PARITY OF THE PARTITION FUNCTION

KEN ONO

ABSTRACT. Although much is known about the partition function, little is known about its parity. For the polynomials $\mathfrak{D}(x) := (Dx^2+1)/24$, where $D \equiv 23 \pmod{24}$, we show that there are infinitely many m (resp. n) for which $p(\mathfrak{D}(m))$ is even (resp. $p(\mathfrak{D}(n))$ is odd) if there is at least one such m (resp. n). We bound the first m and n (if any) in terms of the class number $h(-D)$. For prime D we show that there are indeed infinitely many even values. To this end we construct new modular generating functions using *generalized Borcherds products*, and we employ Galois representations and locally nilpotent Hecke algebras.

1. INTRODUCTION AND STATEMENT OF RESULTS

A *partition* of a non-negative integer n is a non-increasing sequence of positive integers that sum to n. Ramanujan investigated [43, 44] $p(n)$, the number of partitions of n, and he discovered congruences in certain very special arithmetic progressions such as:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5},\\ p(7n+5) &\equiv 0 \pmod{7},\\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{1.1}$$

Ramanujan's congruences have inspired many works (for example, see [2, 3, 4, 5, 6, 7, 8, 9, 15, 19, 20, 24, 26, 28, 31, 32, 39, 51] to name a few). Atkin [6] and Watson [51] notably proved Ramanujan's conjectures concerning congruences mod powers of 5, 7 and 11, while Andrews and Garvan unearthed [5] Dyson's [19] elusive *crank*, thereby providing a procedure for dividing the partitions into 5, 7, and 11 groups of equal size in (1.1).

In the 1960s, Atkin [7] surprisingly discovered further congruences. His work revealed some multiplicative properties which imply monstrous congruences such as:

$$p(1977147619n + 815655) \equiv 0 \pmod{19}.$$

Ten years ago the author revisited Atkin's examples in the context of the Deligne-Serre theory of ℓ-adic Galois representations and Shimura's theory of half-integral weight modular forms [39]. Armed with these powerful tools, the author established, for primes $m \geq 5$, that there always are such congruences mod m. Subsequently, Ahlgren and the author [2, 3] extended this to include all moduli m coprime to 6. Mahlburg [29] has recently explained the role of Dyson's crank within this framework.

Surprisingly, there do not seem to be any such congruences modulo 2 or 3. In fact, the parity of $p(n)$ seems to be quite random, and it is widely believed that the partition function is "equally often" even and odd. More precisely, Parkin and Shanks [42] conjectured that

$$\#\{n \leq X \ : \ p(n) \text{ is even}\} \sim \frac{1}{2}X.$$

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Although there have been many works (for example, see [1, 10, 11, 14, 21, 22, 23, 27, 30, 33, 34, 35, 36, 37, 38, 42, 49], to name a few) on the parity of $p(n)$, we are very far from proving this conjecture. Indeed, the best published estimates do not even preclude the possibility that

$$\lim_{X\to+\infty} \frac{\#\{n \leq X \ : \ p(n) \text{ is even (resp. odd)}\}}{X^{\frac{1}{2}+\epsilon}} = 0.$$

It is simple to identify one serious obstacle when it comes to parity. Virtually every result on congruences has relied on the reductions mod powers of m of the generating functions

$$F(m,k;q) := \sum_{\substack{D\geq 0 \\ m^kD\equiv -1 \pmod{24}}} p\left(\frac{Dm^k+1}{24}\right) q^D. \tag{1.2}$$

Indeed, Ramanujan initiated this line of reasoning with his identities:

$$F(5,1;q) := \sum_{n=0}^{\infty} p(5n+4)q^{24n+19} = 5q^{19}\prod_{n=1}^{\infty}\frac{(1-q^{120n})^5}{(1-q^{24n})^6},$$
$$F(7,1;q) := \sum_{n=0}^{\infty} p(7n+5)q^{24n+17} = 7q^{17}\prod_{n=1}^{\infty}\frac{(1-q^{168n})^3}{(1-q^{24n})^4} + 49q^{41}\prod_{n=1}^{\infty}\frac{(1-q^{168n})^7}{(1-q^{24n})^8}$$

which immediately imply the congruences mod 5 and 7 in (1.1). For primes $m \geq 5$, the $F(m,k;q)$ are congruent modulo powers of m to half-integral weight cusp forms [2, 3, 39]. Unfortunately, $F(2,k;q)$ is identically zero, and this fact has sabotaged attempts to include parity in the general framework.

We construct new generating functions using the *generalized Borcherds products* of the author and Bruinier [16]. If $1 < D \equiv 23 \pmod{24}$ is square-free and $q := e^{2\pi i z}$, then we let

$$\widehat{F}(D;z) := \sum_{\substack{m\geq 1 \\ \gcd(m,6)=1}} p\left(\frac{Dm^2+1}{24}\right) \sum_{\substack{n\geq 1 \\ \gcd(n,D)=1}} q^{mn}. \tag{1.3}$$

Three remarks.

(i) The generating functions $F(m,k;q)$ and $\widehat{F}(D;z)$ are quite different. The series $F(m,k;q)$ are power series over D, while the $\widehat{F}(D;z)$ are power series in m and n. The roles of D and m are switched, and a new parameter n occurs in $\widehat{F}(D;z)$.

(ii) Each partition number $p(n)$, apart from $p(0) = 1$, appears in a single $\widehat{F}(D;z)$ since $24n - 1 = Dm^2$ uniquely determines the square-free $1 < D \equiv 23 \mod 24$.

