THE PARTITION FUNCTION AND HECKE OPERATORS

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Abstract. The theory of congruences for the partition function $p(n)$ depends heavily on the properties of half-integral weight Hecke operators. The subject has been complicated by the absence of closed formulas for the Hecke images $P(z) \mid T(\ell^2)$, where $P(z)$ is the relevant modular generating function. We obtain such formulas using Euler's Pentagonal Number Theorem and the denominator formula for the Monster Lie algebra. As a corollary, we obtain congruences for certain powers of Ramanujan's Delta-function.

1. Introduction and statement of results

A *partition* of an integer n is a non-increasing sequence of positive integers that sum to n. Ramanujan investigated [17, 18] $p(n)$, the number of partitions of n, and he proved that

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5},\\ p(7n+5) &\equiv 0 \pmod{7},\\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

These congruences have inspired many works (for example, see [1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 19, 20, 21] to name a few). In particular, Atkin [5] and Watson [19] proved Ramanujan's conjectures concerning congruences modulo powers of 5, 7 and 11.

In the 60s, Atkin [6] surprisingly discovered congruences modulo some primes $M \geq 13$ by making use of half-integral weight Hecke operators. For example, he proved that

$$p(1977147619n + 815655) \equiv 0 \pmod{19}. \tag{1.1}$$

In the late 90s, the author revisited Atkin's work using ℓ-adic Galois representations and Shimura's theory of half-integral weight modular forms [15], and he proved that there are such congruences modulo every prime $M \geq 5$. Ahlgren and the author [1, 2] later extended this to include all moduli M coprime to 6. Other recent works by Weaver and Yang [20, 21] provide further results along these lines.

Despite these works, little is known about the action of the Hecke operators on the partition generating function. To make this precise, we begin by recalling Dedekind's eta-function $\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ (note. $q := e^{2\pi i z}$ throughout). The modular partition generating function is the weight $-1/2$ modular form

$$1/\eta(24z) = P(z) := \sum_{n=0}^{\infty} p(n) q^{24n-1}. \tag{1.2}$$

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For primes $\ell \geq 5$, we have the normalized Hecke action (for example, see §3.1 of [16])

$$\left(\sum_{n\gg-\infty} a(n)q^n\right) \mid T(\ell^2) := \sum_{n\gg-\infty}\left(\ell^3 a(n\ell^2) + \ell\left(\frac{-3n}{\ell}\right)a(n) + a(n/\ell^2)\right)q^n. \tag{1.3}$$

The general theory of partition congruences depends on the properties of $P(z) \mid T(\ell^2)$, and in the absence of a closed formula, researchers have been required to design special arguments which, under very special circumstances, yield congruences such as (1.1).

Here we consider the seemingly difficult problem of obtaining closed formulas for $P(z) \mid T(\ell^2)$. We obtain a simple solution to this problem by making use of Euler's Pentagonal Number Theorem and the denominator formula for the Monster Lie algebra.

To state these formulas, let $E_4(z)$ and $E_6(z)$ be the usual Eisenstein series

$$E_4(z) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \quad \text{and} \quad E_6(z) := 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{1.4}$$

where $\sigma_v(n) := \sum_{d|n} d^v$. Let $\Delta(z)$ be Ramanujan's weight 12 cusp form

$$\Delta(z) := \eta(z)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24}, \tag{1.5}$$

and let $j(z)$ be Klein's modular function

$$j(z) := E_4(z)^3/\Delta(z) = q^{-1} + 744 + 196884q + \dots. \tag{1.6}$$

Finally, let $(q;q)_\infty$ be Euler's Pentagonal Number generating function

$$(q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n) = \sum_{k\in\mathbb{Z}}(-1)^k q^{(3k^2+k)/2}. \tag{1.7}$$

Using these q-series, we define polynomials $A(m;x) \in \mathbb{Z}[x]$ as the coefficients of the series

$$\begin{aligned}\mathcal{A}(q) = \sum_{m=0}^{\infty} A(m;x)q^m &:= (q;q)_\infty \cdot \frac{E_4(z)^2E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z)-x} \\ &= 1 + (x-745)q + (x^2 - 1489x + 160511)q^2 + \dots.\end{aligned} \tag{1.8}$$

Remark. Each $A(m;x)$ is a monic degree m polynomial with integer coefficients.

We show that $P(z) \mid T(\ell^2)$ is obtained by multiplying $P(z)$ with $A((\ell^2-1)/24; j(24z))$.

Theorem 1.1. *If $\ell \geq 5$ is prime and $\delta_\ell := (\ell^2-1)/24$, then*

$$P(z) \mid T(\ell^2) = P(z)\cdot\left(\ell\left(\frac{3}{\ell}\right) + A(\delta_\ell; j(24z))\right).$$

Remark. For fixed $\ell \geq 5$, this gives a method (see Example 3.3) for computing $p\left(\frac{N\ell^2+1}{24}\right)$. One needs $A(\delta_\ell;x)$ and short initial segments of $j(z)$ and $P(z)$. It suffices to compute

$$P(z)\cdot\left(\ell\cdot\left(\frac{3}{\ell}\right) + A(\delta_\ell; j(24z))\right) = q^{-\ell^2} + \dots + O(q^{N+1}).$$

Theorem 1.1 also gives the following congruences for powers of the Delta-function.

