More on Chorded Cycle

Ron Gould
Emory University

AMS Meeting - Knoxville

March 22-23, 2014

Over the years there have been many results that find conditions sufficient for cycles (often with various properties like containing a set of vertices, or a set of edges, etc.).

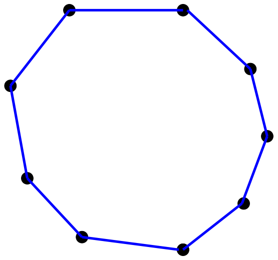
But the one property that was greatly ignored was the following:

An Old Question by Posa, 1960

Question

What conditions imply a graph contains a cycle with a chord?

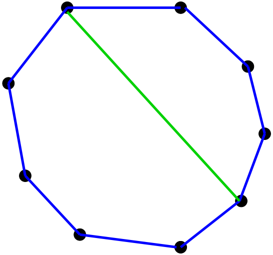
Here a **chord** is an edge between two vertices on the cycle that is not an edge of the cycle.



Question

What conditions imply a graph contains a cycle with a chord?

Here a **chord** is an edge between two vertices on the cycle that is not an edge of the cycle.



First answer by J. Czipzer 1963 - using min deg $\delta(G)$

Theorem

If G has minimum degree at least 3, then G contains a chorded cycle.

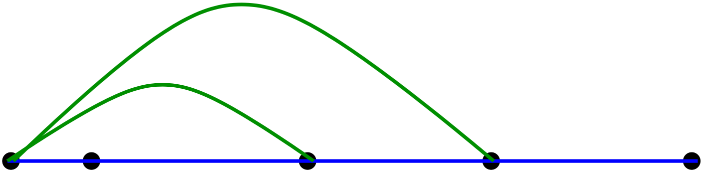
longest path in G

First answer by J. Czipzer 1963 - using min deg $\delta(G)$

Theorem

If G has minimum degree at least 3, then G contains a chorded cycle.

longest path in G



What questions were first asked?

- Other conditions for a chorded cycle.

What questions were first asked?

- Other conditions for a chorded cycle.
- Some specified number of disjoint chorded cycles.

What questions were first asked?

- Other conditions for a chorded cycle.
- Some specified number of disjoint chorded cycles.
- Some specified number of disjoint doubly chorded cycles.

What questions were first asked?

- Other conditions for a chorded cycle.
- Some specified number of disjoint chorded cycles.
- Some specified number of disjoint doubly chorded cycles.
- Cycles with a designated minimum number of chords.

Some History (45 years later) - D. Finkel, 2008

Theorem

If G is a graph on $n \geq 4k$ vertices with minimum degree $\delta(G) \geq 3k$, then G contains at least k independent chorded cycles.

Note: This can be viewed as a generalization of the Corradi-Hajnal Theorem.

Some History (45 years later) - D. Finkel, 2008

Theorem

If G is a graph on $n \geq 4k$ vertices with minimum degree $\delta(G) \geq 3k$, then G contains at least k independent chorded cycles.

Note: This can be viewed as a generalization of the Corradi-Hajnal Theorem.

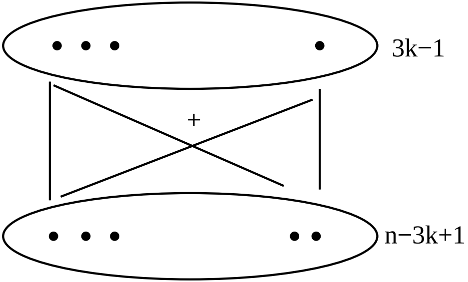
Theorem

Let G be a graph of order $n \geq 3k$ with minimum degree $\delta(G) \geq 2k$, then G contains k disjoint cycles.

Sharpness

Clearly, $n \geq 4k$ is needed as the cycles need at least 4 vertices each.

For $n \geq 6k$, the graph $K_{3k-1,n-3k+1}$ has $\delta = 3k - 1$ and no collection of k independent chorded cycles, as chorded cycles here require 3 vertices from each partite set.



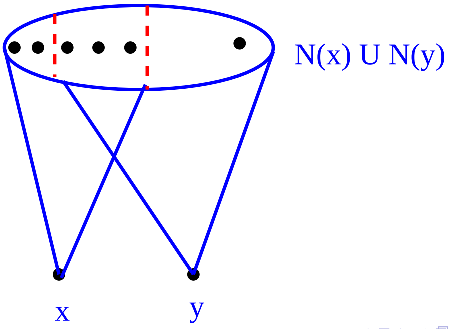
$K_{3k-1,\, n-3k+1}$

Theorem

If G is a graph on $n \geq 4k$ vertices such that for any pair of non-adjacent vertices x, y,

$$|N(x,y)| \geq 4k+1,$$

then H contains at least k independent chorded cycles.