(iii) If we let $U(r)$ be Atkin's U-operator

$$\left(\sum a(n)q^n\right) \mid U(r) := \sum a(rn)q^n,$$

then for $r = 2$ and 3 we have

$$\widehat{F}(D;z) \mid U(2) = \widehat{F}(D;z) \mid U(3) = \widehat{F}(D;z).$$

Throughout we assume that $D \equiv 23 \pmod{24}$ is a positive square-free integer. To accompany the modular generating functions $F(m,k;q)$, we prove the following for $\widehat{F}(D;z)$.

Theorem 1.1. *We have that $\widehat{F}(D;z)$ is congruent mod 2 to a weight 2 meromorphic modular form on $\Gamma_0(6)$ with integer coefficients whose poles are simple and are supported on discriminant $-D$ Heegner points.*

Using these modular forms, we obtain the following theorem on the parity of $p(n)$.

Theorem 1.2. *If $h(-D)$ is the class number of $\mathbb{Q}(\sqrt{-D})$, then the following are true:*

(1) *There are infinitely many m coprime to 6 for which $p\left(\frac{Dm^2+1}{24}\right)$ is even if there is at least one such m. Furthermore, the first one (if any) is bounded by*
$$(12h(-D)+2)\prod_{p\mid D \ \text{prime}}(p+1).$$
(2) *There are infinitely many n coprime to 6 for which $p\left(\frac{Dn^2+1}{24}\right)$ is odd if there is at least one such n. Furthermore, the first one (if any) is bounded by $12h(-D)+2$.*

We have been unable to preclude the possibility that $p\left(\frac{Dm^2+1}{24}\right)$ is even (resp. odd) for every m coprime to 6. Although these possibilities seem to be consistent with the theory of modular forms, we make the following conjecture.

Conjecture. *There are infinitely many m (resp. n) coprime to 6 for which $p\left(\frac{Dm^2+1}{24}\right)$ is even (resp. $p\left(\frac{Dn^2+1}{24}\right)$ is odd).*

Remark. We have confirmed this conjecture for all $D < 25000$.

In the direction of this conjecture, we offer the following results.

Theorem 1.3. *Suppose that D_0 is a square-free integer which is coprime to 6, and suppose that $\ell \nmid D_0$ is a prime which satisfies:*

(1) *We have that $D_0\ell \equiv 23 \pmod{24}$.*
(2) *We have that*
$$\frac{\ell}{12h(-D_0\ell)+2} > \prod_{p\mid D_0 \ \text{prime}}(p+1).$$
Then there are infinitely many integers m coprime to 6 for which $p\left(\frac{D_0\ell m^2+1}{24}\right)$ is even.

For a fixed D_0 in Theorem 1.3, the bound
$$h(-D_0\ell) \ll (D_0\ell)^{\frac{1}{2}}\log(D_0\ell)$$
implies that the conclusion of Theorem 1.3 holds for all but finitely many primes ℓ for which $D_0\ell \equiv 23 \pmod{24}$. For $D_0 = 1$ we have the following corollary.

Corollary 1.4. *If $\ell \equiv 23 \pmod{24}$ is prime, then there are infinitely many m coprime to 6 for which $p\left(\frac{\ell m^2+1}{24}\right)$ is even. Moreover, the first such m is bounded by $12h(-\ell)+2$.*

Remark. The proof of Corollary 1.4 implies that there is an

$$m \leq 12(h(-\ell)+2) \ll \ell^{\frac{1}{2}} \log(\ell)$$

for which $p\left(\frac{\ell m^2+1}{24}\right)$ is even. This then implies the following estimate:

$$\#\{N \leq X \ : \ p(N) \text{ is even}\} \gg X^{\frac{1}{2}}/\log(X).$$

This paper is organized as follows. In §2 we consider the combinatorial properties of the generating function for $p(n)$, Ramanujan's mock theta functions, and a certain vector-valued harmonic Maass form. We then give generalized Borcherds products which arise from this harmonic Maass form, and we prove Theorem 1.1. In §3 we combine the local nilpotency of certain Hecke algebras with the Chebotarev Density Theorem, the properties of 2-adic Galois representations, and some combinatorial arguments to prove Theorem 1.2. We then conclude with the proof of Theorem 1.3 and Corollary 1.4 using bounds for class numbers.

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2. Generalized Borcherds products and the New generating functions

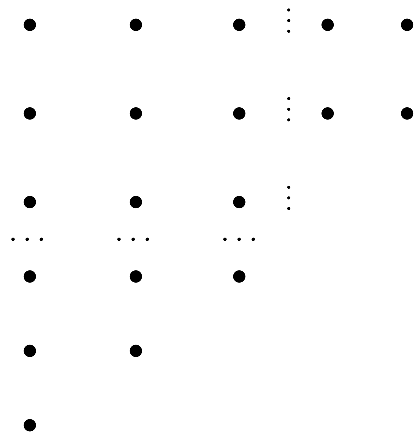
Here we relate the generating functions $\widehat{F}(D;z)$ to meromorphic modular forms produced from generalized Borcherds products, and we prove Theorem 1.1.

2.1. **Combinatorial considerations.** Euler is credited for observing that

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{q^{1/24}}{\eta(z)},$$

where $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's weight 1/2 modular form. The generating functions $F(m,k;q)$ arise naturally from this modularity.