Corollary 1.2. *If $\ell \geq 5$ is prime, then we have that*

$$\Delta(z)^{\delta_\ell} \equiv \frac{1}{A(\delta_\ell; j(z))} \pmod{\ell}.$$

In Section 2 we recall the denominator formula for the Monster Lie algebra, and we then use a classical lemma due to Atkin on $P(z)$ to then prove Theorem 1.1. In Section 3 we give some examples of Theorem 1.1 and Corollary 1.2.

2. Proofs

Here we prove Theorem 1.1 and Corollary 1.2. We begin by recalling Faber polynomials, a sequence of polynomials whose generating function is essentially equivalent to the denominator formula for the Monster Lie algebra.

2.1. Faber polynomials.

If $p := e^{2\pi i\tau}$, then the denominator formula for the Monster Lie algebra is

$$j(\tau) - j(z) = p^{-1} \prod_{m>0 \text{ and } n\in\mathbb{Z}} (1 - p^m q^n)^{c(mn)}.$$

Here the exponents $c(n)$ are the coefficients of $j(z)$. This identity may be reformulated in terms of a sequence of modular functions $j_m(z)$. We let $j_0(z) := 1$ and $j_1(z) := j(z) - 744$. For $m \geq 2$ we let $j_m(z)$ be the unique modular function on $SL_2(\mathbb{Z})$ with an expansion of the form

$$j_m(z) = q^{-m} + \sum_{n=1}^{\infty} c_m(n) q^n. \tag{2.1}$$

It is not difficult to show that the denominator formula is equivalent to

$$j(\tau) - j(z) = p^{-1} \cdot \exp\left(- \sum_{n=1}^{\infty} j_n(z) \cdot \frac{p^n}{n} \right).$$

The modular functions $j_m(z)$ are specializations of polynomials $J_m(x)$ which were previously defined by Faber [9] (also see [4]). These polynomials are defined by the generating function

$$\sum_{m=0}^{\infty} J_m(x) q^m := \frac{E_4(z)^2 E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z) - x} = 1 + (x - 744)q + (x^2 - 1488x + 159768)q^2 + \dots. \tag{2.2}$$

Here we recall some of the main properties of these polynomials (see [4, 9, 22]).

Theorem 2.1. *Assuming the notation above, the following are true.*

(1) *If $m \geq 0$, then $j_m(z) = J_m(j(z))$.*
(2) *If $m \geq 2$, then*

$$j_m(z) = J_1(j(z)) \mid T_0(m),$$

where $T_0(m)$ is the normalized mth weight 0 Hecke operator.

2.2. **Proof of Theorem 1.1.** If $\ell \geq 5$ is prime, then define $F_\ell(z)$ by

$$F_\ell(24z) := \eta(24z) \cdot \left(P(z) \mid T(\ell^2)\right). \tag{2.3}$$

The nonzero coefficients are supported on exponents which are multiples of 24. After letting $z \to z/24$, a standard argument involving the definition of $T(\ell^2)$ and the transformation law for Dedekind's eta-function implies that $F_\ell(z)$ is a modular function on $\mathrm{SL}_2(\mathbb{Z})$. (note. This fact was previously observed by Atkin (see Lemma 2 of [6])). Since $F_\ell(z)$ is holomorphic on the upper half of the complex plane, it is a polynomial in $j(z)$.

By direct calculation, we have that

$$P(z) \mid T(\ell^2) = q^{-\ell^2} + \ell \left(\frac{3}{\ell}\right) q^{-1} + O(q^{23}).$$

Euler's Pentagonal Number Theorem then gives

$$F_\ell(24z) = q^{1-\ell^2} + \ell \left(\frac{3}{\ell}\right) + \sum_{k=1}^{\infty} (-1)^k \cdot \left(q^{1-\ell^2+24\omega(k)} + q^{1-\ell^2+24\omega(-k)}\right) + O(q^{23}),$$

where $\omega(k) := (3k^2+k)/2$. By letting $z \to z/24$, we obtain

$$F_\ell(z) = q^{-\delta_\ell} + \ell \left(\frac{3}{\ell}\right) + \sum_{k=1}^{\infty} (-1)^k \left(q^{-\delta_\ell+\omega(k)} + q^{-\delta_\ell+\omega(-k)}\right) + O(q).$$

We now show that this polynomial in $j(z)$ is $A(\delta_\ell; j(z))$. By Euler's Pentagonal Number Theorem, Theorem 2.1, (1.8), (2.1) and (2.2), it follows that $A(\delta_\ell; j(z))$ is a modular function on $\mathrm{SL}_2(\mathbb{Z})$ with the property that

$$\ell \left(\frac{3}{\ell}\right) + A(\delta_\ell; j(z)) - F_\ell(z) = O(q).$$

This modular function must then be a polynomial in $j(z)$. Since every nonconstant modular function on $\mathrm{SL}_2(\mathbb{Z})$ has a pole, and since this function does not have a pole at infinity, we have

$$\ell \left(\frac{3}{\ell}\right) + A(\delta_\ell; j(z)) = F_\ell(z).$$

After letting $z \to 24z$, the theorem follows from (2.3) by dividing $A(\delta_\ell; j(24z))$ by $\eta(24z) = q(q^{24}; q^{24})_\infty$. The proof is complete because of the fact that $P(z) = \frac{1}{\eta(24z)}$.

