EICHLER-SHIMURA THEORY FOR MOCK MODULAR FORMS

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Abstract. We use *mock modular forms* to compute generating functions for the critical values of modular L-functions, and we answer a generalized form of a question of Kohnen and Zagier by deriving the "extra relation" that is satisfied by even periods of weakly holomorphic cusp forms. To obtain these results we derive an Eichler-Shimura theory for weakly holomorphic modular forms and mock modular forms. This includes two "Eichler-Shimura isomorphisms", a "multiplicity two" Hecke theory, a correspondence between *mock modular periods* and classical periods, and a "Haberland-type" formula which expresses Petersson's inner product and a related antisymmetric inner product on $M_k^!$ in terms of periods.

1. Introduction and statement of results

The recent works of Zwegers [32, 33] on Ramanujan's mock theta functions, combined with the important seminal paper of Bruinier and Funke [5], have catalyzed considerable research on harmonic Maass forms (see § 2 for the definition and basic facts). This research is highlighted by applications to a wide variety of subjects: additive number theory, algebraic number theory, Borcherds products, knot theory, modular L-functions, mathematical physics, representation theory, to name a few (for example, see [21, 22, 31] and the references therein). Here we consider fundamental questions concerning periods and harmonic Maass forms.

Every harmonic Maass form $\mathcal{F}(z)$ has a unique natural decomposition

$$\mathcal{F}(z) = \mathcal{F}^-(z) + \mathcal{F}^+(z),$$

where $\mathcal{F}^-$ (resp. $\mathcal{F}^+$) is nonholomorphic (resp. holomorphic) on the upper-half of the complex plane $\mathbb{H}$. The holomorphic part $\mathcal{F}^+$ has a Fourier expansion

$$\mathcal{F}^+(z) = \sum_{n \gg -\infty} a^+_{\mathcal{F}}(n) q^n$$

($q := e^{2\pi i z}$, $z \in \mathbb{H}$ throughout) which, following Zagier[1], we call a *mock modular form*. The differential operator $\xi_w := 2iy^w \cdot \overline{\frac{\partial}{\partial \bar{z}}}$, which plays a central role in the theory, only sees the nonholomorphic parts of such forms. If $\mathcal{F}$ has weight $2-k$, then $\xi_{2-k}(\mathcal{F}) = \xi_{2-k}(\mathcal{F}^-)$.

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[1]We note that this definition implies that weakly holomorphic modular forms are mock modular forms. Several papers require mock modular forms to correspond to those harmonic Maass forms for which $\mathcal{F}^- \neq 0$.

The most important feature of ξ_{2-k} is that it defines surjective maps

$$\xi_{2-k}:\begin{cases} H_{2-k}(N) \twoheadrightarrow M_k^!(N) \\ H_{2-k}^*(N) \twoheadrightarrow S_k(N),\end{cases}$$

where $H_{2-k}^*(N) \subseteq H_{2-k}(N)$ are spaces of harmonic Maass forms, and where $M_k^!(N)$ (resp. $S_k(N)$) denotes the weight k weakly holomorphic modular (resp. cusp) forms on $\Gamma_0(N)$.

Shimura's work [26] on half-integral weight modular forms, for $k \in 2\mathbb{Z}^+$, provides further maps which interrelate different spaces of modular forms. He defined maps

$$Sh : S_{\frac{k+1}{2}}(4N) \longrightarrow S_k(N),$$

which when combined with the preceding discussion, gives the following diagram:

$$\begin{array}{ccccc} H^*_{\frac{3-k}{2}}(4N) & \xrightarrow{\xi_{\frac{3-k}{2}}} & S_{\frac{k+1}{2}}(4N) & \xrightarrow{Sh} & S_k(N) \\ & & & & \uparrow {\scriptstyle \xi_{2-k}} \\ & & & & H^*_{2-k}(N) \end{array} \tag{1.1}$$

It is natural to study the arithmetic of this diagram. Since $\xi_{\frac{3-k}{2}}$ and ξ_{2-k} only use the non-holomorphic parts of harmonic Maass forms, the main problem then is that of determining the arithmetic content of the holomorphic parts of these forms. What do they encode?

For newforms $f \in S_2(N)$, Bruinier and the fourth author [6] investigated this problem for the horizontal row of (1.1). Using important works of Gross and Zagier [9], of Kohnen and Zagier [16, 17], and of Waldspurger [28], they essentially proved that there is a form $\mathcal{F} = \mathcal{F}^- + \mathcal{F}^+ \in H^*_{\frac{1}{2}}(4N)$, satisfying $Sh(\xi_{\frac{1}{2}}(\mathcal{F})) = f$, which has the property that the coefficients of the mock modular form $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) determine the nonvanishing of the central derivatives (resp. values) of the quadratic twist L-functions $L(f, \chi_D, s)$.

In this paper we study the vertical map in (1.1), and we show that forms[2] $\mathcal{F} \in H^*_{2-k} := H^*_{2-k}(1)$ beautifully encode the critical values of L-functions arising from S_k. This statement is very simple to prove (we give two straightforward proofs), and it provides our inspiration for extending Eichler-Shimura theory and work of Haberland to the setting of mock modular forms.

We begin by stating this elementary connection between mock modular forms and critical values of modular L-functions. For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, we define the *$\gamma$-mock modular period function* for $\mathcal{F}^+$ by

$$\mathbb{P}\left(\mathcal{F}^+, \gamma; z\right) := \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \cdot \left(\mathcal{F}^+ - \mathcal{F}^+|_{2-k}\gamma\right)(z), \tag{1.2}$$

where for any function g, we let $\left(g|_w\gamma\right)(z) := (cz+d)^{-w} g\left(\frac{az+b}{cz+d}\right)$. The map

$$\gamma \mapsto \mathbb{P}\left(\mathcal{F}^+, \gamma; z\right)$$

gives an element in the first cohomology group of $SL_2(\mathbb{Z})$ with polynomial coefficients, and we shall see that they are intimately related to classical "period polynomials".

[2]Throughout we omit the dependence on the level in the case of $SL_2(\mathbb{Z})$.

For positive c, let $\zeta_c := e^{2\pi i/c}$, and for $0 \leq d < c$, let $\gamma_{c,d} \in SL_2(\mathbb{Z})$ be any matrix satisfying $\gamma_{c,d} := \left(\begin{smallmatrix} * & * \\ c' & d' \end{smallmatrix}\right)$. Here the integers $0 \leq d' < c'$ are chosen so that $\frac{d}{c} = \frac{d'}{c'}$ in lowest terms.

Theorem 1.1. *Suppose that $4 \leq k \in 2\mathbb{Z}$, and suppose that $\mathcal{F} \in H^*_{2-k}$ and $f = \xi_{2-k}(\mathcal{F})$. Then we have that*

$$\overline{\mathbb{P}\left(\mathcal{F}^+, \gamma_{1,0}; \overline{z}\right)} = \sum_{n=0}^{k-2} \frac{L(f, n+1)}{(k-2-n)!} \cdot (2\pi i z)^{k-2-n}.$$

Moreover, if $\chi \pmod{c}$ is a Dirichlet character, then

$$\frac{1}{c} \sum_{m \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(m) \sum_{d=0}^{c-1} \zeta_c^{md} \cdot \overline{\mathbb{P}\left(\mathcal{F}^+, \gamma_{c,d}; \overline{z} - \frac{d}{c}\right)} = \sum_{n=0}^{k-2} \frac{L(f, \chi, n+1)}{(k-2-n)!} \cdot (2\pi i z)^{k-2-n}.$$

Here $L(f,s)$ (resp. $L(f,\chi,s)$) is the usual L-function (resp. twisted by χ) for f.

Remark. In Theorem 1.1 and throughout the remainder of the paper we assume that $k \geq 4$ is even. Theorem 1.1, which can be generalized to arbitrary levels, is related to Manin's observation [20] that twisted L-values may be given as expressions involving periods. These expressions are typically quite complicated. The theory underlying Theorem 1.1 relates the mock modular periods to such periods, and then gives nice generating functions.

Our first proof of Theorem 1.1 follows from the fact that the non-holomorphic part $\mathcal{F}^-$ can be described in terms of a "period integral" of f (see Section 2). In particular, it then suffices to consider the integral

$$\int_{-\overline{z}}^{i\infty} f^c(\tau)(z+\tau)^{k-2} d\tau.$$

Theorem 1.1 then follows from the standard fact, for $0 \leq n \leq k-2$, that

$$L(f, n+1) = \frac{(2\pi)^{n+1}}{n!} \cdot \int_0^\infty f(it)t^n dt.$$

We leave the details to the reader.

Theorem 1.1 also follows easily from the principle that the obstruction to modularity determines period functions, which, in turn, are generating functions for critical L-values (see Section 5). This principle appears prominently in the framework of the "Eichler-Shimura theory" of periods. The pioneering work of Eichler [8] and Shimura [25], expounded upon by Manin [20], is fundamental in the theory of modular forms, and it has deep implications for elliptic curves and critical values of L-functions. Therefore, in view of Theorem 1.1, we are motivated here to extend this theory to the context of mock modular forms and weakly holomorphic modular forms.

One of the main features of the theory is the Eichler-Shimura isomorphism, which relates spaces of cusp forms to the first parabolic cohomology groups for $SL_2(\mathbb{Z})$ with polynomial coefficients.

Remark. Knopp, and his collaborators (for example, see [14, 15]) have investigated Eichler cohomology groups more generally, with a special emphasis on the automorphic properties of various families of Poincaré series.

We recall the Eichler-Shimura isomorphism following the discussion in [18, 30]. Define S, T, and U by

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad U := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

and let

$$\mathbf{V} := \mathbf{V}_{k-2}(\mathbb{C}) = \mathrm{Sym}^{k-2}(\mathbb{C} \oplus \mathbb{C})$$

be the linear space of polynomials of degree $\leq k-2$ in z. Let

$$\mathbf{W} := \left\{ P \in \mathbf{V} : P + P|_{2-k}S = P + P|_{2-k}U + P|_{2-k}U^2 = 0 \right\}. \tag{1.3}$$

The space $\mathbf{V}$ splits as a direct sum $\mathbf{V} = \mathbf{V}^+ \oplus \mathbf{V}^-$ of even and odd polynomials. Putting $\mathbf{W}^\pm := \mathbf{W} \cap \mathbf{V}^\pm$ one obtains the splitting $\mathbf{W} = \mathbf{W}^+ \oplus \mathbf{W}^-$.

There are two period maps $r^\pm : S_k \longrightarrow \mathbf{W}^\pm$

$$r^+(f; z) := \sum_{\substack{0 \leq n \leq k-2 \\ n \text{ even}}} (-1)^{\frac{n}{2}} \binom{k-2}{n} \cdot r_n(f) \cdot z^{k-2-n},$$

$$r^-(f; z) := \sum_{\substack{0 \leq n \leq k-2 \\ n \text{ odd}}} (-1)^{\frac{n-1}{2}} \binom{k-2}{n} \cdot r_n(f) \cdot z^{k-2-n},$$

where, for each integer $0 \leq n \leq k-2$, the nth *period* of f is defined by

$$r_n(f) := \int_0^\infty f(it) t^n dt. \tag{1.4}$$

Notice that if we let $r(f; z) := r^-(f; z) + ir^+(f; z)$, then

$$r(f; z) = \sum_{n=0}^{k-2} i^{-n+1} \binom{k-2}{n} \cdot r_n(f) \cdot z^{k-2-n} = \int_0^{i\infty} f(\tau)(z-\tau)^{k-2} d\tau.. \tag{1.5}$$

The Eichler-Shimura isomorphism theorem asserts that r^- (resp. r^+) is an isomorphism onto $\mathbf{W}^-$ (resp. $\mathbf{W}_0^+ \subseteq \mathbf{W}^+$, the codimension 1 subspace not containing $z^{k-2} - 1$). Therefore $\mathbf{W}_0 \subseteq \mathbf{W}$, the corresponding codimension 1 subspace, represents two copies of S_k.

