GAPS IN THE SATURATION SPECTRUM OF TREES

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Abstract. A graph G is H-saturated if H is not a subgraph of G but the addition of any edge from the complement of G to G results in a copy of H. The minimum number of edges (the size) of an H-saturated graph on n vertices is denoted $\mathbf{sat}(n, H)$, while the maximum size is the well studied extremal number, $\mathbf{ex}(n, H)$. The saturation spectrum for a graph H is the set of sizes of H-saturated graphs between $\mathbf{sat}(n, H)$ and $\mathbf{ex}(n, H)$. In this paper we show that paths and also trees with a vertex adjacent to many leaves, have a gap in the saturation spectrum.

The *broom* $B_{\ell-s,s}$, for $2 \leq s \leq \ell - 3$ is obtained by taking a K_{s+1} and subdividing one edge $\ell - s - 1$ times. Note that we explicitly exclude the star $K_{\ell-1}$ which would correspond to $B_{2,\ell-2}$ from the family.

Theorem 0.1. *Brooms $B_{\ell-s,s}$ are strongly ES-embeddable, for $2 \leq s \leq \ell - 3$.*

Proof. Note that if $s \geq \lfloor \frac{\ell}{2} \rfloor$, then $B_{\ell-s,s}$ is a scrub-grass tree and already covered, so without loss of generality $s < \lfloor \frac{\ell}{2} \rfloor$. Let us fix a $B_{\ell-s,s}$, and we shall show that it embeds into a graph satisfying the conditions.

Let G be a vertex minimal counterexample: that G is a graph with as few vertices as possible satsfying $delta(G) \geq \lfloor \frac{\ell}{2} \rfloor$ and $\overline{d}(G) > \ell - 3$, with at least ℓ vertices, and that G contains no B_s.

First consider the case where G is 2-connected. Then, by Dirac's theorem, G contains a cycle C on at least $2\lfloor \frac{\ell}{2} \rfloor \geq \ell - 1$ vertices. Furthermore G by the average degree condition, G contains a vertex v of degree at least $\ell - 2$.

If v is off of the cycle, let r denote the number of neighbors on the cycle. If $r = 0$, we are done by connectivity: which includes a portion of the cycle along with the shortest path from the cycle to v as the path, and some of v's other neighbors as the leaves, which is possible as the shortest path contains at most one of v's neighbors. Thus we may assume $r \geq 0$. If there exists a sequence of $\ell - s - 2$ consecutive vertices on the cycle with so that v has at least $s + 1$ neighbors *not* in those consecutive vertices we have found our broom. Indeed, let P denote a sequence of $\ell - s - 2$ vertices of C containing a minimum number of v's neighbors. v has at at least one neighbor on the cycle not in P, so take one, v' whose cycle distance to P is minimum. Then the path including P extending v' along the cycle (in a shortest way) and then to v has at least $\ell - s$ vertices and then there are at least s additional neighbors of v to form the broom. That such a sequence exists can be established by averaging: There exists a sequence of vertices with at least

$$N = d(v) - (\ell - s - 2) \cdot \frac{r}{|C|}$$

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This is a decreasing function of r and $r \le \min\{d(v), |C|\}$. If $\ell - 1 \le |C| \le d(v)$, then one immediately verifies that this is at least $s+1$, by considering the case where $r = |C|$ and $d(v) = \ell - 1$. If, otherwise, $\ell - 2 \le d(v) < |C|$, then one observes that

$$N \ge \left\lceil (\ell-2) - (\ell - s - 2) \cdot \frac{\ell-2}{\ell-1} \right\rceil = \left\lceil s + 1 - \frac{s+1}{\ell-1} \right\rceil \ge s+1,$$

since $s \le \ell - 3$.

Now assume that every such vertex lies on the cycle. It is easy to see that if such a vertex v either has degree at least $\ell - 1$ or v is not complete to the remainder of the cycle then the broom is contained in the graph by using v and cycle vertices (in the direction of it's nearest non-neighbor along the cycle, if such exists) as the path, and the remainder of v's neighbors as the star vertices. (Quite importantly, this count relies on the fact that $s < \lfloor \frac{\ell}{2}$, so the path is guaranteed to include the non-neighbor of v if the degree of v is $\ell - 2$.)