A less well known identity for $P(q)$ leads to $\widehat{F}(D;z)$. Although it is well known to specialists, we recall it to highlight the special combinatorial properties of certain harmonic Maass forms. As usual, represent a partition $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ by a *Ferrers diagram*, a left justified array of dots consisting of k rows in which there are λ_i dots in the ith row. The *Durfee square* is the largest square of nodes in the upper left hand corner of the diagram. Its boundary divides a partition into a square and two partitions whose parts do not exceed the side length of the square. For example, consider the partition $5 + 5 + 3 + 3 + 2 + 1$:



This partition decomposes as a square of size 9, and the two partitions 2+2, and 3+2+1.

We have the following alternate identity for $P(q)$, which in turn provides a crucial congruence between $P(q)$ and Ramanujan's third order mock theta function

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}. \tag{2.1}$$

Lemma 2.1. *The following combinatorial identity is true:*

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^m)^2}.$$

In particular, we have that $P(q) \equiv f(q) \pmod 2$.

Proof. For every positive integer m, the q-series

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)} = \sum_{n=0}^{\infty} a_m(n)q^n$$

is the generating function for $a_m(n)$, the number of partitions of n whose summands do not exceed m. Therefore by the discussion above, the q-series

$$\frac{q^{m^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^m)^2} = \sum_{n=0}^{\infty} b_m(n)q^n$$

is the generating function for $b_m(n)$, the number of partitions of n with a Durfee square of size m^2. The identity follows by summing in m, and the claimed congruence follows trivially. $\square$

Remark. It is also easy to see that $f(q) \equiv P(q) \pmod 4$.

2.2. **A generalized Borcherds product.** In their work on derivatives of modular L-functions, the author and Bruinier [16] produced *generalized Borcherds Products* arising from weight 1/2 harmonic Maass forms. These products give generalizations of some of the automorphic infinite products obtained by Borcherds [12, 13]. The general result (see Theorems 6.1 and 6.2 of [16]) gives modular forms with twisted Heegner divisor whose infinite product expansions arise from harmonic Maass forms.

Ramanujan's mock theta functions, which are special examples of harmonic Maass forms of weight 1/2 (for example, see [40, 52, 53, 54]), can be used to construct such products. We recall one example which involves the third order mock theta functions $f(q)$ (see (2.1)) and

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} = \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} + \frac{q^{12}}{(1-q)^2(1-q^3)^2(1-q^5)^2} + \cdots. \tag{2.2}$$

It is important to note that $f(q)$ and $\omega(q)$ have integer coefficients.

For $0 \leq j \leq 11$ we define the functions

$$H_j(z) = \sum_{n \geq n_j} C(j;n) q^n$$

by

$$H_j(z) := \begin{cases} 0 & \text{if } j = 0, 3, 6, 9, \\ q^{-1} f(q^{24}) & \text{if } j = 1, 7, \\ -q^{-1} f(q^{24}) & \text{if } j = 5, 11, \\ 2q^8 \left(-\omega(q^{12}) + \omega(-q^{12})\right) & \text{if } j = 2, \\ -2q^8 \left(\omega(q^{12}) + \omega(-q^{12})\right) & \text{if } j = 4, \\ 2q^8 \left(\omega(q^{12}) + \omega(-q^{12})\right) & \text{if } j = 8, \\ 2q^8 \left(\omega(q^{12}) - \omega(-q^{12})\right) & \text{if } j = 10. \end{cases} \tag{2.3}$$

For each D (recall that $1 < D \equiv 23 \pmod{24}$ is square-free), we have the function

$$P_D(X) := \prod_{b \bmod D} (1 - e(-b/D)X)^{\left(\frac{-D}{b}\right)}, \tag{2.4}$$

where $e(\alpha) := e^{2\pi i \alpha}$ and $\left(\frac{-D}{b}\right)$ is the Kronecker character for the negative fundamental discriminant $-D$. We define the *generalized Borcherds product* $\Psi_D(z)$ by

$$\Psi_D(z) := \prod_{m=1}^{\infty} P_D(q^m)^{C(\overline{m};Dm^2)}. \tag{2.5}$$

Here $\overline{n}$ denotes the canonical residue class of n modulo 12.

Theorem 2.2. [§8.2 of [16]] *The function $\Psi_D(z)$ is a weight 0 meromorphic modular form on $\Gamma_0(6)$ with a discriminant -D twisted Heegner divisor (see §5 of* [16] *for the explicit divisor).*

2.3. **Proof of Theorem 1.1.** If we let $\Theta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$, then it is well known that $\Theta(\Psi_D(z))$ is a weight 2 meromorphic modular form on $\Gamma_0(6)$. It follows that the logarithmic derivative $\frac{\Theta(\Psi_D(z))}{\Psi_D(z)}$ is a meromorphic modular form on $\Gamma_0(6)$ whose poles are simple and are supported at the $\Gamma_0(6)$-Heegner points of discriminant $-D$.

A simple calculation using (2.4) shows that

$$\frac{\Theta(P_D(q^m))}{P_D(q^m)} = \sqrt{-D} m \sum_{n=1}^{\infty} \left(\frac{-D}{n}\right) q^{mn}.$$

Therefore, (2.5) implies that

$$\mathfrak{F}(D;z) := \frac{1}{\sqrt{-D}} \cdot \frac{\Theta(\Psi_D(z))}{\Psi_D(z)} = \sum_{m=1}^{\infty} mC(\overline{m}; Dm^2) \sum_{n=1}^{\infty} \left(\frac{-D}{n}\right) q^{mn} \tag{2.6}$$

is a weight 2 meromorphic modular form on $\Gamma_0(6)$ with integer coefficients whose poles are simple and are supported at some $\Gamma_0(6)$-Heegner points of discriminant $-D$.