Concerning $\mathbf{W}_0$ and $z^{k-2} - 1$, Kohnen and Zagier ask (see p. 201 of [18]):

Question. *What extra relation is satisfied by the even periods of cusp forms besides the relations defining* $\mathbf{W}$*?*

In §4 of [18], they give formulas, involving Bernoulli numbers, which answer this question.

Here we clarify the nature of this problem by making explicit the roles of $\mathbf{W}$ and $\mathbf{W}_0$ in the general theory of periods. It turns out that both naturally arise when considering periods of weakly holomorphic modular forms. We derive Eichler-Shimura isomorphism theorems for both $\mathbf{W}_0$ and $\mathbf{W}$, ones which involve *weakly holomorphic cusp forms.* Let $M_k^!$ be the space of weight k *weakly holomorphic modular forms* on $SL_2(\mathbb{Z})$, those meromorphic modular forms whose poles (if any) are supported at the cusp infinity. A form $F \in M_k^!$ is a *weakly holomorphic cusp form* if its constant term vanishes. In other words, F has a Fourier expansion of the form

$$F(z) = \sum_{\substack{n \geq n_0 \\ n \neq 0}} a_F(n) q^n.$$

Let $S_k^!$ denote the space of weakly holomorphic cusp forms.

Our work depends on an extension to $M_k^!$ of the map $r = r^- + ir^+$. Since the integrals in (1.4) diverge for forms with poles, the extension must be obtained by other means. To define it, suppose that $F(z) = \sum_{n \gg -\infty} a_F(n) q^n \in M_k^!$. Its *Eichler integral* [19] is

$$\mathcal{E}_F(z) := \sum_{\substack{n \gg -\infty \\ n \neq 0}} a_F(n) n^{1-k} q^n. \tag{1.6}$$

We define the *period function* for F by

$$r(F; z) := c_k \left(\mathcal{E}_F - \mathcal{E}_F|_{2-k} S \right)(z), \tag{1.7}$$

where $c_k := -\frac{\Gamma(k-1)}{(2\pi i)^{k-1}}$. If F is a cusp form, then one easily sees that

$$\mathcal{E}_F(z) = \frac{1}{c_k} \cdot \int_z^{i\infty} F(\tau)(z - \tau)^{k-2} d\tau, \tag{1.8}$$

and so, by a change of variable, (1.5) implies that (1.7) indeed extends the classical period map $r = r^- + ir^+$.

Remark. We have that $r(F; z) = \alpha(z^{k-2} - 1)$ if and only if $\mathcal{E}_F(z) + \frac{\alpha}{c_k}$ is in $M_{2-k}^!$.

The period functions $r(F; z)$ are essentially polynomials in z with degree $\leq k-2$. The contribution from the constant term $a_F(0)$, which is a multiple of $z^{k-1} + \frac{1}{z}$, poses the only obstruction. Therefore, in analogy with (1.5), we define $r_n(F)$, the periods of F, by

$$r(F; z) := \frac{a_F(0)}{k-1} \cdot \left(z^{k-1} + \frac{1}{z} \right) + \sum_{n=0}^{k-2} i^{-n+1} \binom{k-2}{n} \cdot r_n(F) \cdot z^{k-2-n}. \tag{1.9}$$

The extended period function r, restricted to $S_k^!$, defines the maps:

$$r : S_k^! \to \mathbf{W} \quad \text{and} \quad \widetilde{r} : S_k^! \to \mathbf{W}_0$$

where the second map is the composition of r and the projection from $\mathbf{W}$ to $\mathbf{W}_0$. Furthermore, there are maps $r^\pm : S_k^! \to \mathbf{W}^\pm$ which extend the classical even and odd period maps on S_k. We obtain "Eichler-Shimura" isomorphisms for these two maps. To compute their kernels, we use the differential operator $D := \frac{1}{2\pi i} \cdot \frac{d}{dz}$ which, by a well known (extended) identity of Bol (see Theorem 1.2 of [7]), satisfies

$$D^{k-1} : M_{2-k}^! \longrightarrow S_k^! \quad \text{and} \quad D^{k-1} : H_{2-k}^* \longrightarrow S_k^!.$$

We prove the following isomorphisms.

Theorem 1.2. *The following sequences are exact*

$$0 \longrightarrow D^{k-1}(M^!_{2-k}) \longrightarrow S^!_k \xrightarrow{\widetilde{r}} \mathbf{W}_0 \longrightarrow 0$$

and

$$0 \longrightarrow D^{k-1}(S^!_{2-k}) \longrightarrow S^!_k \xrightarrow{r} \mathbf{W} \longrightarrow 0.$$

The first exact sequence from Theorem 1.2 tells us that

$$S^!_k / D^{k-1}(M^!_{2-k}) \cong \mathbf{W}_0, \tag{1.10}$$

and the second exact sequence explains the role of the codimension one subspace $\mathbf{W}_0$ in the classical setting. The presence of non-zero constant terms of (cf. Proposition 3.5 below) weight $2-k$ weakly holomorphic modular forms gives

$$r : D^{k-1}(M^!_{2-k})/D^{k-1}(S^!_{2-k}) \xrightarrow{\sim} \mathbf{W}/\mathbf{W}_0 \cong \left\langle z^{k-2} - 1 \right\rangle.$$

Theorem 1.2 sheds further light on the classical Eichler-Shimura isomorphism, where the maps

$$r^{\pm} : S_k \longrightarrow \mathbf{W}_0 \tag{1.11}$$

each give one copy of S_k inside $\mathbf{W}_0$ so that $\mathbf{W}_0 \cong S_k \oplus S_k$. Equations (1.10) and (1.11) tell us that $S^!_k/D^{k-1}(M^!_{2-k}) \cong S_k \oplus S_k$. We directly explain this isomorphism. We have that D^{k-1} only sees the holomorphic parts $\mathcal{F}^+$ of harmonic Maass forms $\mathcal{F} \in H^*_{2-k}$ (i.e. $D^{k-1}(\mathcal{F}) = D^{k-1}(\mathcal{F}^+)$), and we shall show that the two copies of S_k arise from the quotient space $H^*_{2-k}/M^!_{2-k}$ and the inclusion of $S_k \subseteq S^!_k$. In particular, we will show that

$$\mathbf{W}_0 \cong \widetilde{r}(D^{k-1}(H^*_{2-k})) \oplus \widetilde{r}(\xi_{2-k}(H^*_{2-k})).$$

We also revisit the question of Kohnen and Zagier on the "extra relation" satisfied by even periods of cusp forms. The second exact sequence in Theorem 1.2,

$$S^!_k/D^{k-1}(S^!_{2-k}) \cong \mathbf{W},$$

shows that there are no *extra relations* in the setting of weakly holomorphic cusp forms. Therefore, it is natural to ask the following reformulation of the question of Kohnen and Zagier.

Question. *If $F \in S^!_k$, then (as a function of its principal part) what extra relation is satisfied by the even periods of F?*

Remark. The original question pertains to forms in $S^!_k$ with trivial principal part.

The following theorem answers this question in terms of Bernoulli numbers B_k and divisor functions $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$.

Theorem 1.3. *Define rational numbers $\lambda_{k,n}$ ($k \geq 4$ even, $0 \leq n \leq k-2$, n even) by*

$$\lambda_{k,n} := B_k \cdot \left(1 + \binom{k-1}{n} - \binom{k-1}{n+1}\right) + 2\sum_{r=1}^{k/2} \binom{2r-1}{n}\binom{k}{2r} B_{2r} B_{k-2r}.$$

If $F \in S^!_k$ has principal part

$$F_{\mathrm{prin}}(q) := \sum_{n>0} a(n) q^{-n},$$

then

$$\sum_{\substack{0\leq n\leq k-2\\ \text{even}}} (-1)^{\frac{n}{2}}\lambda_{k,n}\cdot r_n(F) = -6(2i)^k\frac{k!}{(4\pi)^{k-1}}\sum_{n>0}\frac{a(n)\sigma_{k-1}(n)}{n^{k-1}}. \tag{1.12}$$

Three remarks

1) We note that here $a(n)$ in the principal part is the coefficient of q^{-n}, instead of q^n, which is a departure from the convention adopted throughout this paper.
2) If F is a cusp form, then we have that $F_{\text{prin}}(q) = 0$. The relation in Theorem 1.3 then reduces to the solution offered by Kohnen and Zagier on the extra relation satisfied by the even periods of cusp forms.
3) The work of Kohnen and Zagier [18] is largely about cusp forms with rational periods (see also the forthcoming paper by Popa [23]). Theorem 1.3 implies that if a weakly holomorphic cusp form has rational even periods, then

$$\frac{1}{\pi^{k-1}}\sum_{n>0}\frac{a(n)\sigma_{k-1}(n)}{n^{k-1}}\in\mathbb{Q}.$$

To obtain Theorems 1.2 and 1.3, we must understand the interrelationships between the three period functions $r(\xi_{2-k}(\mathcal{F}^-); z)$, $r(D^{k-1}(\mathcal{F}^+); z)$, and $\mathbb{P}(\mathcal{F}^+, \gamma_{1,0}; z)$. We show that these functions are essentially equal up to complex conjugation and the change of variable $z \to \overline{z}$. Strictly speaking, our functions are not defined for $\bar{z}$. However, since we apply complex conjugation these period functions are well defined. We obtain the following period relations on H_{2-k}.

Theorem 1.4. *If $\mathcal{F}\in H_{2-k}$, then we have that*

$$r\left(\xi_{2-k}(\mathcal{F}); z\right) \equiv -\frac{(4\pi)^{k-1}}{\Gamma(k-1)}\cdot\overline{r\left(D^{k-1}(\mathcal{F}); \overline{z}\right)} \pmod{z^{k-2}-1}$$

where equivalence modulo $z^{k-2}-1$ means that the difference of the two functions is a constant multiple of $z^{k-2}-1$. Moreover, there is a function $\widehat{\mathcal{F}}\in H_{2-k}$ for which $\xi_{2-k}(\widehat{\mathcal{F}}) = \xi_{2-k}(\mathcal{F})$ and

$$r\left(\xi_{2-k}(\mathcal{F}); z\right) = -\frac{(4\pi)^{k-1}}{\Gamma(k-1)}\cdot\overline{r\left(D^{k-1}\left(\widehat{\mathcal{F}}\right); \overline{z}\right)}.$$

Three remarks

1) If $\mathcal{F}\in H^*_{2-k}$ has constant term 0, then we have the following mock modular period identity:

$$r(D^{k-1}(\mathcal{F}); z) = r(D^{k-1}(\mathcal{F}^+); z) = c_k\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\cdot\mathbb{P}(\mathcal{F}^+, \gamma_{1,0}; z). \tag{1.13}$$

This follows from (1.2) and (1.7). Moreover, we shall show in Proposition 3.5 that there always are forms $F\in M^!_{2-k}$ for which $\mathcal{F}+F\in H^*_{2-k}$ has constant term zero.
2) Since D^{k-1} annihilates constants, one cannot avoid the $z^{k-2}-1$ ambiguity in Theorem 1.4.
3) Many of the technical difficulties in this paper arise from the need to carefully take into account the constant terms of Maass-Poincaré series and their corresponding Eisenstein series.

This issue is even more complicated in the setting of congruence subgroups. This is why we are content to work in the setting of the full modular group $SL_2(\mathbb{Z})$.