Thus we have reduced ourselves to the case where all vertices v of degree $\ell - 2$ lie on a cycle and are adjacent to *all* other vertices in the cycle (which necessarily has length $\ell - 1$). Since G has at least ℓ vertices, and the graph is two connected, at least two vertices on the cycle have neighbors off the cycle – and if we can find a way to one of them to a vertex of degree $\ell - 2$ on our cycle by a path of length $\ell - s - 1$ we easily find our broom (by extending the path using the vertex off the cycle, and using v's neighbors to complete the star potion of the broom). For instance, if a neighbor of a degree $\ell - 2$ vertex is off the cycle, it is easy to build such a path (using an edge off of the degree $\ell - 2$ vertex to travel an appropriate distance around the cycle, then following the cycle to the neighbor.) While we cannot guarantee this in general, we can come close enough: Since the $\overline{d}(G) > \ell - 3$ and no vertices of degree $\ell - 2$ are off the cycle, $\overline{d}(C) > \ell - 3$, and by averaging two vertices in C, adjacent on the cycle, have average degree $> \ell - 3$. But then one, v, has degree $\ell - 2$ and the other (v') has degree either $\ell - 2$ or $\ell - 3$. In either case: v' is either incident to the other neighbor of v along C or to the next neighbor in the cycle, and it is then easy to build any length path desired from v to the vertex with a neighbor off of the path.

Finally, we treat the case where G is not 2-connected. In this case we consider the block decomposition of the graph. If there is a leaf-block of the block decomposition with at least $\ell - 3$ vertices, then we are easily done: Since each vertex beyond the cut vertex in that component has degree $\lfloor \frac{\ell}{2} \rfloor$, there is (by the results of Erdős and Gallai of [**?**]) a path of length $\ell - 3$ in the component starting at the cut vertex. Taking a vertex in a different one of the blocks, and some of its neighbors to be the head of the broom and connecting it to the path yields the desired broom.

Note that if removal of the internal vertices of a leaf-block leaves at least ℓ vertices, then an easy computation and minimality of the counterexample, shows that the block must have at least $\ell - 3$ vertices. Indeed, since at least ℓ vertices remain, the reason this does not yield a smaller counterexample must be that the density is too low. That is, if G' denotes the graph with the block removed,

$$\frac{\sum_{v \in G'} \deg(v) - \sum_{v \in G \setminus G'} \deg(v)}{|G'|} \le \ell - 3,$$

so

$$\sum_{v \in G \setminus G'} \deg(v) \ge \sum_{v \in G} \deg(v) - |G'|(\ell - 3) > |G \setminus G'|(\ell - 3)$$

Thus the average degree in the block $(G \setminus G')$ is at least $\ell - 3$, so there is some vertex of such a degree in the component necessitating that the component is at least so large. We say that a block is *non-trivial* if it has at least 3 vertices; a leaf block is necessarily non-trivial because of the minimum degree requirement. We next note that if there are at least 3 non-trivial blocks, then there exists a leaf block whose removal leaves at least ℓ vertices. Indeed ignoring one leaf block, each block contains an internal vertex of degree $\lfloor \frac{\ell}{2} \rfloor$ sending no edges to the component to be deleted. Then there are at least $2(\lfloor \frac{\ell}{2} \rfloor + 1) - 1 \geq \ell$ between the two blocks (with the -1 arising as the blocks may share a cut vertex). Thus we may reduce to the case of having exactly 2 non-trivial blocks, which are connected either by an edge or at a cut vertex. The case where the two non-trivial blocks are joined by an edge can be easily discarded by considering a vertex of degree $\geq \ell - 2$, which must send at least $\ell - 3$ edges into one of the blocks and ensures that it has size at least $\ell - 3$. Otherwise, the average of the degrees *non*-cut vertices is at most one less than the total average degree. Thus there is a vertex within one of the blocks of degree $\ell - 3$ within a block, and it guarantees the existence of a leaf block of at least $\ell - 3$ vertices. □

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