. **601**, (1977), pages 17-51.

[11] J.-P. Serre, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. No. 54 (1981), 323–401.

I.R.M.A., UMR 7501, Université Louis Pasteur et CNRS, 7 rue René-Descartes, F-67084 Strasbourg, France

E-mail address: guoniu@math.u-strasbg.fr

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

E-mail address: ono@math.wisc.edu