time). Hence, Prim's algorithm requires $O(|V|^2)$ time.

At this stage we must point out that the corresponding problem of finding minimum weight spanning trees in digraphs is much harder. In fact, there is no known polynomial algorithm for solving such a problem.

Section 3.3 Counting Trees

Let's turn our attention now to problems involving counting trees. Although there is no simple formula for determining the number of nonisomorphic spanning trees of a given order, if we place labels on the vertices, we are able to introduce a measure of control on the situation. We say two graphs G_1 and G_2 are *identical* if $V(G_1) = V(G_2)$ and $E(G_1) = E(G_2)$. Now we consider the question of determining the number of nonidentical spanning trees of a given graph (that is, on a given number of vertices). Say $G = (V, E)$ and for convenience we let $V = \{ 1, 2, \ldots, p \}$. For $p = 2$, there is only one tree, namely K_2. For $p = 3$, there are three such trees (see Figure 3.3.1).

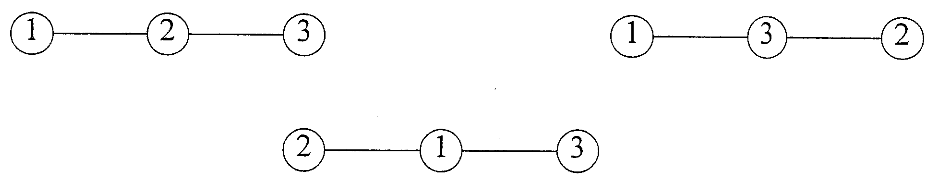


Figure 3.3.1. The spanning trees on $V = \{ 1, 2, 3 \}$.

Cayley [1] determined a simple formula for the number of nonidentical spanning trees on $V = \{ 1, 2, \ldots, p \}$. The proof presented here is from Prüfer [9]. This result is known as Cayley's tree formula.

Theorem 3.3.1 (Cayley's tree formula). The number of nonidentical spanning trees on p distinct vertices is p^{p-2}.

Proof. The result is trivial for $p = 1$ or $p = 2$ so assume $p \geq 3$. The strategy of this proof is to find a one-to-one correspondence between the set of spanning trees of G and the p^{p-2} sequences of length $p - 2$ with entries from the set $\{ 1, 2, \ldots, p \}$. We demonstrate this correspondence with two algorithms, one that finds a sequence corresponding to a tree and one that finds a tree corresponding to a sequence. In what follows, we will identify each vertex with its label. The algorithm for finding the sequence that corresponds to a given tree is:

1. Let $i \leftarrow 1$.
2. Let $j \leftarrow$ the end vertex of the tree with smallest label. Remove j and its incident edge $e = jk$. The ith term of the sequence is k.
3. If $i = p - 2$ then halt; else $i \leftarrow i + 1$ and go to 2.

Since every tree of order at least 3 has two or more end vertices, step 2 can always be performed. Thus, we can produce a sequence of length $p - 2$. Now we must show that no sequence is produced by two or more different trees and that every possible sequence is produced from some tree. To accomplish these goals, we show that the mapping that assigns sequences to trees also has an inverse.

Let $w = n_1, n_2, \ldots, n_{p-2}$ be an arbitrary sequence of length $p - 2$ with entries from the set V. Each time (except the last) that an edge incident to vertex k is removed from the tree, k becomes the next term of the sequence. The last edge incident to vertex k may never actually be removed if k is one of the final two vertices remaining in the tree. Otherwise, the last time that an edge incident to vertex k is removed, it is because vertex k has degree 1, and, hence, the other end vertex of the edge was the one inserted into the sequence. Thus, $deg_T\ k = 1 +$ (the number of times k appears in w). With this observation in mind, the following algorithm produces a tree from the sequence w:

1. Let $i \leftarrow 1$.
2. Let j be the least vertex such that $deg_T\ j = 1$. Construct an edge from vertex j to vertex n_i and set $deg_T\ j \leftarrow 0$ and $deg_T\ n_i \leftarrow deg_T\ n_i - 1$.
3. If $i = p - 2$, then construct an edge between the two vertices of degree 1 and halt; else set $i \leftarrow i + 1$ and go to step 2.