By (2.3), we find that

$$H_j(z) = \sum_{n \geq n_j} C(j;n) q^n \equiv \begin{cases} q^{-1} f(q^{24}) \pmod 2 & \text{if } j = 1,5,7,11, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

Using Lemma 2.1, the fact that $\left(\frac{-D}{n}\right) = 0$ if $\gcd(n,D) \neq 1$, and (2.6), we find that

$$\mathfrak{F}(D;z) \equiv \sum_{\substack{m \geq 1 \\ \gcd(m,6)=1}} p\left(\frac{Dm^2+1}{24}\right) \sum_{\substack{n \geq 1 \\ \gcd(n,D)=1}} q^{mn} \pmod 2.$$

This is $\widehat{F}(D;z) \pmod 2$, and this completes the proof.

3. Modular forms mod 2 and the proof of Theorem 1.2

We give facts about locally nilpotent Hecke algebras, and we give results on the distribution of the odd coefficients of modular forms. We conclude with proofs of Theorems 1.2 and 1.3.

3.1. **Local nilpotency of the Hecke algebra on modular forms mod 2.** Serre suggested, and Tate proved (see page 115 of [45], page 251 of [46], and [50]) that the action of Hecke algebras on level 1 modular forms mod 2 is *locally nilpotent*. This means that if $f(z)$ is an integer weight holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ with integer coefficients, then there is a positive integer i with the property that

$$f(z) \mid T_{p_1} \mid T_{p_2} \mid \cdots \mid T_{p_i} \equiv 0 \pmod 2$$

for every collection of odd primes $p_1, p_2, \dots, p_i$.

Recently, the author and Taguchi [41] listed further levels where this nilpotency holds. They used the fact that the phenomenon coincides with the non-existence of non-trivial mod 2 Galois representations which are unramified outside 2 and the primes dividing the level.

We make this precise here. For a subring $\mathcal{O}$ of $\mathbb{C}$, we let $S_k(\Gamma;\mathcal{O})$ be the $\mathcal{O}$-module of cusp forms of integer weight k with respect to Γ whose coefficients lie in $\mathcal{O}$. If $\Gamma = \Gamma_0(N)$ and $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is a Dirichlet character, we denote by $S_k(\Gamma_0(N), \chi; \mathcal{O})$ the $\mathcal{O}$-module of cusp forms of weight k on $\Gamma_0(N)$ and Nebentypus character χ whose Fourier coefficients lie in $\mathcal{O}$. We suppress χ from the notation when it is the trivial character. Finally, if l is a prime of a number field L, then we let $\mathcal{O}_{L,l}$ be the localization of the integer ring $\mathcal{O}_L$ at l.

Theorem 3.1. [Theorem 1.3 of [41]] *Let a be a non-negative integer, and let N and k be positive integers. Suppose that $\chi : (\mathbb{Z}/2^aN)^\times \to \mathbb{C}^\times$ is a Dirichlet character with conductor $\mathfrak{f}(\chi)$, and suppose that L is a number field containing the coefficients of all the integer weight k newforms in the spaces $S_k(\Gamma_0(M), \chi)$, for every M with $M \mid 2^aN$ and $\mathfrak{f}(\chi) \mid M$. Let l be a* *prime of L lying above 2. If $N = 1, 3, 5, 15$ or 17, then there is an integer $c \geq 0$ such that for every $f(z) \in S_k(\Gamma(2^a N), \chi; \mathcal{O}_{L,l})$ and every $t \geq 1$ we have*

$$f(z) \mid T_{p_1} \mid T_{p_2} \mid \cdots \mid T_{p_{c+t}} \equiv 0 \pmod{l^t},$$

whenever $p_1, p_2, \ldots, p_{c+t}$ are odd primes not dividing N.

Remark. Theorem 3.1 is the first case of Theorem 1.3 of [41].

Remark. Theorem 3.1 applies to forms whose coefficients are in a subfield K of L. One replaces l by the prime l_0 of K lying below l. If e is the ramification index of l/l_0, then

$$f(z) \mid T_{p_1} \mid T_{p_2} \mid \cdots \mid T_{p_{c+et}} \equiv 0 \pmod{l_0^t}.$$

Suppose that $f(z)$ is an integer weight modular form on $\Gamma_0(N)$ with integer coefficients. If $f(z) \not\equiv 0 \pmod 2$, then we say that $f(z)$ has *degree of nilpotency* i if there are primes $p_1, p_2, \ldots, p_{i-1}$ not dividing $2N$ for which

$$f(z) \mid T_{p_1} \mid T_{p_2} \mid \cdots \mid T_{p_{i-1}} \not\equiv 0 \pmod 2,$$

and if for every set of primes $\ell_1, \ell_2, \ldots, \ell_i$ not dividing $2N$ we have

$$f(z) \mid T_{\ell_1} \mid T_{\ell_2} \mid \cdots \mid T_{\ell_i} \equiv 0 \pmod 2.$$

Lemma 3.2. *Suppose that $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(N); \mathbb{Z})$ has degree of nilpotency $i > 0$. Then the following are true:*

(1) *There are primes $p_1, p_2, \ldots, p_{i-1}$ not dividing $2N$, and an integer n_0 such that*

$$a(n_0 M^2 p_1 p_2 \cdots p_{i-1}) \equiv 1 \pmod 2$$

for every integer $M \geq 1$ that is coprime to $2p_1 p_2 \cdots p_{i-1} N$.