There is a theory of Hecke operators on $S_k^!/D^{k-1}(M_{2-k}^!)$. For any positive integer $m \geq 2$, let $T(m)$ be the usual weight k index m Hecke operator. We say that $F \in S_k^!$ is a *Hecke eigenform* with respect to $S_k^!/D^{k-1}(M_{2-k}^!)$ if for every Hecke operator $T(m)$ there is a complex number $b(m)$ for which

$$\left(F \mid_k T(m) - b(m)F\right)(z) \in D^{k-1}\left(M_{2-k}^!\right).$$

This definition includes the usual notion of Hecke eigenforms for (holomorphic) cusp forms. Indeed, in this case we simply have

$$\left(F \mid_k T(m) - b(m)F\right)(z) = 0.$$

It is natural to determine the dimension of those subspaces which correspond to a system of Hecke eigenvalues. We prove the following "multiplicity two" theorem.

Theorem 1.5. *Let $d = \dim S_k$, and let $f_i(z) = \sum b_i(n)q^n \in S_k$ $(1 \leq i \leq d)$ be a basis consisting of normalized Hecke eigenforms. The $2d$-dimensional space $S_k^!/D^{k-1}(M_{2-k}^!)$ splits into a direct sum*

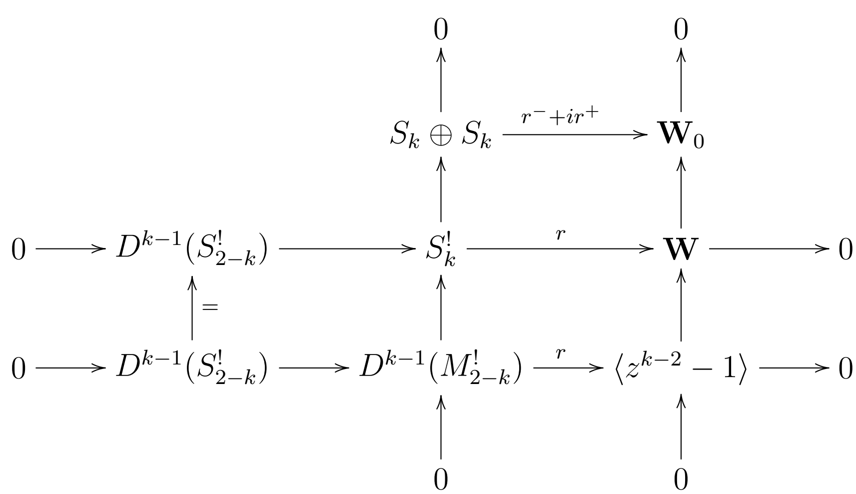
$$S_k^!/D^{k-1}(M_{2-k}^!) = \bigoplus_{i=1}^{d} \mathbb{T}_i$$

of two-dimensional spaces $\mathbb{T}_i$ such that $f_i \in \mathbb{T}_i$, and every element of $\mathbb{T}_i$ is a Hecke eigenform with respect to $S_k^!/D^{k-1}(M_{2-k}^!)$ with the same Hecke eigenvalues as f_i.

Two remarks

1) This multiplicity two phenomenon first appeared in a paper by the second author [10].

2) To clarify the results proved in this paper, we offer the following commutative diagram which clearly illustrates the relationships between the various spaces of modular forms and period polynomials, and describes the multiplicity two phenomenon.



We conclude with a study of Petersson's inner product, and a related inner product of Bruinier and Funke [5]. The Petersson inner product of cusp forms $f_1, f_2 \in S_k$ is the hermitian (i.e. $(f_1, f_2) = \overline{(f_2, f_1)}$) scalar product defined by $(z = x + iy)$

$$(f_1, f_2) := \int_{\mathbb{H}/SL_2(\mathbb{Z})} f_1(z)\overline{f_2(z)}y^k \cdot \frac{dxdy}{y^2}. \tag{1.14}$$

It is natural to seek an extension of this inner product to $M_k^!$. Obviously, one faces problems related to the convergence of the defining integral. Zagier [29, 30] extended the product to Eisenstein series using Rankin's method. More generally, Borcherds [1] (see [7] for a discussion) defined an extension to $M_k^!$ using regularized integrals, when at least one of the forms is holomorphic at the cusps. Here we give a closed formula for Borcherds's extension using periods of weakly holomorphic modular forms.

We relate Petersson's inner product to the "inner product" $\{\bullet, \bullet\}$ on $M_k^!$ which is defined as follows (also see discussions in [5, 7]). If $F, G \in M_k^!$ have expansions

$$F(z) = \sum_{n \gg -\infty} a_F(n)q^n \quad \text{and} \quad G(z) = \sum_{n \gg -\infty} a_G(n)q^n,$$

then define $\{F, G\}$ by

$$\{F, G\} := \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{a_F(-n)a_G(n)}{n^{k-1}}. \tag{1.15}$$

This pairing is antisymmetric (i.e. $\{F, G\} = -\{G, F\}$), bilinear, and is Hecke equivariant (i.e. $\{F \mid_k T(m), G\} = \{F, G \mid_k T(m)\}$). We show that it dissects $D^{k-1}(M_{2-k}^!)$ from $S_k^!$.

Theorem 1.6. *Let $F \in S_k^!$. The following conditions are equivalent:*

(i) $F \in D^{k-1}(M_{2-k}^!)$,
(ii) $r(F; z) \equiv 0 \pmod{z^{k-2} - 1}$,
(iii) $\{F, G\} = 0$*, for every* $G \in S_k^!$.

We now explain how to compute the extended $(\bullet, \bullet)$ in terms of $\{\bullet, \bullet\}$. Suppose that $F, G \in M_k^!$, and that $\mathcal{G} \in H_{2-k}$ has the property that $\xi_{2-k}(\mathcal{G}) = G$. As a consequence of Proposition 4.1 in § 4 it follows that

$$(F, G) = \{F, D^{k-1}(\mathcal{G})\} + a_F(0) \cdot a_{\mathcal{G}}^+(0), \tag{1.16}$$

whenever one of the forms F or G is holomorphic and where $a_{\mathcal{G}}^+(0)$ is the constant term of the mock modular form $\mathcal{G}^+$. Computing (F, G) then reduces to the task of computing $\{\bullet, \bullet\}$ on $M_k^!$.

Two Remarks.

1) Formula (1.16) gives an extension of the Petersson scalar product, one which works even when other "regularizations" fail.

2) Although there is ambiguity in the choice of $\mathcal{G} \in H_{2-k}$ such that $\xi_{2-k}(\mathcal{G}) = G$, we stress that the right-hand side of (1.16) does not depend on this choice.

Generalizing an argument of Kohnen and Zagier [18], we obtain the following closed formula for these products, which is an analog of a classical result of Haberland [11, 18].

Theorem 1.7. *For $F, G \in M_k^!$ we have*

$$\{F, G\} = \frac{(2\pi)^{k-1}}{3\cdot(k-2)!} \sum_{\substack{0\le m<n\le k-2\\ m\not\equiv n \pmod 2}} i^{(n+1+m)} \binom{k-2}{n}\binom{n}{m} r_n(F) r_{k-2-m}(G)$$
$$+ \frac{2\cdot(2\pi)^{k-1}}{3\cdot(k-2)!} \sum_{\substack{0\le n\le k-2\\ n\equiv 0 \pmod 2}} i^{(k-n)} \binom{k-1}{n+1} \left(r_n(G)\frac{a_F(0)}{k-1} - r_n(F)\cdot\frac{a_G(0)}{k-1}\right).$$

Remark. In a recent paper [13], the third author extended many of the results in this paper to include Eisenstein series.

In § 2 we recall definitions and basic facts about harmonic Maass forms, and we construct harmonic Maass-Poincaré series which map to the holomorphic Eisenstein series under ξ_{2-k} and D^{k-1}. In § 3 we derive some fundamental properties of the period functions and certain auxiliary integrals, and we conclude with a proof of Theorem 1.4. In § 4, we study Borcherds's extension of the Petersson inner product, and we conclude with proofs of Theorems 1.2, 1.3, 1.5, 1.6, and 1.7. In § 5 we recall some crucial analytic number theory which relates Eichler integrals to critical values of L-functions, and we prove Theorem 1.1.

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2. Harmonic Maass forms

Here we briefly recall basic facts about harmonic (weak) Maass forms (see [3, 5, 7, 21] for more details), we decompose $S_k^!$ (see Proposition 2.3), and we construct Maass-Poincaré series which naturally correspond to the classical Eisenstein series.

2.1. **Basic facts.** We let $z = x + iy \in \mathbb{H}$, the complex upper-half plane, with $x, y \in \mathbb{R}$, and suppose throughout that $k \geq 4$ is even. The *weight* $2-k$ *hyperbolic Laplacian* is defined by

$$\Delta_{2-k} := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i(2-k)y\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

A *harmonic Maass form of weight* $2-k$ is any smooth function $\mathcal{F} : \mathbb{H} \to \mathbb{C}$ satisfying:

(i) $\mathcal{F}(z) = (\mathcal{F}|_{2-k}\gamma)(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$;
(ii) $\Delta_{2-k}(\mathcal{F}) = 0$;
(iii) The function $\mathcal{F}(z)$ has at most linear exponential growth at infinity.

We denote the space of such forms by H_{2-k}. We also require the subspace H^*_{2-k} of H_{2-k}, which consists of those $\mathcal{F} \in H_{2-k}$ with the property that if $\mathcal{F} \neq 0$, then there is a nonzero polynomial $P_{\mathcal{F}} \in \mathbb{C}[q^{-1}]$ for which $\mathcal{F}(z) - P_{\mathcal{F}}(q) = O(e^{-\epsilon y})$, as $y \to +\infty$, for some $\epsilon > 0$.

The following describes the Fourier expansions of harmonic Maass forms (see [5]).

Proposition 2.1. *If $\mathcal{F} \in H_{2-k}$, then*

$$\mathcal{F}(z) = a_{\mathcal{F}}^{-}(0)y^{k-1} + \sum_{\substack{n \ll \infty \\ n \neq 0}} a_{\mathcal{F}}^{-}(n)h_{2-k}(2\pi ny)e(nx) + \sum_{n \gg -\infty} a_{\mathcal{F}}^{+}(n)q^n,$$

where $e(\alpha) := e^{2\pi i\alpha}$ *and* $h_{2-k}(w) := e^{-w}\int_{-2w}^{\infty} e^{-t}t^{k-2}dt$.

Therefore, we have that $\mathcal{F} = \mathcal{F}^- + \mathcal{F}^+$, where the *nonholomorphic part* $\mathcal{F}^-$ (resp. *holomorphic part* $\mathcal{F}^+$) is defined by

$$\begin{aligned} \mathcal{F}^-(z) &:= a_{\mathcal{F}}^{-}(0)y^{k-1} + \sum_{\substack{n \ll \infty \\ n \neq 0}} a_{\mathcal{F}}^{-}(n)h_{2-k}(2\pi ny)e(nx), \\ \mathcal{F}^+(z) &:= \sum_{n \gg -\infty} a_{\mathcal{F}}^{+}(n)q^n. \end{aligned} \tag{2.1}$$

Remark. If $n < 0$ and $\Gamma(\alpha, \beta) := \int_{\beta}^{\infty} e^{-t}t^{\alpha-1}dt$ is the incomplete Gamma-function, then we have

$$h_{2-k}(2\pi ny)e(nx) = \Gamma(k-1, 4\pi|n|y)q^n.$$

The following proposition (see [5, 7]) gives the main features of the differential operators $\xi_{2-k} := 2iy^{2-k}\overline{\frac{\partial}{\partial \bar{z}}}$ and $D := \frac{1}{2\pi i} \cdot \frac{d}{dz}$.

Proposition 2.2. *The following are true:*

(1) *The operator ξ_{2-k} defines the surjective maps*

$$\xi_{2-k} : H_{2-k}^* \twoheadrightarrow S_k,$$
$$\xi_{2-k} : H_{2-k} \twoheadrightarrow M_k^!.$$

(2) *The operator D^{k-1} defines maps*

$$D^{k-1} : H_{2-k}^* \to S_k^!,$$
$$D^{k-1} : H_{2-k} \to M_k^!.$$

Moreover, the map $D^{k-1} : H_{2-k} \twoheadrightarrow M_k^!$ is surjective.

