Suitable Permutations and Covering Arrays

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Let t, v be positive integers with $t < v$. A set Π of permutations of $\{0, \ldots, v-1\}$ is *t-suitable* if for every t-subset $X \subset \{0, \ldots, v-1\}$ and every $\sigma \in X$, there is at least one permutation $\pi \in \Pi$ with $\pi^{-1}(\sigma) < \pi^{-1}(y)$ for all $y \in X \setminus \{\sigma\}$. Here is a 4-suitable set for $v = 9$:

3	8	5	7	6	2	4	1	0
4	2	0	6	7	5	1	8	3
1	7	6	5	0	8	2	3	4
0	5	6	2	8	7	3	1	4
8	2	6	7	4	5	3	1	0
7	2	4	5	3	6	8	1	0
6	2	5	3	8	0	1	7	4

The objective is to determine the size $S(t, v)$ of a smallest t-suitable set of permutations, given the strength t and length v. In 1950, Dushnik proposed this problem and determined $S(t, v)$ precisely when $t > 2\sqrt{v} - 1$. In 1972, Spencer showed that $S(t, v)$ is $\Theta(\log \log v)$ when $t > 2$ is fixed.

A second combinatorial object plays a role. Let N, k, t, and v be positive integers. Let C be an $N \times k$ array with entries from an alphabet Σ of size v; we typically take $\Sigma = \{0, \ldots, v-1\}$. When $(\nu_1, \ldots, \nu_t)$ is a t-tuple with $\nu_i \in \Sigma$ for $1 \leq i \leq t$, $(c_1, \ldots, c_t)$ is a tuple of t column indices ($c_i \in \{1, \ldots, k\}$), and $c_i \neq c_j$ whenever $\nu_i \neq \nu_j$, the t-tuple $\{(c_i, \nu_i) : 1 \leq i \leq t\}$ is a *t-way interaction*. The array *covers* the t-way interaction $\{(c_i, \nu_i) : 1 \leq i \leq t\}$ if, in at least one row ρ of C, the entry in row ρ and column c_i is ν_i for $1 \leq i \leq t$. Array C is a *covering array* $\mathrm{CA}(N; t, k, v)$ of *strength* t when every t-way interaction is covered.

Spencer's proof uses a surprising connection between suitable sets of permutations and covering arrays, which we explore in this talk. We focus on open questions that arise from this connection.