(2) *If $\ell_1, \ell_2, \ldots, \ell_i$ are primes not dividing $2N$, then*

$$a(n\ell_1 \ell_2 \cdots \ell_i) \equiv 0 \pmod 2$$

for every n coprime to $\ell_1, \ell_2, \ldots, \ell_i$.

Proof. For brevity we prove (1). The proof of (2) is similar. By definition, there are primes $p_1, p_2, \ldots, p_{i-1}$ not dividing $2N$ for which

$$f_{i-1}(z) = \sum_{n=1}^{\infty} b_{i-1}(n) q^n := f(z) \mid T_{p_1} \mid T_{p_2} \mid \cdots \mid T_{p_{i-1}} \not\equiv 0 \pmod 2. \tag{3.1}$$

Suppose that $b_{i-1}(n_0)$ is odd. Without loss of generality, we may assume that n_0 is coprime to $p_1 p_2 \cdots p_{i-1}$. This follows since the coefficients of cusp forms are dictated by the 2-adic Galois representations associated to the weight k newforms with level dividing N (for example, see the proof of Theorem 3.3). More precisely, the Chebotarev Density Theorem implies that a positive proportion of the $(i-1)$-tuples of primes give the same $f_{i-1}(z) \pmod 2$ when replaced in (3.1), and so one clearly may choose tuples using primes which do not divide n_0.

By the definition of i, if $p \nmid 2N$ is prime, then

$$f(z) \mid T_{p_1} \mid T_{p_2} \mid \cdots \mid T_{p_{i-1}} \mid T_p \equiv 0 \pmod 2.$$

By the definition of the Hecke operators, this means that

$$f_{i-1}(z) \mid T_p \equiv \sum_{n=1}^{\infty} \left(b_{i-1}(pn) + b_{i-1}(n/p)\right) q^n \equiv 0 \pmod{2},$$

and so it follows that $b_{i-1}(n_0M^2) \equiv 1 \pmod{2}$ for every integer $M \geq 1$ coprime to $2N$.

Define integers $b_j(n)$ by

$$\begin{aligned}
\sum_{n=1}^{\infty} b_1(n)q^n &:= f(z) \mid T_{p_1}, \\
\sum_{n=1}^{\infty} b_2(n)q^n &:= f(z) \mid T_{p_1} \mid T_{p_2}, \\
&\vdots \\
\sum_{n=1}^{\infty} b_{i-1}(n)q^n &:= f(z) \mid T_{p_1} \mid T_{p_2} \mid \cdots \mid T_{p_{i-1}}.
\end{aligned}$$

For every M coprime to $2Np_1p_2\cdots p_{i-1}$, we then have

$$1 \equiv b_{i-1}(n_0M^2) \equiv b_{i-2}(n_0M^2p_{i-1}) \equiv \cdots \equiv b_1(n_0M^2p_2p_3\cdots p_{i-1}) \pmod{2}.$$

The proof follows since $b_1(n_0M^2p_2p_3\cdots p_{i-1}) \equiv a(n_0M^2p_1p_2\cdots p_{i-1}) \pmod{2}$. □

3.2. **Distribution of the "odd" coefficients of modular forms.** The following theorem describes the distribution of the odd coefficients of integer weight modular forms.

Theorem 3.3. *If $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(N); \mathbb{Z})$, then the following are true:*

(1) *A positive proportion of the primes $p \equiv -1 \pmod{2N}$ have the property that*

$$f(z) \mid T_p \equiv 0 \pmod{2}.$$

(2) *Suppose that n_0 is an integer coprime to N with the property that*

$$a(n_0p_1p_2\cdots p_{i-1}) \equiv 1 \pmod{2},$$

where $p_1, p_2, \ldots, p_{i-1}$ are primes which do not divide $2n_0N$. If M is an integer coprime to $2N$ and $\gcd(r, M) = 1$, then

$$\#\{m \leq X \ : \ a(n_0m) \equiv 1 \pmod{2} \ \text{ and } \ m \equiv r \pmod{M}\} \gg \frac{X}{\log X}(\log\log X)^{i-2}.$$

Proof. Claim (1) is a well known result due to Serre [46].

We now prove (2). By the theory of newforms, every $F \in S_k(\Gamma_0(N))$ can be uniquely expressed as a linear combination

$$F(z) = \sum_{j=1}^{s} \alpha_j A_j(z) + \sum_{j=1}^{t} \beta_j B_j(\delta_j z),$$

where $A_j(z)$ and $B_j(z)$ are newforms of weight k and level a divisor of N, and where each δ_j is a divisor of N with $\delta_j > 1$. Let

$$F^{\text{new}}(z) := \sum_{j=1}^{s} \alpha_j A_j(z) \quad \text{and} \quad F^{\text{old}}(z) := \sum_{j=1}^{t} \beta_j B_j(\delta_j z) \tag{3.2}$$

be, respectively, the *new part of* F and the *old part of* F.