The following proposition, whose proof uses Theorem 1.4, allows us to decompose a form $F \in S_k^!$ uniquely into a cusp form and an element in $D^{k-1}(H_{2-k}^*)$.

Proposition 2.3. *Each $F \in S_k^!$ has a unique representation of the form*

$$F(z) = \phi(z) + \psi(z),$$

where $\phi \in S_k$ and $\psi \in D^{k-1}(H_{2-k}^*)$.

Proof. First we show that such a representation, if it exists, is unique. Suppose on the contrary that $\widehat{\psi_1}, \widehat{\psi_2} \in H_{2-k}^*$ have the property that

$$F(z) = \phi_1(z) + D^{k-1}\left(\widehat{\psi_1}(z)\right) = \phi_2(z) + D^{k-1}\left(\widehat{\psi_2}(z)\right),$$

where $\phi_1, \phi_2 \in S_k$. Then $D^{k-1}\left(\widehat{\psi}_1 - \widehat{\psi}_2\right)$ is a cusp form, thus the holomorphic part of the function $\widehat{\psi}_1 - \widehat{\psi}_2$ has (up to the constant term) no principal part. Since this function is also in H^*_{2-k} it must be 0.

Now we establish the existence of the desired representation. By the modularity of F, it follows that $r(F;z) = r^-(F;z) + ir^+(F;z) \in \mathbf{W}$. The classical Eichler-Shimura isomorphism guarantees the existence of cusp forms $g_1, g_2 \in S_k$ such that

$$r^-(F;z) = r^-(g_1;z) \quad \text{and} \quad r^+(F;z) \equiv r^+(g_2;z) \pmod{z^{k-2}-1}.$$

By Proposition 2.2 (1), the operator ξ_{2-k} maps H^*_{2-k} onto S_k. Therefore there are harmonic Maass forms $\mathcal{G}_1, \mathcal{G}_2 \in H^*_{2-k}$ for which $\xi_{2-k}(\mathcal{G}_i) = (2i)^{k-1} g_i^c$, which one checks are also in S_k. Here we define for $G \in M_k^!$, the involution G^c as

$$G^c(z) := \overline{G(-\bar{z})}. \tag{2.2}$$

We find that that $\mathcal{E}_{G^c}(z) = \overline{\mathcal{E}_G(-\bar{z})}$, which in turn implies that

$$r(G^c;z) = -\overline{r(G;-\bar{z})}. \tag{2.3}$$

The fundamental theorem of calculus (with respect to $\bar{z}$) then implies that

$$\mathcal{G}_i(z) = \mathcal{G}_i^+(z) + \int_{-\bar{z}}^{i\infty} g_i(\tau)(\tau+z)^{k-2}\, d\tau,$$

where the $\mathcal{G}_i^+$ are holomorphic functions on $\mathbb{H}$.

The proof of Theorem 1.4 (see expression (3.8)) then implies that

$$r\left(D^{k-1}(\mathcal{G}_i); -z\right) \equiv -c_k \cdot r(g_i;z) \pmod{z^{k-2}-1}.$$

We let

$$\phi(z) := \frac{g_1(z)+g_2(z)}{2} \quad \text{and} \quad \Psi(z) := \frac{D^{k-1}(\mathcal{G}_1)(z) - D^{k-1}(\mathcal{G}_2)(z)}{2c_k},$$

and obtain that

$$r(F;z) \equiv r(\phi+\Psi;z) \pmod{z^{k-2}-1}. \tag{2.4}$$

Now define

$$h(z) := F(z) - \phi(z) - \Psi(z) \in S_k^!,$$

and observe that by (2.4), we have that $r(h;z) = \alpha(z^{k-2}-1)$ for some $\alpha \in \mathbb{C}$. Of course, this then means that $\mathcal{E}_h(z) + \frac{\alpha}{c_k} \in M^!_{2-k}$. Consequently, we then have that

$$h = D^{k-1}(\mathcal{E}_h) = D^{k-1}\left(\mathcal{E}_h + \frac{\alpha}{c_k}\right) \in D^{k-1}(M^!_{2-k}).$$

Letting $\psi = \Psi + h$ we obtain the desired decomposition. $\square$

2.2. **Maass-Poincaré series and Eisenstein series.** To obtain our results on the extended Petersson inner product, we must pay careful attention to constant terms of harmonic Maass forms and weakly holomorphic modular forms. To this end, we require weight $2-k$ harmonic Maass forms whose image under ξ_{2-k} are the classical Eisenstein series

$$E_k(z) := \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} (1|_k\gamma)(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \tag{2.5}$$

where $\Gamma_\infty := \{\pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) \ : \ n\in\mathbb{Z}\}$, B_k is the kth Bernoulli number, and $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$.

Remark. The Maass-Poincaré series $P_{E_k}(z)$ constructed below should not be confused with the Maass-Poincaré series which have been employed to study H_{2-k}^* (for example, see [3, 7, 21]). Those harmonic Maass forms project to cusp forms under ξ_{2-k}.

We define P_{E_k} by

$$P_{E_k}(z) := \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \left(y^{k-1}|_{2-k}\gamma\right)(z). \tag{2.6}$$

The following theorem provides the main properties of these Poincaré series.

Theorem 2.4. *If $k\geq 4$ is even, then the following are true:*

(1) *The function P_{E_k} is a harmonic Maass form of weight $2-k$ which satisfies*

$$D^{k-1}\left(P_{E_k}\right)(z) = -\frac{(k-1)!}{(4\pi)^{k-1}}E_k(z),$$
$$\xi_{2-k}\left(P_{E_k}\right)(z) = (k-1)E_k(z).$$

(2) *The Fourier expansion of P_{E_k} is given by*

$$\begin{aligned} P_{E_k}(z) \ &= \ \frac{2\cdot k!}{B_k}\cdot\frac{\zeta(k-1)}{(4\pi)^{k-1}} + y^{k-1} + \frac{(k-1)!}{(4\pi)^{k-1}}\frac{2k}{B_k}\sum_{n>0}\frac{\sigma_{k-1}(n)}{n^{k-1}}q^n, \\ &\quad + \frac{(k-1)}{(4\pi)^{k-1}}\frac{2k}{B_k}\sum_{n>0}\frac{\sigma_{k-1}(n)}{n^{k-1}}\Gamma(k-1,4\pi ny)q^{-n}. \end{aligned}$$

Remark. Theorem 1.4 for P_{E_k} and E_k follows immediately from Theorem 2.4 (1).

Proof. We first consider (1). By the standard theory of Poincaré series, one easily checks that P_{E_k} is a harmonic Maass form. The claimed images under ξ_{2-k} and D^{k-1} are obtained by applying these operators summand by summand. Straightforward calculations, combined with (2.5), gives (1).

We now consider (2). By Proposition 2.1 and part (1), we have that P_{E_k} has a Fourier expansion of the form

$$P_{E_k}(z) = a^+(0) + a^-(0)y^{k-1} + \sum_{n>0}a^+(n)q^n + \sum_{n>0}a^-(n)\Gamma(k-1,4\pi ny)q^{-n}.$$

The exact values for all Fourier coefficients but $a^+(0)$ can be determined by computing the action of D^{k-1} and ξ_{2-k} on P_{E_k} and comparing coefficients using (1).

The constant term $a^+(0)$ is then computed in a standard manner, but for the reader's convenience we add the proof. Using the notation $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and the fact that $\mathrm{Im}(\gamma(z)) = \frac{y}{|cz+d|^2}$, we obtain

$$P_{E_k}(z) = y^{k-1} \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} (cz+d)^{-1}(c\overline{z}+d)^{1-k} = y^{k-1} + y^{k-1} \sum_{c \geq 1} c^{-k} \sum_{\substack{d=0 \\ \gcd(c,d)=1}}^{c-1} V\left(z + \frac{d}{c}\right), \tag{2.7}$$

where

$$V(z) := \sum_{n \in \mathbb{Z}} v(z+n) \quad \text{with} \quad v(z) := z^{-1}\overline{z}^{1-k}.$$

The sum $V(z)$ is now explicitly evaluated on page 84 of [27], and has the term $\frac{\pi}{(2i)^{k-2}} y^{1-k}$. Using (2.7), it follows that the constant term $a^+(0)$ of $P_{E_k}(z)$ is given by

$$a^+(0) = \frac{\pi}{(2i)^{k-2}} \sum_{c \geq 1} \frac{\phi(c)}{c^k} = \frac{\pi}{(2i)^{k-2}} \frac{\zeta(k-1)}{\zeta(k)},$$

where ϕ is Euler's totient function and we used that it is multiplicative. Finally, we may simplify further using the classical evaluation $\zeta(k) = -\frac{(2\pi i)^k B_k}{2 \cdot k!}$ to obtain the desired form of the constant term. ☐

3. Properties of period functions

Here we consider auxiliary functions related to period functions, and we then give some consequences for the period functions of harmonic Maass forms and weakly holomorphic modular forms. We then conclude with the proof of Theorem 1.4.

3.1. **Some auxiliary functions related to periods.** Here we define auxiliary functions which relate period functions of weakly holomorphic modular forms to Eichler integrals.

Recall again that if $g \in S_k$, then

$$c_k \mathcal{E}_g(z) = \int_z^{i\infty} g(\tau)(\tau - z)^{k-2}\, d\tau.$$

Although such integrals do not converge for $G \in S_k^!$ with a pole at infinity, for $\rho := \frac{1+\sqrt{-3}}{2}$ we have the convergent integral

$$\mathcal{E}_G^\rho(z) := \int_z^\rho G(\tau)(\tau - z)^{k-2}\, d\tau. \tag{3.1}$$

An induction argument shows that, for any integer $n \geq 0$,

$$\int_z^\rho G(\tau)(\tau - z)^n\, d\tau = n! \int_z^\rho \int_{z_n}^\rho \cdots \int_{z_1}^\rho G(z_0)\, dz_0 \cdots dz_{n-1}\, dz_n.$$

It follows that

$$D^{k-1}\left(\mathcal{E}_G^\rho(z)\right) = c_k G(z),$$

and by (1.6) we have that

$$\mathcal{E}_G^{\rho}(z) = c_k \mathcal{E}_G(z) + q_G(z), \tag{3.2}$$

where $q_G(z)$ is a polynomial of degree $\leq k-2$.

Remark. The discussion above holds if ρ is replaced by any point in $\mathbb{H}$. However, the subsequent discussion will make important use of the fact that ρ is an elliptic fixed point. We could have chosen ρ^2 or i in its place.

We also require the auxiliary function

$$H_G(z) := \int_{\rho^2}^{\rho} G(\tau)(z-\tau)^{k-2}\, d\tau. \tag{3.3}$$

We note that z in this setting is not required to be an element of $\mathbb{H}$. In the next proposition we record some properties of the functions $r(G; z)$, q_G, and H_G involving the action of the matrices S and T.