2.3. **Proof of Corollary 1.2.** If $\ell \geq 5$ is prime, then Theorem 1.1 implies that

$$P(z) \mid T(\ell^2) \equiv P(z) \cdot A(\delta_\ell; j(24z)) \pmod{\ell}.$$

By (1.3), we have that

$$P(z) \mid T(\ell^2) \equiv \sum_{n=0}^{\infty} p(n) q^{(24n-1)\ell^2} \equiv P(\ell^2 z) \pmod{\ell}.$$

Putting these together, we find that

$$\frac{1}{A(\delta_\ell; j(24z))} \equiv \frac{P(z)}{P(\ell^2 z)} \pmod{\ell}.$$

By direct calculation we have

$$\frac{P(z)}{P(\ell^2 z)} = \frac{\eta(24\ell^2 z)}{\eta(24z)} \equiv \eta(24z)^{\ell^2-1} = \Delta(24z)^{\delta_\ell} \pmod{\ell},$$

and so the corollary follows by letting $z \to z/24$.

3. Examples

Here we illustrate the results described in the introduction.

Example 3.1. Here we illustrate Theorem 1.1 for $\ell = 7$. Then we have that

$$A(\delta_7; x) = A(2; x) = x^2 - 1489x + 160511.$$

Therefore, we find that

$$7 \cdot \left(\frac{3}{7}\right) + A(2; j(24z)) = q^{-48} - q^{-24} - 8 + 42790636q^{24} + 40470415636q^{48} + \dots,$$

which in turn gives

$$P(z) \cdot \left(7 \cdot \left(\frac{3}{7}\right) + A(2; j(24z)\right) = q^{-49} - 7q^{-1} + 42790629q^{23} + 40513206258q^{47} + \dots.$$

This illustrates Theorem 1.1 since one directly finds that

$$P(z) \mid T(7^2) = q^{-49} - 7q^{-1} + 42790629q^{23} + 40513206258q^{47} + \dots.$$

Example 3.2. Here we illustrate Corollary 1.2 for $\ell = 13$. We have that

$$\begin{aligned} A(\delta_{13}; x) = A(7; x) = x^7 &- 5209x^6 + 10250531x^5 - 9444792416x^4 + 4084546595190x^3 \\ &- 721470585282643x^2 + 35089738412615282x - 104996593133311511, \end{aligned}$$

which in turn gives

$$\begin{aligned} \frac{1}{A(7; j(z))} &= q^7 + q^8 + 2q^9 + \dots - 2854208487467q^{15} - \dots \\ &\equiv q^7 + q^8 + 2q^9 + 3q^{10} + 5q^{11} + 7q^{12} + 11q^{13} + 2q^{14} + 9q^{15} + 4q^{16} + \dots \pmod{13}. \end{aligned}$$

This illustrates Corollary 1.2 because

$$\begin{aligned} \Delta(z)^{\delta_{13}} &= \Delta(z)^7 = q^7 - 168q^8 + 13860q^9 - 748160q^{10} + \dots \\ &\equiv q^7 + q^8 + 2q^9 + 3q^{10} + 5q^{11} + 7q^{12} + 11q^{13} + 2q^{14} + 9q^{15} + 4q^{16} + \dots \pmod{13}. \end{aligned}$$

Example 3.3. Here we illustrate how one may efficiently compute partition numbers of the form $p\left(\frac{N\ell^2+1}{24}\right)$. We consider the simple case where $N = 71$ and $\ell = 5$, and so our aim is to calculate $p(74)$. We compute $p(74)$ using $p(0), \dots, p(4)$, the first five coefficients of $j(z)$, and the polynomial $A(\delta_5; x) = x - 745$. By (1.2) and (1.3), if we let

$$\sum_{n \gg -\infty} a_5(n) q^n := P(z) \mid T(5^2),$$

then we find that

$$a_5(71) = 5^3 \cdot p\left(\frac{71 \cdot 5^2 + 1}{24}\right) - 5p\left(\frac{71+1}{24}\right) = 5^3 \cdot p(74) - 5p(3).$$

By Theorem 1.1, we have that

$$\begin{aligned} P(z) \mid T(5^2) &= P(z) \cdot (j(24z) - 750) \\ &= \left(q^{-1} + q^{23} + 2q^{47} + 3q^{71} + 5q^{95} + \dots\right) \cdot \left(q^{-24} - 6 + 196884q^{24} + \dots\right) \\ &= q^{-25} - 5q^{-1} + 196880q^{23} + 21690635q^{47} + 886187485q^{71} + \dots. \end{aligned}$$

Since $a_5(71) = 886187485$ and $p(3) = 3$, we then find that $p(74) = 7089500$.

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