If $F(z) := \sum_{n=1}^{\infty} c(n)q^n$ is a newform, then the $c(n)$'s are algebraic integers which generate a finite extension of $\mathbb{Q}$, say K_F. If K is any finite extension of $\mathbb{Q}$ containing K_F, and if $\mathcal{O}_{K,l_0}$ is the completion of the ring of integers of K at any finite place l_0 with residue characteristic, say ℓ, then by the work of Deligne, Serre, and Shimura [17, 18, 47] there is a (not necessarily unique) continuous representation

$$\rho_{F,l_0} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{K,l_o})$$

for which

$$\rho_{F,l_0} \text{ is unramified at all primes } p \nmid N\ell, \tag{3.3}$$

$$\mathrm{Trace}(\rho_{F,l_0}(\mathrm{Frob}_p)) = c(p) \text{ for all primes } p \nmid N\ell. \tag{3.4}$$

Here Frob_p denotes any Frobenius element for the prime p.

Now we let $F(z) = f(z)$, and we let l_0 be a place over 2 in a number field K which contains the coefficients of all the newforms with level dividing N. Write $f = f^{\text{new}} + f^{\text{old}}$. Since n_0 is coprime to N, it follows that f^{new} is not identically zero. Write

$$f^{\text{new}} = \sum_{j=1}^{h} \alpha_j \widehat{f}_j(z), \quad \alpha_j \neq 0,$$

where each $\widehat{f}_j(z) := \sum_{n=1}^{\infty} \widehat{c}_j(n)q^n$ is a newform. Let L be a finite extension of $\mathbb{Q}$ containing K, the Fourier coefficients of each $\widehat{f}_j$, and the α_j's. Let l be a place of L over l_0, let e be the ramification index of l over l_0, let $\mathcal{O}_{L,l}$ be the completion of the ring of integers of L at the place l, and let λ be a uniformizer for $\mathcal{O}_{L,l}$. Let

$$E = \max_{1 \leq j \leq h} |\mathrm{ord}_l(4\alpha_j)|, \tag{3.5}$$

and let $\rho_{\widehat{f}_j,l} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{L,l})$ be a representation satisfying (3.3) and (3.4) for $\widehat{f}_j(z)$. Consider the representation

$$\rho = \bigoplus_{j=1}^{h} \rho_{\widehat{f}_j,l} \mod \lambda^{E+1}. \tag{3.6}$$

Since the primes $p_1, p_2, \ldots, p_{i-1}$ do not divide $2N$, the Chebotarev Density Theorem implies, for each $1 \leq r \leq i-1$, that there are $\gg X/\log X$ primes q less than X for which $\rho(\mathrm{Frob}_q) = \rho(\mathrm{Frob}_{p_r})$. By (3.4), for such a prime, $\widehat{c}_j(q) \equiv \widehat{c}_j(p_r) \mod \lambda^{E+1}$ for all j. It follows from these observations and the multiplicativity of the Fourier coefficients of newforms that there are $\gg \frac{X}{\log X}(\log\log X)^{i-2}$ integers $n_0 q_1 \cdots q_{i-1} < X$, such that

$$\widehat{c}_j(n_0 p_1 p_2 \cdots p_{i-1}) \equiv \widehat{c}_j(n_0 q_1 q_2 \cdots q_{i-1}) \bmod \lambda^{E+1}.$$

This in turn implies that $a(n_0 q_1 q_2 \cdots q_{i-1}) \equiv 1 \pmod 2$.

To obtain the claim concerning the congruence class modulo M, one simply modifies the construction of ρ in (3.6) to account for the cyclotomic characters giving the action of Galois on wth power roots of unity for the primes w dividing M. For example, since $\gcd(M, 2N) = 1$, one can insist, again by the Chebotarev Density Theorem and Dirichlet's Theorem on Primes in Arithmetic Progressions, that $q_1, q_2, \cdots, q_{i-2} \equiv 1 \pmod{M}$, and that $q_{i-1} \equiv r \pmod{M}$. This then completes the proof of (2). $\square$

3.3. **Producing some modular forms mod 2.** Here we relate $\widehat{F}(D; z)$ to holomorphic modular forms. Let $H_D(x)$ be the usual discriminant $-D$ Hilbert class polynomial, let $h(-D)$ be the class number of the ideal class group $CL(-D)$, and let

$$j(z) = q^{-1} + 744 + 196884q + \cdots$$

be the usual elliptic modular j-function. Finally, let

$$\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + \cdots$$

be the unique weight 12 normalized cusp form on $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 3.4. *There is a holomorphic modular form of weight* $12h(-D) + 2$ *on* $\Gamma_0(6)$ *with integer coefficients which is congruent to*

$$\widehat{F}(D; z)H_D(j(z))\Delta(z)^{h(-D)} \pmod{2}.$$

Proof. Let $\mathfrak{F}(D; z)$ be the meromorphic modular form in the proof of Theorem 1.1. This form has integer coefficients and satisfies the congruence

$$\mathfrak{F}(D; z) \equiv \widehat{F}(D; z) \pmod{2}.$$

The form $\mathfrak{F}(D; z)$ has the additional property that its poles are simple, and are supported at some discriminant $-D$ Heegner points under the action of $\Gamma_0(6)$. Since $H_D(j(z))$ is the modular function which has a simple zero at each discriminant $-D$ Heegner point with respect to $\mathrm{SL}_2(\mathbb{Z})$, it follows that $\mathfrak{F}(D; z)H_D(j(z))\Delta(z)^{h(-D)}$ is a weight $12h(-D)+2$ holomorphic modular form on $\Gamma_0(6)$. The factor $\Delta(z)^{h(-D)}$ is required to compensate for the poles introduced by $H_D(j(z))$. The proof follows from Theorem 1.1. $\square$