Proposition 3.1. *Suppose that $G \in S_k^!$. Then the following are true:*

(1) *We have that*
$$H_G(z) = \left(\mathcal{E}_G^{\rho}|_{2-k}(1-S)\right)(z) = \left(\mathcal{E}_G^{\rho}|_{2-k}(1-T)\right)(z).$$
(2) *We have that*
$$\left(H_G|_{2-k}(1+S)\right)(z) = 0.$$
(3) *We have that*
$$H_G(z) = \left(q_G|_{2-k}(1-T)\right)(z) = r(G; z) + \left(q_G|_{2-k}(1-S)\right)(z).$$
(4) *We have that*
$$r(G; z) = \left(q_G|_{2-k}(S-T)\right)(z).$$

Proof. Claim (1) follows from the fact that $-\rho^{-1} = \rho - 1 = \rho^2$, and claim (2) follows by (1). Claim (3) is obtained by applying $(1-S)$ and $(1-T)$ to (3.2), and (4) follows immediately from (3). □

We also require a nonholomorphic analog of $\mathcal{E}_G^{\rho}$, namely the function

$$\Phi_G(z) := \int_{-\bar{z}}^{\rho} G(\tau)(\tau+z)^{k-2}\, d\tau. \tag{3.4}$$

Proposition 3.2. *Suppose that $G \in S_k^!$. Then the following are true:*

(1) *We have that*
$$\left(\Phi_G|_{2-k}T^{-1}\right)(z) = \left(\Phi_G|_{2-k}S\right)(z) = \int_{-\overline{z}}^{\rho^2} G(\tau)(\tau+z)^{k-2} d\tau.$$
(2) *We have that*
$$H_G(-z) = \left(\Phi_G|_{2-k}(1-T^{-1})\right)(z) = \left(\Phi_G|_{2-k}(1-S)\right)(z).$$

Proof. Claim (1), which follows by substitution, immediately implies (2). □

3.2. **The role of harmonic Maass forms.** Here we obtain relations between Φ_G and harmonic Maass forms. As in the proof of Proposition 2.3, we make use of the involution (2.2) on $M_k^!$ which preserves the space $S_k^!$. Suppose that $G \in S_k^!$ is fixed. By Proposition 2.2 (1), let $\mathcal{F} \in H_{2-k}$ be a harmonic Maass form for which $\xi_{2-k}(\mathcal{F})(z) = (2i)^{k-1}G^c(z)$. The fundamental theorem of calculus (with respect to $\bar{z}$), then implies that

$$\mathcal{F}(z) = \int_{-\bar{z}}^{\rho^2} G(\tau)(\tau+z)^{k-2}\, d\tau + C_G(z), \tag{3.5}$$

where C_G is holomorphic on $\mathbb{H}$. The next proposition relates Φ_G and C_G.

Proposition 3.3. *Assume the notation and hypotheses above. Then the following are true:*

(1) *We have that*

$$\Phi_G(z) = \mathcal{F}(z) - \left(C_G|_{2-k}T\right)(z) = \mathcal{F}(z) - \left(C_G|_{2-k}S\right)(z).$$

(2) *We have that*

$$\left(C_G|_{2-k}T\right)(z) = \left(C_G|_{2-k}S\right)(z).$$

Proof. By (3.5) and Proposition 3.2 (1), we have that

$$\left(\Phi_G|_{2-k}T^{-1}\right)(z) = \left(\Phi_G|_{2-k}S\right)(z) = \mathcal{F}(z) - C_G(z).$$

We obtain (1) by applying T and S to $\mathcal{F}$, and (2) follows immediately from (1). □

To prove Theorem 1.4, we shall make use of the following elementary proposition.

Proposition 3.4. *For polynomials $p(z)$ of degree at most $-\ell \in 2\mathbb{N}$, let $\widetilde{p}(z) := p(-z)$. Then*

$$\left(\widetilde{p|_l S}\right)(z) = \left(\widetilde{p}|_l S\right)(z) \quad \text{and} \quad \left(\widetilde{p|_l T}\right)(z) = \left(\widetilde{p}|_l T^{-1}\right)(z).$$

3.3. **The proof of Theorem 1.4.** We require the following proposition.

Proposition 3.5. *There are forms in $M_{2-k}^!$ with nonzero constant terms.*

Proof. Suppose that f_1 and f_2 are holomorphic modular forms with leading coefficient 1 for which f_1/f_2 has weight $2-k$. One easily finds a polynomial (in variable x and dependent upon f_1 and f_2), say $M(f_1, f_2; x)$, for which

$$\widehat{M}(f_1, f_2; z) := \frac{f_1(z)}{f_2(z)} \cdot M(f_1, f_2; j(z)) \tag{3.6}$$

is in $M_{2-k}^!$. Here $j(z) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n$ is the usual Hauptmodul for $SL_2(\mathbb{Z})$. This polynomial is chosen to cancel poles in $\mathbb{H}$. For convenience, suppose that

$$\widehat{M}(f_1, f_2; z) = \sum_{n=-m}^{\infty} a(n)q^n.$$

For every prime p, let $j_p(z)$ be the modular function

$$j_p(z) := p\left((j(z) - 744)|T(p)\right) = q^{-p} + \sum_{n=1}^{\infty} c_p(n)q^n,$$

where $T(p)$ is the usual Hecke operator. Define a weight $2-k$ form in $M_{2-k}^!$ by

$$\widehat{M}_p(f_1, f_2; z) := \widehat{M}(f_1, f_2; z) \cdot j_p(z) = \sum_{n=-m-p}^{\infty} a_p(n) q^n.$$

Obviously, we have that the constant term is given by

$$a_p(0) = a(p) + c_p(m) + \sum_{n=1}^{m-1} a(-n) c_p(n).$$

Using the definition of the Hecke operators, for primes $p > m$, we have that $a_p(0)$ vanishes if and only if

$$a(p) = -pc(pm) - p \sum_{n=1}^{m-1} a(-n) c(pn). \tag{3.7}$$

Using the "circle method", or the method of Poincaré series (for example, see [4]), it follows that there are nonzero constants κ_1, κ_2 such that, for $1 \leq n \leq m$, we have

$$a(p) \sim \kappa_1 p^{\frac{k-1}{2}} \cdot I_{k-1}(4\pi\sqrt{mp}),$$
$$pc(pn) \sim \kappa_2 \sqrt{\frac{p}{n}} \cdot I_1(4\pi\sqrt{np}).$$

Here $I_\alpha(x)$ is the usual I-Bessel function of order α. Using the asymptotics for $I_1(x)$, the right hand side of (3.7) satisfies

$$-pc(pm) - p \sum_{n=1}^{m-1} a(-n) c(pn) \sim \kappa_2 \sqrt{\frac{p}{m}} \cdot I_1(4\pi\sqrt{mp}),$$

as $p \to +\infty$ among primes. Since $\lim_{x\to+\infty} \frac{I_{k-1}(x)}{I_1(x)} = 1$, and since $k \geq 4$ is even, this asymptotic and the one for $a(p)$ are not compatible with (3.7). Therefore, the constant terms of $\widehat{M}_p(f_1, f_2; z)$ are nonvanishing for all large primes p. ☐

We now prove Theorem 1.4.

Proof of Theorem 1.4. We begin by proving the first claim in Theorem 1.4. We continue using the notation and hypotheses on $\mathcal{F}$ and G from the previous subsection. For the case when G is a constant multiple E_k, the result follows easily from the work in §2.2. Otherwise, we fix $G \in S_k^!$ and assume that $\mathcal{F} \in H_{2-k}$ satisfies

$$\xi_{2-k}(\mathcal{F}) = (2i)^{k-1} G^c(z).$$

Now let $F := D^{k-1}(\mathcal{F})$.

Making use of (2.3), we find that it suffices to prove that

$$\widetilde{r(F; z)} \equiv -c_k \cdot r(G; z) \pmod{z^{k-2} - 1}. \tag{3.8}$$

Let p_G be the holomorphic function given by

$$p_G(z) := C_G(z) - \mathcal{E}_F(z).$$

Since we have that

$$D^{k-1}(p_G(z)) = D^{k-1}(C_G(z)) - D^{k-1}(\mathcal{E}_F(z)) = F(z) - F(z) = 0,$$

it follows that p_G is a polynomial of degree $\leq k-2$. By definition (1.7), we obtain, by applying S to the definition of p_G, that

$$\left(p_G|_{2-k}(1-S)\right)(z) = \left(C_G|_{2-k}(1-S)\right)(z) - \frac{1}{c_k} r(F;z).$$

Moreover, applying T to the definition of p_G gives

$$\left(p_G|_{2-k}(1-T)\right)(z) = \left(C_G|_{2-k}(1-T)\right)(z). \tag{3.9}$$

By Proposition 3.3 (2), we then find that

$$\frac{1}{c_k} r(F;z) = \left(p_G|_{2-k}(S-T)\right)(z). \tag{3.10}$$

We now relate the polynomials H_G and p_G. Combining Proposition 3.2 (2) and Proposition 3.3 (1) with the modularity of $\mathcal{F}$ and (3.9), we find that

$$\widetilde{H_G}(z) = \left(\Phi_G|_{2-k}(1-T^{-1})\right)(z) = \left(C_G|_{2-k}(1-T)\right)(z) = \left(p_G|_{2-k}(1-T)\right)(z).$$

Proposition 3.4 then implies that

$$H_G(z) = \left(\widetilde{p_G}|_{2-k}(1-T^{-1})\right)(z) = -\left(\widetilde{p_G}|_{2-k}T^{-1}(1-T)\right)(z),$$

and Proposition 3.1 (3) in turn implies that

$$\left(\left(q_G + \widetilde{p_G}|_{2-k}T^{-1}\right)|_{2-k}(1-T)\right)(z) = 0.$$

This means that the polynomial $\left(q_G + \widetilde{p_G}|_{2-k}T^{-1}\right)(z)$ is a constant, say α. Applying TS to the resulting identity

$$q_G(z) = -\left(\widetilde{p_G}|_{2-k}T^{-1}\right)(z) + \alpha, \tag{3.11}$$

we obtain

$$\left(q_G|_{2-k}TS\right)(z) = -\left(\widetilde{p_G}|_{2-k}S\right)(z) + \alpha z^{k-2}. \tag{3.12}$$

We now compare $c_k \cdot r(G;z)$ and $\widetilde{r(F;z)}$. By (3.10) and Proposition 3.4, we have

$$\frac{1}{c_k}\widetilde{r(F;z)} = \left(\widetilde{p_G}|_{2-k}(S-T^{-1})\right)(z).$$

Combining this with Proposition 3.1 (4), and making use of (3.11) and (3.12), we then obtain

$$\begin{aligned}\frac{1}{c_k}\widetilde{r(F;z)} + r(G;z) &= \left(\widetilde{p_G}|_{2-k}S\right)(z) - \left(\widetilde{p_G}|_{2-k}T^{-1}\right)(z) + \left(q_G|_{2-k}S\right)(z) - \left(q_G|_{2-k}T\right)(z)\\ &= -\left(q_G|_{2-k}TS\right)(z) + \alpha z^{k-2} + q_G(z) - \alpha + \left(q_G|_{2-k}S\right)(z) - \left(q_G|_{2-k}T\right)(z)\\ &= \left(q_G|_{2-k}(1-T)(1+S)\right)(z) + \alpha\left(z^{k-2}-1\right).\end{aligned}$$

Since Proposition 3.1 gives the identities

$$\left(q_G|_{2-k}(1-T)\right)(z) = H_G(z) \quad \text{and} \quad \left(H_G|_{2-k}(1+S)\right)(z) = 0,$$

we conclude that

$$\frac{1}{c_k}\widetilde{r(F;z)} + r(G;z) = \alpha\left(z^{k-2}-1\right).$$

This proves (3.8), and it completes the proof of the first claim of the theorem.

To prove the second claim, it suffices to produce a weakly holomorphic form $\mathcal{W} \in M_{2-k}^!$ with nonzero constant term β. It is easy to see that $r(D^{k-1}(\mathcal{W});z) = \beta c_k(z^{k-2}-1)$ by modularity of $\mathcal{W}$. Since $\xi_{2-k}(\mathcal{W}) = 0$ and $r(D^{k-1}(\mathcal{F});z) - r(D^{k-1}(\mathcal{F}+\mathcal{W});z)$ is a nonzero constant multiple of $z^{k-2}-1$, the claimed second identity follows easily. The existence of such a form is guaranteed by Proposition 3.5. □

4. The extended Petersson inner product

We now apply the results of the last section to prove Theorems 1.2, 1.3, 1.5, 1.6, and 1.7.