It is easy to show that this algorithm selects the same vertex j as the algorithm for producing the sequence from the tree (Chapter 3, exercise 17). It is also easy to see that a tree is constructed. Note that at each step of the algorithm, the selection of the next vertex is forced and, hence, only one tree can be produced. Thus, the inverse mapping is produced and the result is proved. □

Example 3.3.1. Prüfer mappings. We demonstrate the two mappings determined in the proof of Cayley's theorem. Suppose we are given the tree T of Figure 3.3.2.

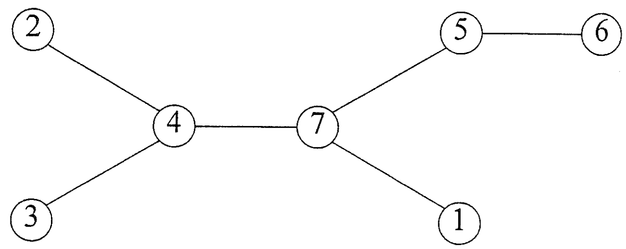


Figure 3.3.2. The tree T.

Among the leaves of T, vertex 1 has the minimum label, and it is adjacent to 7; thus, $n_1 = 7$. Our next selection is vertex 2, adjacent to vertex 4, so $n_2 = 4$. Our tree now appears as in Figure 3.3.3.

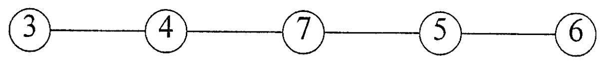


Figure 3.3.3. The tree after the first two deletions.

The third vertex selected is 3, so $n_3 = 4$. We then select vertex 4; thus, $n_4 = 7$. Finally, we select vertex 6, setting $n_5 = 5$. What remains is just the edge from 5 to 7; hence, we halt. The sequence corresponding to the tree T of Figure 3.3.2 is 74475.

To reverse this process, suppose we are given the sequence $s = 74475$. Then we note that:

$$\deg 1 = 1,\ \deg 2 = 1, \deg 3 = 1, \deg 4 = 3,$$
$$\deg 5 = 2,\ \deg 6 = 1, \deg 7 = 3.$$

According to the second algorithm, we select the vertex of minimum label with degree 1; hence, we select vertex 1. We then insert the edge from 1 to $n_1 = 7$. Now set $deg\ 1 = 0$ and $deg\ 7 = 2$ and repeat the process. Next, we select vertex 2 and insert the edge from 2 to $n_2 = 4$:

Figure 3.3.4. The reconstruction after two passes.

Again reducing the degrees, $deg\ 2 = 0$ and $deg\ 4 = 2$. Next, we select vertex 3 and insert the edge from 3 to $n_3 = 4$ (see Figure 3.3.5).

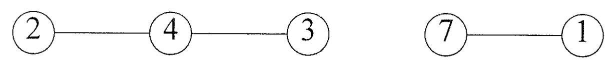


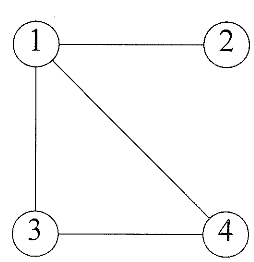
Figure 3.3.5. The reconstruction after three passes.

Now, select vertex 4 and insert the edge from 4 to $n_4 = 7$. This is followed by the selection of vertex 6 and the insertion of the edge from 6 to $n_5 = 5$. Finally, since $i = p - 2$, we end the construction by inserting the edge from 5 to 7, which completes the reconstruction of T. □

An alternate expression for the number of nonidentical spanning trees of a graph is from Kirchhoff [5]. This result uses the $p \times p$ *degree matrix* $C = [\, c_{ij} \,]$ of G, where $c_{ii} = deg\ v_i$ and $c_{ij} = 0$ if $i \neq j$. This result is known as the matrix-tree theorem. For each pair (i, j), let the matrix B_{ij} be the $n - 1 \times n - 1$ matrix obtained from the $n \times n$ matrix B by deleting row i and column j. Then det B_{ij} is called the *minor of B* at position (i, j) and, $(-1)^{i+j}$ det B_{ij} is called the *cofactor of B* at position (i, j).

Theorem 3.3.2 (The matrix-tree theorem) Let G be a nontrivial graph with adjacency matrix A and degree matrix D. Then the number of nonidentical spanning trees of G is the value of any cofactor of $D - A$.

Example 3.3.2. Consider the following graph:



We can use the matrix-tree theorem to calculate the number of nonidentical spanning trees of this graph as follows. The matrices