3.4. **Proof of Theorem 1.2.** First we prove (1). Let $E_2(z) := 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$ be the usual weight 2 Eisenstein series, where $\sigma_1(n) := \sum_{d|n} d$. Although $E_2(z)$ is not modular, it is well known that if $t \geq 2$, then $E_2(z) - tE_2(tz)$ is a weight 2 modular form on $\Gamma_0(t)$. Therefore, we have that

$$\mathcal{E}(z) := \frac{(E_2(z) - 3E_2(3z)) - 2(E_2(z) - 2E_2(2z))}{24} = q - q^2 + 7q^3 - 5q^4 - \cdots \tag{3.7}$$

is a holomorphic weight 2 modular form on $\Gamma_0(6)$. One checks that

$$\mathcal{E}(z) \equiv \sum_{\substack{m\geq 1 \\ \gcd(m,6)=1}} \sum_{n\geq 1} q^{mn} \pmod{2}.$$

A straightforward calculation then shows that

$$\mathcal{E}(D;z) := \sum_{1\leq \delta | D} \mathcal{E}(\delta z) \equiv \sum_{\substack{m\geq 1\\ \gcd(m,6)=1}} \sum_{\substack{n\geq 1\\ \gcd(n,D)=1}} q^{mn} \pmod 2. \tag{3.8}$$

Therefore if $p\left(\frac{Dm^2+1}{24}\right)$ is odd for every m coprime to 6, then $\mathcal{E}(D;z) \equiv \widehat{F}(D;z) \pmod 2$. Lemma 3.4, combined with the fact that $\mathcal{E}(D;z)$ is a holomorphic modular form of weight 2 on $\Gamma_0(6D)$, now implies that

$$\mathcal{E}(D;z)H_D(j(z))\Delta(z)^{h(-D)} \equiv \widehat{F}(D;z)H_D(j(z))\Delta(z)^{h(-D)} \pmod 2$$

in $M_{12h(-D)+2}(\Gamma_0(6D))$. A theorem of Sturm [48] shows that this holds if and only if the first $(12h(-D)+2)\prod_p (p+1)$ coefficients are congruent mod 2, where the product is over the primes p dividing D. This proves the bound concerning the first (if any) even value.

Suppose that there are only finitely many m coprime to 6, say $m_1, m_2, \ldots, m_s$, for which $p\left(\frac{Dm^2+1}{24}\right)$ is even. Let m_1 be the smallest of these numbers. By (3.8), we have that

$$\mathcal{E}(D;z) - \widehat{F}(D;z) \equiv \sum_{i=1}^{s} \sum_{\substack{n\geq 1\\ \gcd(n,D)=1}} q^{m_i n} \pmod 2. \tag{3.9}$$

Let $T(D;z)$ be the level 1 cusp form

$$T(D;z) = \sum_{n=5}^{\infty} t(D;n)q^n := H_D(j(z))\Delta(z)^{h(-D)+5} = q^5 + \cdots. \tag{3.10}$$

By Theorem 3.1, the Hecke algebra acts locally nilpotently on $T(D;z)$. Since $t(D;5)$ is odd, it follows that its degree of nilpotency is an integer $\eta = \eta_D \geq 2$. Lastly, Theorem 3.3 (2) implies in every arithmetic progression $r \pmod M$, where $\gcd(r,M) = 1$ and M is odd, that

$$\#\{p \leq X \ : \ t(D;p) \equiv 1 \pmod 2 \ \text{ and } \ p \equiv r \pmod M\} \gg X/\log X. \tag{3.11}$$

Using $\mathcal{E}(D;z)$ and Lemma 3.4, there is a cusp form $S(D;z)$ on $\Gamma_0(6D)$ for which

$$S(D;z) = \sum_{n=1}^{\infty} s(D;n)q^n \equiv (\mathcal{E}(D;z) - \widehat{F}(D;z))T(D;z) \pmod 2. \tag{3.12}$$

If N is a positive integer, then (3.9) implies that

$$s(D;N) \equiv \sum_{i=1}^{s} \sum_{\substack{n\geq 1\\ \gcd(n,D)=1}} t(D;N-m_i n) \pmod 2. \tag{3.13}$$

Since $t(D;\alpha) = 0$ when $\alpha \leq 0$, these sums are finite.

By Theorem 3.3 (1), there are infinitely many primes p for which

$$s(D;n) \equiv 0 \pmod 2 \tag{3.14}$$

when $p||n$. Let $p_0 \nmid m_1m_2\dots m_sD$ be such a prime. Now let $Y := p_0^2 D\cdot \mathrm{lcm}(m_1, m_2, \dots, m_s)$. Using (3.13), it follows by construction that

$$s(D; N+Y) - s(D;N) \equiv \sum_{i=1}^{s} \sum_{\substack{1\le n\le \frac{Y}{m_i}\\ \gcd(n,D)=1}} t(D; N+Y-m_in) \pmod 2. \tag{3.15}$$

Now consider the arithmetic progression

$$N \equiv p_0 \pmod{p_0^2}. \tag{3.16}$$

For each (i,n) with $1\le i\le s$ and $1\le n\le \frac{Y}{m_i}$, except $(m_1, 1)$, we consider progressions of the form

$$N \equiv -(Y-m_in) + \ell_1(i,n)\ell_2(i,n)\cdots\ell_\eta(i,n) \pmod{\ell_1(i,n)^2\ell_2(i,n)^2\cdots\ell_\eta(i,n)^2}. \tag{3.17}$$