4.1. **General considerations.**

We first recall the extension of $(\bullet, \bullet)$ to $M_k^!$, and we obtain a closed formula for it in terms of periods. Denote by D_T the truncated fundamental domain $(\tau = x+iy)$

$$D_T := \left\{\tau \in \mathbb{H} \ : \ |\tau| \geq 1,\ |x| \leq \frac{1}{2},\ y \leq T\right\}. \tag{4.1}$$

Write $F, G \in M_k^!$ as

$$F(z) = \sum_{n \gg -\infty} a_F(n)q^n \quad \text{and} \quad G(z) = \sum_{n \gg -\infty} a_G(n)q^n.$$

Then we may define an extension of Petersson's inner product as

$$(F,G) := \lim_{T\to\infty}\left(\int_{D_T} F(\tau)\overline{G(\tau)}y^{k-2}\,dx\,dy - \frac{a_F(0)\overline{a_G(0)}}{k-1}T^{k-1}\right) \tag{4.2}$$

when the limit exists.
Identity (1.16) is an immediate consequence of the following proposition.

Proposition 4.1. *In the following cases*

(i) $F \in M_k$ *and* $G \in M_k^!$
(ii) $F \in M_k^!$ *and* $G \in M_k$

the extended Petersson product is well defined, and is given by

$$(F,G) = \text{constant term of } F\mathcal{G}^+,$$

where $\mathcal{G} \in H_{2-k}$ such that $\xi_{2-k}(\mathcal{G}) = G$. Moreover, we have that

$$(F,G) = \frac{1}{3\cdot 2^{k-1}} \sum_{\substack{0\le m<n\le k-2\\ m\not\equiv n \pmod 2}} i^{(n+1-m)} \binom{k-2}{n}\binom{n}{m} r_n(F)\overline{r_{k-2-m}(G)}$$
$$+ \frac{2}{3\cdot 2^{k-1}} \sum_{\substack{0\le n\le k-2\\ n\equiv 0 \pmod 2}} i^{(k-n)} \binom{k-1}{n+1} \left(\overline{r_n(G)}\frac{a_F(0)}{k-1} + r_n(F)\frac{\overline{a_G(0)}}{k-1}\right). \tag{4.3}$$

Proof. The existence of an appropriate harmonic Maass form $\mathcal{G}$ in every case follows from Proposition 2.2 (1). That (4.2) is well defined can be proved using an argument of Bruinier and Funke (see Proposition 3.5 of [5]). It is easy to see that the restrictions imposed in their work may be relaxed to obtain

$$(F,G) = \lim_{T\to\infty}\left(\int_{D_T} F(\tau)\overline{G(\tau)}y^{k-2}\ dx\ dy - \frac{a_F(0)\overline{a_G(0)}}{k-1}T^{k-1}\right) = \text{constant term of } F\mathcal{G}^+.$$

To complete the proof, we need to prove formula (4.3). Due to the Hermitian properties of the extended Petersson scalar product, it suffices to consider the following three cases:

Case (1): $F = G = E_k$.
Case (2): $F \in S_k^!$ and $G \in S_k$.
Case (3): $F \in S_k^!$ and $G = E_k$.

For Case (1), we begin by recalling the values of the periods for E_k (see page 240 of [18]):

$$\begin{aligned}
r_0(E_k) &= \frac{k}{B_k}\frac{(-1)^{\frac{k}{2}}(k-2)!}{(2\pi)^{k-1}}\cdot\zeta(k-1),\\
r_{k-2}(E_k) &= \frac{k}{B_k}\frac{(k-2)!}{(2\pi)^{k-1}}\cdot\zeta(k-1),\\
r_n(E_k) &= 0 && (\text{ for } 0<n<k-2,\ n \text{ even}),\\
r_n(E_k) &= -\frac{k}{B_k}(-1)^{\frac{n+1}{2}}\frac{B_{n+1}}{n+1}\frac{B_{k-1-n}}{k-1-n} && (\text{ for } 0<n<k-2,\ n \text{ odd}).
\end{aligned}\tag{4.4}$$

We substitute these values into the right hand side of (4.3), and make use of Euler's identity for Bernoulli numbers

$$\sum_{m=2}^{k-2}\binom{k}{m}B_mB_{k-m} = -(k+1)B_k \tag{4.5}$$

(for integers $k\ge 4$). Noting that $\mathcal{G} = \frac{P_{E_k}}{k-1}$ now easily gives the claim computing the constant term of $E_k\mathcal{G}^+$ using Theorem 2.4. We note that this result matches Zagier's calculation [29] for (E_k, E_k).

Case (2) is proven by modifying an argument of Kohnen and Zagier (see pp. 244-246 of [18]) which they used to prove the Haberland identity for cusp forms. We begin by considering the given contour integral, and recall the well known fact (i.e. Stokes' Theorem) that

$$(F,G) = -\lim_{T\to\infty}\int_{\partial D_T} F(\tau)\mathcal{G}(\tau)\ d\tau = -\lim_{T\to\infty}\int_{\partial D_T} F(\tau)\mathcal{G}^-(\tau)\ d\tau,$$

since the function $F\mathcal{G}^+$ is holomorphic on D_T. Therefore we have that $\int_{\partial D_T} F(\tau)\mathcal{G}^+(\tau)\ d\tau = 0$. The function $F\mathcal{G}^-$ is periodic with period 1 in x, because both F and $\mathcal{G}^-$ are. Thus the integrals along the vertical lines cancel. Moreover, as in the proof of Proposition 3.5 in [5], we can show that

$$\lim_{T\to\infty}\int_{-1/2}^{1/2} F(x+iT)\mathcal{G}^-(x+iT)\ dx = 0,$$

and so

$$(F,G) = -\int_C F(\tau)\mathcal{G}^-(\tau)\ d\tau,$$

where C is the arc of the unit circle from ρ^2 to ρ which bounds the fundamental domain from the bottom. Note that $F\mathcal{G}^-\ d\tau$ is not invariant under S, so this integral may be non-zero. Also, S maps C into itself with orientation reversed, so we have

$$2(F,G) = -\int_C F(\tau)(\mathcal{G}^-|_{2-k}(1-S))(\tau)\ d\tau.$$

Now, because $\mathcal{G} = \mathcal{G}^+ + \mathcal{G}^-$ is of weight $2-k$, by Theorem 1.4 and (1.7) we have

$$\begin{aligned}\left(\mathcal{G}^-|_{2-k}(1-S)\right)(z) = \left(-\mathcal{G}^+|_{2-k}(1-S)\right)(z) &\equiv -\frac{1}{c_k} r(D^{k-1}(\mathcal{G}); z)\\ &\equiv \frac{1}{c_k}\frac{(k-2)!}{(4\pi)^{k-1}}\overline{r(G;\bar{z})} \pmod{z^{k-2}-1}.\end{aligned}$$

Thus we have that

$$\begin{aligned}2(2i)^{k-1}(F,G) &= -(2i)^{k-1}\int_C F(\tau)(\mathcal{G}^-|_{2-k}(1-S))(\tau)\ d\tau\\ &= -(2i)^{k-1}\frac{1}{c_k}\frac{(k-2)!}{(4\pi)^{k-1}}\int_C F(\tau)\overline{r(G;\bar{\tau})}\ d\tau = -\int_C F(\tau)\overline{r(G;\bar{\tau})}\ d\tau.\end{aligned} \tag{4.6}$$

Equality holds in each of the above steps since

$$\int_{\rho^2}^{\rho} F(\tau)(\tau^{k-2}-1)\ d\tau = 0,$$

which in turn follows since F is modular of weight k without a constant term.

We now proceed as in [18] and define a pairing on polynomials in $\mathbf{V}$ (degree at most $k-2$) as follows

$$\left\langle \sum_{n=0}^{k-2} a_n z^n, \sum_{n=0}^{k-2} b_n z^n \right\rangle := \sum_{n=0}^{k-2}(-1)^n \binom{k-2}{n}^{-1} a_n b_{k-2-n}.$$

By a straightforward but lengthy calculation, this pairing is symmetric and $SL_2(\mathbb{Z})$-invariant (i.e. for all $p, q \in \mathbf{V}$ and $\gamma \in SL_2(\mathbb{Z})$ we have $\langle p|_{2-k}\gamma, q|_{2-k}\gamma\rangle = \langle p, q\rangle$). We may rewrite (4.6) as

$$2(2i)^{k-1}(F,G) = -\left\langle H_F(z), \overline{r(G;\overline{z})}\right\rangle \tag{4.7}$$

where H_F was defined in (3.3). Making use of Proposition 3.1 (2), (3), and (4), along with the relations defining the space $\mathbf{W}$, and the properties of the pairing $\langle \bullet, \bullet\rangle$, we have the following:

$$\begin{aligned}
\left\langle H_F(z), \overline{r(G;\overline{z})}\right\rangle &= \left\langle \left(q_F|_{2-k}\left(1-T\right)\right)(z), \overline{r(G;\overline{z})}\right\rangle \\
&= \left\langle q_F(z), \overline{r(G;\overline{z})}\ |_{2-k}\left(1-T^{-1}\right)\right\rangle \\
&= \left\langle q_F(z), \overline{r(G;\overline{z})}\ |_{2-k}\left(1+ST^{-1}\right)\right\rangle \\
&= \left\langle q_F(z), \overline{r(G;\overline{z})}\ |_{2-k}\left(1+U^2\right)\right\rangle \\
&= \frac{1}{3}\left\langle q_F(z), \overline{r(G;\overline{z})}\ |_{2-k}\left(U^2-U\right)\left(1-U^{-1}\right)\right\rangle \\
&= \frac{1}{3}\left\langle \left(q_F\ |_{2-k}\left(1-U\right)\right)(z), \overline{r(G;\overline{z})}\ |_{2-k}\left(U^2-U\right)\right\rangle \\
&= \frac{1}{3}\left\langle -r(F;z), \overline{r(G;\overline{z})}\ |_{2-k}\left(ST^{-1}-TS\right)\right\rangle \\
&= \frac{1}{3}\left\langle r(F;z)|_{2-k}\left(T-T^{-1}\right), \overline{r(G;\overline{z})}\right\rangle.
\end{aligned}$$

Note that we used the fact that $\overline{r(G;\overline{z})} \in \mathbf{W}$. This follows by conjugating the period relations (see page 199 of [18]) defining $r(G;z) \in \mathbf{W}$. Identity (4.3) follows by combining the above calculation with (4.7) to obtain

$$-6(2i)^{k-1}(F,G) = \left\langle r(F;z)|_{2-k}(T-T^{-1}), \overline{r(G;\overline{z})}\right\rangle.$$

Case (3) may be broken into three subcases by making use of the Hermitian properties of the extended Petersson product along with Propositions 2.3 and 3.5, i.e.

Case (3a): $F \in S_k$.
Case (3b): $F = D^{k-1}(\Psi)$ with $\Psi \in H^*_{2-k}$ whose constant term vanishes.
Case (3c): $F = D^{k-1}(\Psi)$ with $\Psi \in M^!_{2-k}$.

Case (3a) is proven in [18], so we focus instead of cases (3b) and (3c). In both cases, we let $\mathcal{G} = \frac{P_{E_k}}{k-1}$.