Choose distinct odd primes $\ell_j(i,n)$ coprime to $p_0m_1m_2\cdots m_sD$ so that the system (3.16) and (3.17) has, by the Chinese Remainder Theorem, a solution of the form $N \equiv r_0 \pmod{M_D}$, where $\gcd(r_0+Y-m_1, M_D) = 1$. By (3.11), there are then infinitely many primes p of the form $N_p + Y - m_1$ for which $t(D; N_p+Y-m_1) \equiv 1 \pmod 2$. For these p, we have:

1. We have that $p_0||N_p$ and $p_0||(N_p+Y)$ (since $p_0^2 \mid Y$).
2. For each pair (i,n) with $1\le i\le s$ and $1\le n\le \frac{Y}{m_i}$, except $(m_1,1)$, we have
$$N_p + Y - m_in \equiv \ell_1(i,n)\ell_2(i,n)\cdots\ell_\eta(i,n) \pmod{\ell_1(i,n)^2\ell_2(i,n)^2\cdots\ell_\eta(i,n)^2}.$$

By (3.14), both $s(D;N_p+Y)$ and $s(D;N_p)$ are even. By Lemma 3.2 (2), the fact that $T(D;z)$ has degree of nilpotency η implies that each summand in (3.15), except $t(D;N_p+Y-m_1)$, is even. Since $t(D;N_p+Y-m_1)$ is odd, (3.15) gives the contradiction $0\equiv 1 \pmod 2$. Consequently, there must be an infinite number of m coprime to 6 for which $p\left(\frac{Dm^2+1}{24}\right)$ is even unless there are no such m. This completes the proof of (1).

To prove (2), first notice that $\widehat{F}(D;z)H_D(j(z))\Delta(z)^{h(-D)}$ is trivial mod 2 if and only if $\widehat{F}(D;z)\equiv 0 \pmod 2$. Lemma 3.4 and Sturm's theorem [48] then imply that this triviality occurs if and only if the first $12h(-D)+2$ coefficients are even.

Suppose now that $\widehat{F}(D;z)\not\equiv 0 \pmod 2$, and that there are only finitely many m coprime to 6, say $m_1, m_2, \dots, m_s$, for which $p\left(\frac{Dm^2+1}{24}\right)$ is odd. Then we have that

$$\widehat{F}(D;z) \equiv \sum_{i=1}^{s}\sum_{\substack{n\ge 1\\ \gcd(n,D)=1}} q^{m_in} \pmod 2.$$

Since this is the series in (3.9), the proof of (2) follows *mutatis mutandis* by replacing $(\mathcal{E}(D;z) - \widehat{F}(D;z))T(D;z)$ with $\widehat{F}(D;z)T(D;z)$ in (3.12).

Remark. In the proof of Theorem 1.2 (2), notice that $\widehat{F}(D;z)T(D;z)$ is in the space of modular forms mod 2 on $\Gamma_0(6)$, and so local nilpotency applies (i.e. $a=1$ and $N=3$ in Theorem 3.1). We can arrange the progressions so that the resulting $s(D;N_p+Y)$ and $s(D,N_p)$ are even due to nilpotency instead of Theorem 3.3 (1) which we used in the proof of (1). This is not possible for the proof of (1) since $\mathcal{E}(D;z)-\widehat{F}(D;z) \pmod 2$ is on $\Gamma_0(6D)$, and the Hecke algebra is not locally nilpotent in general.

3.5. **Proof of Theorem 1.3.** Suppose on the contrary that $p\left(\frac{D_0\ell m^2+1}{24}\right)$ is odd for every m coprime to 6. As in the proof of Theorem 1.2, we then have that

$$\widehat{F}(D_0\ell;z) \equiv \mathcal{E}(D_0\ell;z) = \sum_{1\leq\delta|D_0\ell} \mathcal{E}(\delta z) \pmod{2}. \tag{3.18}$$

Since $\mathcal{E}(\delta z)$ is a weight 2 holomorphic modular form on $\Gamma_0(6\delta)$, Lemma 3.4 implies that

$$\sum_{n\geq 0} b(n)q^n := \left(\widehat{F}(D_0\ell;z) - \sum_{1\leq\delta|D_0} \mathcal{E}(\delta z)\right) H_{D_0\ell}(j(z))\Delta(z)^{h(-D_0\ell)}$$

is congruent mod 2 to a weight $12h(-D_0\ell)+2$ holomorphic modular form on $\Gamma_0(6D_0)$. By (3.18), we then have that

$$\sum_{n\geq 0} b(n)q^n \equiv H_{D_0\ell}(j(z))\Delta(z)^{h(-D_0\ell)} \sum_{1\leq\delta|D_0} \mathcal{E}(\delta\ell z) \equiv q^\ell + \cdots \pmod{2}.$$

This contradicts Sturm's bound [48] for the first odd coefficient, which implies that

$$\ell \leq (12h(-D_0\ell)+2) \prod_{p|D_0 \text{ prime}} (p+1).$$

3.6. **Proof of Corollary 1.4.** Here we let $D_0 := 1$ in Theorem 1.3. If $\ell \equiv 23 \pmod{24}$ is a prime for which

$$\ell > 12h(-\ell)+2, \tag{3.19}$$

then Theorem 1.3 gives the conclusion. By Dirichlet's class number formula it is known that

$$h(-\ell) < \frac{1}{\pi}\sqrt{\ell}\log(\ell),$$

which in turn implies that (3.19) holds for all $\ell \geq 599$. The corollary follows by applying Theorem 1.2 (1) for each prime $\ell < 599$.

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Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706
E-mail address: ono@math.wisc.edu