For case (3b), we make use of the Fourier expansion of $\mathcal{G}$ as given in Theorem 2.6 to obtain

$$\begin{aligned}(D^{k-1}(\Psi), G) &= \text{ constant term of } D^{k-1}(\Psi)\mathcal{G}^+ \\ &= \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \text{ constant term of } G\Psi^+ \\ &= \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \cdot (G, \xi_{2-k}(\Psi))\end{aligned}$$

We must then show that the right hand side of (4.3) is the same. Using Theorem 1.4 and the fact that the constant term of Ψ vanishes, we are able to deduce that

$$r(\xi_{2-k}(\Psi); z) = -\frac{(4\pi)^{k-1}}{\Gamma(k-1)} \cdot \overline{r(D^{k-1}(\Psi); \overline{z})},$$

from which it then follows that for integers $0 \leq n \leq k-2$, we have

$$r_n(D^{k-1}(\Psi)) = (-1)^n \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \cdot \overline{r_n(\xi_{2-k}(\Psi))}.$$

Using these period relations and the periods of G in (4.4), we find that the right hand side of (4.3) is also equal to $\frac{\Gamma(k-1)}{(4\pi)^{k-1}}(G, \xi_{2-k}(\Psi))$, which completes this case.

For case (3c), we let $\Psi = \sum_{n \gg -\infty} b_n q^n \in M^!_{2-k}$. We begin by noting that $\Psi G \in M^!_2$ has a vanishing constant term since it is a derivative of a polynomial in the j-function. Combining this fact with the Fourier expansion of $\mathcal{G}$ in Theorem 2.6, we find that

$$b_0 = -\frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) b_{-n}.$$

It then follows that

$$\begin{aligned}(D^{k-1}(\Psi), G) &= \text{constant term of } D^{k-1}(\Psi)\mathcal{G}^+ \\ &= -\frac{\Gamma(k-1)}{(4\pi)^{k-1}} b_0.\end{aligned}$$

To show that the right hand side of (4.3) coincides with this, we begin by noting that since $\Psi \in M^!_{2-k}$ we have $r(D^{k-1}(\Psi); z) = c_k b_0 \left(z^{k-2} - 1\right)$. From this and (1.9), it follows that $r_0(D^{k-1}(\Psi)) = ic_k b_0$ and $r_n(D^{k-1}(\Psi)) = i^{k-1} c_k b_0$. Finally, we substitute these values and the periods of G given in (4.4) into the right hand side of (4.3), and make use of (4.5) to obtain the result. □

4.2. **Proof of Theorem 1.3.** Let $F \in S^!_k$ and $G = E_k$. Then we may use Proposition 4.1 and simplify (4.3). For even $0 < m < k-2$, we have $r_m(E_k) = 0$ by (4.4), so the summation over even m reduces to case when $m = 0$. For $m = 0$, we have that

$$\sum_{\substack{1 \leq n \leq k-1 \\ n \text{ odd}}} i^{(n+1)} \binom{k-2}{n} r_n(F) = 0. \tag{4.8}$$

The last equality follows from $r^-(F)|(1+U+U^2)=0$ (see [18, p.199]). Making use of the fact that $a_F(0)=0$ and the formulas for the odd-indexed periods of the Eisenstein series E_k in (4.4), we reduce (4.3) to

$$(F,E_k)=-\frac{1}{3B_k(k-1)(2i)^k}\sum_{\substack{0\leq n\leq k-2\\ n\ \text{even}}} i^n\Lambda_{k,n}r_n(F),$$

with

$$\Lambda_{k,n}:=B_k\left[\binom{k-1}{n}-\binom{k-1}{n+1}\right]+\sum_{r=1}^{k/2}\binom{k}{2r}\left[\binom{2r-1}{n}-\binom{2r-1}{k-2-n}\right]B_{2r}B_{k-2r}.$$

Let $\mathcal{G}:=\frac{1}{k-1}P_{E_k}$ so that $\xi_{2-k}(\mathcal{G})=G$. By Theorem 2.4, the constant term of the product $F\mathcal{G}^+$ is equal to the right side of (1.12). The theorem now follows from Proposition 4.1 and the following equality for even n:

$$\Lambda_{k,n}=\lambda_{k,n}.$$

In order to prove the latter identity, we observe by [18, Theorem 9(i)] that, for even n,

$$\lambda_{k,n}-\Lambda_{k,n}=B_k+\sum_{\substack{r=2\\ \text{even}}}^{k}\binom{k}{r}\left[\binom{r-1}{n}+\binom{r-1}{k-2-n}\right]B_{k-r}B_r=\frac{1}{2}\left(\lambda_{k,n}+\lambda_{k,k-2-n}\right)=0.$$

4.3. **Proof of Theorem 1.6.** Let $F\in S_k^!$ be given.

To prove (i) → (ii), we assume that $F=D^{k-1}(\mathcal{F})$ where $\mathcal{F}\in M_{2-k}^!$ has constant term $\alpha\in\mathbb{C}$. Then by (1.6) we have $\mathcal{F}-\alpha=\mathcal{E}_F$ and it follows by modularity of $\mathcal{F}$ and (1.7) that $r(F;z)=\alpha c_k(z^{k-2}-1)$.

For the implication (ii) → (iii), we have that $r(F;z)=\alpha(z^{k-2}-1)$ for some $\alpha\in\mathbb{C}$. By (1.7), $\mathcal{E}_F+\alpha/c_k\in M_{2-k}^!$. This implies that for $G\in S_k^!$, the scalar product $\{F,G\}$ equals the constant term of the weight 2 weakly holomorphic modular form $-(\mathcal{E}_F+\alpha/c_k)G$, and vanishes, because every such form is a derivative of a polynomial in the j-function.

We now prove (iii) → (i). By Proposition 2.3, we may write

$$F=\phi+\psi$$

with $\phi\in S_k$ and $\psi=D^{k-1}(\mathcal{G})$ where $\mathcal{G}\in H_{2-k}^*$. By our hypothesis and Proposition 4.1 (i), for every $h\in S_k$,

$$0=\{h,F\}=\{h,\psi\}=(h,\xi_{2-k}(\mathcal{G})).$$

We conclude that $\xi_{2-k}(\mathcal{G})=0$. Therefore $\mathcal{G}\in M_{2-k}^!$ and $\psi\in D^{k-1}(M_{2-k}^!)$. Now for every $h\in S_k$ there exists $\mathcal{G}_h\in H_{2-k}^*$ such that $\xi_{2-k}(\mathcal{G}_h)=h$. Since $D^{k-1}(\mathcal{G}_h)\in S_k^!$, we can use our hypothesis and Proposition 4.1 (i) again to conclude that for every $h\in S_k$,

$$0=\{F,D^{k-1}(\mathcal{G}_h)\}=\{\phi,D^{k-1}(\mathcal{G}_h)\}+\{\psi,D^{k-1}(\mathcal{G}_h)\}=\{\phi,D^{k-1}(\mathcal{G}_h)\}=(\phi,h).$$

The third equality holds by following the proofs of implications (i) → (ii) → (iii). Therefore $\phi=0$ and so $F=\psi\in D^{k-1}(M_{2-k}^!)$ as required.

4.4. **Proof of Theorem 1.2.** The injectivity of the embedding $D^{k-1}(M^!_{2-k}) \to S^!_k$ is obvious, and the exactness in $S^!_k$, for the first sequence, follows immediately from Theorem 1.6. Therefore, it suffices to establish surjectivity.

The argument closely follows our proof of Proposition 2.3. The Eichler-Shimura isomorphism allows us to write an arbitrary polynomial $r \in \mathbf{W}_0$ as

$$r = r^-(g_1) + ir^+(g_2)$$

with $g_1, g_2 \in S_k$. Using the notation from the proof of Proposition 2.3, we then find that $F(z) := \phi(z) + \Psi(z) \in S^!_k$, and $r(F; z) = r \in \mathbf{W}_0$, which establishes surjectivity.

The exactness of the second sequence now easily follows from

$$r : D^{k-1}(M^!_{2-k})/D^{k-1}(S^!_{2-k}) \tilde{\to} \mathbf{W}/\mathbf{W}_0 \cong \left\langle z^{k-2} - 1 \right\rangle .$$

This is an immediate consequence of Proposition 3.5, the definition (1.7), and the fact that for a form $h \in M^!_{2-k}$ the constant term of h equals $h - \mathcal{E}_{D^{k-1}(h)}$.

4.5. **Proof of Theorem 1.5.** Let $d = \dim(S_k)$, and for $1 \leq i \leq d$, let

$$f_i(z) = \sum_{n>0} b_i(n) q^n \in S_k$$

be a basis of normalized Hecke eigenforms for S_k. For each i, $\{b_i(n)\}_{n>0}$ is a system of Hecke eigenvalues, and $f_i \in S^!_k/D^{k-1}(M^!_{2-k})$.

Let $\mathcal{F}_i \in H^*_{2-k}$ such that $\xi_{2-k}(\mathcal{F}_i) = f_i$. The differential operator ξ_{2-k} and the Hecke operator $T(m)$ obey the following commutation relation

$$\left(\xi_{2-k}\left(\mathcal{F}_i \mid_{2-k} T(m)\right)\right)(z) = m^{1-k}\left(\xi_{2-k}\left(\mathcal{F}_i\right) \ \mid_k T(m)\right)(z).$$

Because $\xi_{2-k}\left(\mathcal{F}_i \mid_{2-k} T(m) - m^{1-k} b_i(m)\mathcal{F}_i\right) = 0$, it follows that there is some $r_m(z) \in M^!_{2-k}$ such that

$$(\mathcal{F}_i \mid_{2-k} T(m))(z) = m^{1-k} b_i(m)\mathcal{F}_i(z) + r_m(z).$$

We apply the operator D to this identity $k-1$ times and use Bol's identity to find that

$$\left(D^{k-1}(\mathcal{F}_i) \mid_k T(m)\right)(z) = b_i(m) D^{k-1}(\mathcal{F}_i)(z) + m^{k-1} D^{k-1}(r_m)(z).$$

Therefore $F_i = D^{k-1}(\mathcal{F}_i) \in S^!_k$ is a weakly holomorphic Hecke eigenform in $S^!_k/D^{k-1}(M^!_{2-k})$.

To complete the proof, we show that the forms F_i are linearly independent. Assume that $\sum_{i=0}^d c_i F_i = 0$. Then for each j such that $0 \leq j \leq d$, we make use of Proposition 4.1 to obtain

$$0 = \left\{ f_j, \sum_{i=0}^d c_i F_i \right\} = \sum_{i=0}^d c_i \left\{ f_j, F_i \right\} = \sum_{i=0}^d c_i \left(f_j, f_i \right).$$

Because each f_i is a Hecke eigenform, we know that $(f_j, f_i) \neq 0$ if and only if $i = j$. Therefore each $c_i = 0$ and the forms F_i are linearly independent.

Now, we may use a dimension argument by combining Proposition 2.3 and Theorem 1.2. This shows that the set of forms f_i together with the set of forms F_i form a basis of $S^!_k/D^{k-1}(M^!_{2-k})$, proving the theorem.

4.6. **Proof of Theorem 1.7.** We make use of Proposition 2.3 to obtain the decomposition

$$G = a_G(0)E_k + \phi_G + \psi_G,$$

with $\phi_G \in S_k$, and $\psi_G \in D^{k-1}(H^*_{2-k})$. Also, let $F_0 := F - a_F(0)E_k$ and $G_0 := G - a_G(0)E_k$. By the obvious linearity we obtain

$$\{F, G\} = a_F(0)a_G(0)\{E_k, E_k\} + a_F(0)\{E_k, G_0\} + a_G(0)\{F_0, E_k\} + \{F_0, \phi_G\} + \{F_0, \psi_G\},$$

and we now need to prove the required identity for each of the five terms separately.

We begin by letting $\mathcal{E}_k = -\frac{(4\pi)^{k-1}}{(k-1)!}P_{E_k} \in H_{2-k}$, $\mathcal{F} \in H_{2-k}$, and $\mathcal{G} \in H^*_{2-k}$ so that $E_k = D^{k-1}(\mathcal{E}_k)$, $F_0 = D^{k-1}(\mathcal{F})$, and $\psi_G = D^{k-1}(\mathcal{G})$. It follows that

$$\begin{aligned}
\{E_k, E_k\} &= 0 = (E_k, \xi(\mathcal{E}_k)) - \text{constant term of } E_k\mathcal{E}_k^+,\\
\{E_k, G_0\} &= -\{G_0, E_k\} = -\text{constant term of } G_0\mathcal{E}_k^+ = -(G_0, \xi(\mathcal{E}_k)),\\
\{F_0, E_k\} &= \text{constant term of } F_0\mathcal{E}_k^+ = (F_0, \xi(\mathcal{E}_k)),\\
\{F_0, \phi_G\} &= -\{\phi_G, F_0\} = -\text{constant term of } \phi_G\mathcal{F}^+ = -(\phi_G, \xi(\mathcal{F})), \text{ and}\\
\{F_0, \psi_G\} &= \text{constant term of } F_0\mathcal{G}^+ = (F_0, \xi(\mathcal{G})).
\end{aligned}$$

In each of these cases, one of the conditions of Proposition 4.1 holds. The desired identity now almost immediately follows from (4.3) and Theorem 1.4. The only difficulty is that Theorem 1.4 leaves ambiguity in the 0th and $(k-2)$nd periods. However, this ambiguity vanishes in (4.3) by making use of (4.8).

5. The period polynomial principle and the proof of Theorem 1.1

Here we prove Theorem 1.1 using the principle that "period polynomials" encode critical values of L-functions. We choose this perspective, instead of working directly with period integrals of cusp forms, to highlight the role that Bol's identity plays in relating pairs of functional equations. This is the analytic process by which one obtains critical L-values (see [24] for similar results).

5.1. **Period polynomials and critical values of L-functions.** If f is a weight k cusp form, then its *critical values* are the numbers

$$C(f) := \{L(f, 1), L(f, 2), \ldots, L(f, k-1)\},$$

where $L(f, s)$ is the usual analytically continued L-function. Here we show that such values arise naturally as the coefficients of "period polynomials", functions in z which measure the obstruction to modularity.

Theorem 5.1. *Suppose that*

$$A(z) = \sum_{n=1}^{\infty} \alpha(n)q^{\frac{n}{\lambda}},$$

$$B(z) = \sum_{n=1}^{\infty} \beta(n)q^{\frac{n}{\lambda}}$$

are holomorphic functions on $\mathbb{H}$ *where* $|\alpha(n)|, |\beta(n)| = O(n^\delta)$, *where* $\lambda, \delta > 0$. *If*

$$A(z) = z^{-k}B(-1/z),$$

where $k \geq 2$ *is even, then*

$$\mathbb{E}_{A,k}(z) - z^{k-2}\mathbb{E}_{B,k}(-1/z) = \sum_{j=0}^{k-2} \frac{L(A, k-1-j)}{j!} \cdot \left(\frac{2\pi i z}{\lambda}\right)^j.$$

Here $L(A,s)$ *is the analytic continuation of*

$$L(A,s) := \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s},$$

and

$$\mathbb{E}_{\phi,k}(z) := \sum_{n=1}^{\infty} \nu(n) n^{1-k} q^{\frac{n}{\lambda}},$$

when $\phi(z) = \sum_{n=1}^{\infty} \nu(n) q^{\frac{n}{\lambda}}$.

Sketch of the proof. The proof depends on the relationship between functional equations for L-functions, Mellin transforms, and inverse Mellin transforms. Since these notions are standard (for example, see §7.2 of [12]) here we provide just a brief sketch of the proof.

Since $A(z) = z^{-k}B(-1/z)$, the analytically continued Dirichlet series for $A(z)$ and $B(z)$, say $L(A,s)$ and $L(B,s)$, satisfy the functional equation

$$\Lambda_A(s) = i^k \Lambda_B(k-s). \tag{5.1}$$

As usual, we have that

$$\Lambda_A(s) := \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L(A,s),$$
$$\Lambda_B(s) := \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L(B,s).$$

Moreover, we have that Λ_A and Λ_B are entire and are bounded in vertical strips.

Differentiating a function $\Phi(z)$ has the effect of taking $L(\Phi,s)$ to $L(\Phi,s-1)$. Such differentiation typically gives more complicated functional equations. However, by Bol's identity we find that (5.1) is naturally linked to the following functional equation for Eichler integrals:

$$\widehat{\Lambda}_A(s) = -i^k \cdot \widehat{\Lambda}_B(2-k-s), \tag{5.2}$$

where

$$\widehat{\Lambda}_A(s) := \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L(A, s+k-1),$$
$$\widehat{\Lambda}_B(s) := \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L(B, s+k-1).$$

By the assumptions on A and B, there is a rational function $\widehat{\Psi}(s)$ for which $\widehat{\Lambda}_A - \widehat{\Psi}$ is holomorphic and bounded in vertical strips. Using the Mellin inversion formula, we have that for $c_1 > 0$ (which we will choose sufficiently small)

$$\mathbb{E}_{A,k}(z) = \frac{1}{2\pi i}\int_{c_1 - i\infty}^{c_1 + i\infty} \widehat{\Lambda}_A(s)\left(\frac{z}{i}\right)^{-s} ds,$$
$$\mathbb{E}_{B,k}(z) = \frac{1}{2\pi i}\int_{c_1 - i\infty}^{c_1 + i\infty} \widehat{\Lambda}_B(s)\left(\frac{z}{i}\right)^{-s} ds.$$

By shifting the line of integration to the left of the line $\mathrm{Re}(s) = 2 - k - c_1$, Cauchy's Residue Theorem, combined with (5.2) (after letting $s \to 2 - k - s$), implies that

$$\mathbb{E}_{A,k}(z) = z^{k-2}\mathbb{E}_{B,k}(-1/z) + \left(\frac{2\pi i}{\lambda}\right)^{k-1} \cdot \sum \mathrm{Res}\left(\widehat{\Psi}(s)\right) \cdot \left(\frac{z}{i}\right)^{-s},$$

where the sum is over the poles of $\widehat{\Psi}(s)$, namely $s = 0, -1, \ldots, -(k-2)$. A residue calculation then shows that

$$\mathbb{E}_{A,k}(z) - z^{k-2}\mathbb{E}_{B,k}\left(-1/z\right) = \sum_{j=0}^{k-2} \frac{L(A, k-1-j)}{j!} \cdot \left(\frac{2\pi i z}{\lambda}\right)^{j}.$$

□

We now apply Theorem 5.1 to modular forms. Throughout this subsection, we suppose that $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k$. A direct calculation for $0 \leq n \leq k-2$ gives that

$$L(f, n+1) = \frac{(2\pi)^{n+1}}{n!} \cdot r_n(f). \tag{5.3}$$

These are the *critical values*. The following immediate application of Theorem 5.1 provides a proof of (5.3), and it also motivates the definition of the period function $r(f; z)$ in (1.9).

Corollary 5.2. *We have that*

$$\mathcal{E}_f(z) - z^{k-2}\mathcal{E}_f(-1/z) = \sum_{n=0}^{k-2} \frac{L(f, n+1)}{(k-2-n)!} \cdot (2\pi i z)^{k-2-n}$$
$$= \frac{1}{c_k} \cdot \sum_{n=0}^{k-2} i^{1-n}\binom{k-2}{n} \cdot r_n(f) \cdot z^{k-2-n} = \frac{1}{c_k} \cdot r(f; z).$$

If $1 \leq d, c$ are coprime integers, then define the *twisted* L-function

$$L\left(f, \zeta_c^{-d}, s\right) := \sum_{n=1}^{\infty} \frac{a(n)\zeta_c^{-dn}}{n^s}. \tag{5.4}$$

Corollary 5.2 has the following generalization for these L-functions.

Corollary 5.3. *If $1 \leq d < c$ are coprime, then let $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then we have that*

$$\mathcal{E}_f(z) - \left(\mathcal{E}_f|_{2-k}\gamma\right)(z) = \sum_{n=0}^{k-2} \frac{L(f, \zeta_c^{-d}, n+1)}{(k-2-n)!} \cdot (2\pi i)^{k-2-n} \cdot \left(z + \frac{d}{c}\right)^{k-2-n}.$$

Proof. If $\eta = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ is a matrix with $C \neq 0$, then let

$$f(\eta; z) := f\left(\frac{z}{|C|} - \frac{D}{C}\right).$$

By modularity, it follows that

$$f(\gamma; z) = z^{-k} f\left(\gamma^{-1}; -\frac{1}{z}\right).$$

We now apply Theorem 5.1 with

$$A(z) := f(\gamma; z) = \sum_{n=1}^{\infty} a(n) \zeta_c^{-dn} q^{\frac{n}{c}},$$

$$B(z) := f\left(\gamma^{-1}; z\right) = f\left(\frac{z}{c} + \frac{a}{c}\right) = \sum_{n=1}^{\infty} a(n) \zeta_c^{an} q^{\frac{n}{c}}.$$

Letting $z \to cz + d$ in the conclusion of Theorem 5.1 gives

$$\begin{aligned}\mathbb{E}_{A,k}(cz+d) - (cz+d)^{k-2}\mathbb{E}_{B,k}\left(-\frac{1}{cz+d}\right) &= \sum_{j=0}^{k-2} \frac{L(A, k-1-j)}{j!} \cdot \left(\frac{2\pi i(cz+d)}{c}\right)^j \\ &= \sum_{n=0}^{k-2} \frac{L(f, \zeta_c^{-d}, n+1)}{(k-2-n)!} \cdot (2\pi i)^{k-2-n} \cdot \left(z + \frac{d}{c}\right)^{k-2-n}.\end{aligned}$$

The claim now follows, using the following two identities

$$\begin{aligned}\mathbb{E}_{A,k}(cz+d) &= \mathcal{E}_f(z), \\ \left(\mathcal{E}_f|_{2-k}\gamma\right)(z) &= (cz+d)^{k-2} \cdot \mathbb{E}_{B,k}\left(-\frac{1}{cz+d}\right).\end{aligned}$$

□

5.2. **Proof of Theorem 1.1.** We prove Theorem 1.1 using Corollaries 5.2 and 5.3. Suppose that $f \in S_k$ and $\mathcal{F} \in H^*_{2-k}$ have the property that $\xi_{2-k}(\mathcal{F}) = f$. In Theorem 1.4, the constant term of $\mathcal{F}$ is the only obstacle which keeps us from obtaining equality between the two period polynomials. The problem is that both polynomials depend upon $\mathcal{F}$ after differentiation, but this operation annihilates the constant term and there is no way to recover it. By working with $\mathcal{F}$ before differentation, Theorem 1.4 actually implies that

$$c_k \cdot \overline{\mathbb{P}(\mathcal{F}^+, \gamma_{1,0}; \overline{z})} = r(f; z). \tag{5.5}$$

The first claim in Theorem 1.1 now follows from Corollary 5.2.

To prove the second claim, we apply Corollary 5.3 using the fact that similarly to (5.5) we have, for any matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, the identity

$$\overline{\mathbb{P}\left(\mathcal{F}^+, \gamma_{c,d}; \bar{z}\right)} = \left(\mathcal{E}_f - \mathcal{E}_f|_{2-k}\gamma\right)(z).$$

We require the standard orthogonality relation for roots of unity which asserts that

$$\sum_{d=0}^{c-1} \zeta_c^{-m_1 d} \cdot \zeta_c^{m_2 d} = \begin{cases} c & \text{if } m_1 \equiv m_2 \pmod{c}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $\gcd(m,c) = 1$, we have that

$$\frac{1}{c} \sum_{d=0}^{c-1} \zeta_c^{md} \cdot L(f, \zeta_c^{-d}, s) = \sum_{\substack{n \geq 1 \\ n \equiv m \pmod{c}}} \frac{a(n)}{n^s}.$$

Summing in m, combined with the discussion above, gives the second claim in the theorem.

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