RG, K. Hirohata, P. Horn

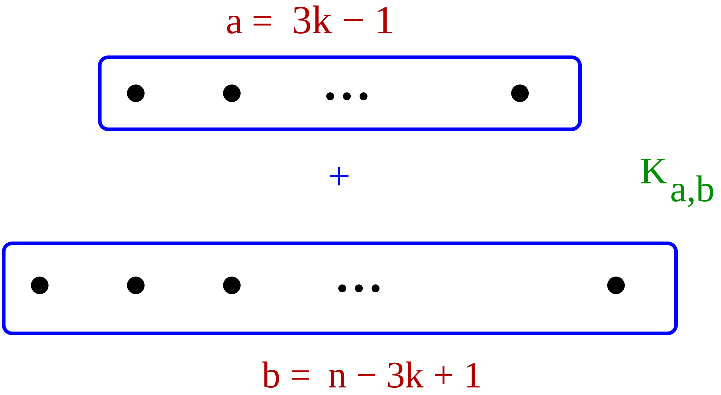
Theorem

If G is a graph on $n \geq 6k$ vertices with

$$\sigma_2(G) \geq 6k - 1,$$

then G contains k vertex disjoint doubly chorded cycles.

Sharpness



Other Natural Questions for Chorded Cycles

- Make a set of edges the chords.

Other Natural Questions for Chorded Cycles

- Make a set of edges the chords.
- Place k vertices on k vertex disjoint chorded cycles.

Other Natural Questions for Chorded Cycles

- Make a set of edges the chords.
- Place k vertices on k vertex disjoint chorded cycles.
- Place k indep. edges on k vertex disjoint chorded cycles.

Other Natural Questions for Chorded Cycles

- Make a set of edges the chords.
- Place k vertices on k vertex disjoint chorded cycles.
- Place k indep. edges on k vertex disjoint chorded cycles.
- Place k-path linear forest on k disjoint chorded cycles.

Other Natural Questions for Chorded Cycles

- Make a set of edges the chords.
- Place k vertices on k vertex disjoint chorded cycles.
- Place k indep. edges on k vertex disjoint chorded cycles.
- Place k-path linear forest on k disjoint chorded cycles.
- Control the order of the chorded cycles.

Other Natural Questions for Chorded Cycles

- Make a set of edges the chords.
- Place k vertices on k vertex disjoint chorded cycles.
- Place k indep. edges on k vertex disjoint chorded cycles.
- Place k-path linear forest on k disjoint chorded cycles.
- Control the order of the chorded cycles.
- Expand our chorded cycle system to span $V(G)$.

Specifing certain edges to be chords

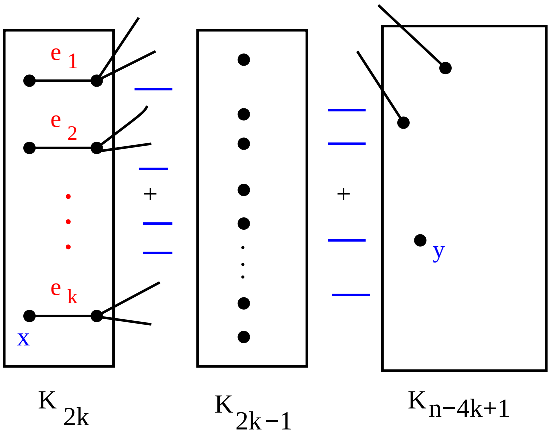
Question

Can we make an independent set of k edges the chords of k vertex disjoint chorded cycles?

Theorem

Let $k \geq 1$ be an integer and G be a graph of order $n \geq 14k$. If $\sigma_2(G) \geq n + 3k - 2$, then for any k independent edges $e_1, e_2, \ldots, e_k$ of G, the graph G contains k vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that e_i is a chord of C_i for all $1 \leq i \leq k$. Furthermore, $4 \leq |V(C_i)| \leq 5$ for each i.

Sharpness Example



Question- Placing vertices on chorded cycles

Question

When can we distribute k vertices on k disjoint chorded cycles?

Placing vertices on chorded cycles

[with M. Cream, R. Faudree and K. Hirohata]

Theorem

Let $k \geq 1$ be an integer and let G be a graph of order $n \geq 16k - 12$. If $\delta(G) \geq n/2$ then for any set of k vertices $\{v_1, v_2, \ldots, v_k\}$ there exists a collection of k vertex disjoint chorded cycles $\{C_1, \ldots, C_k\}$ such that $v_i \in V(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

Question

When can we distribute k independent edges on k disjoint chorded cycles?

Placing Edges on Chorded Cycles

[with M. Cream, R. Faudree and K. Hirohata]

Theorem

Let G be a graph of order $n \geq 18k - 2$ and let $e_1, e_2, \ldots, e_k$ be a set of k independent edges in G. If

$$\delta(G) \geq \frac{n + 2k - 2}{2}$$

then there exists a system of k chorded cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

[with M. Cream, R. Faudree and K. Hirohata]

As a Corollary to the proof we obtain the fact the edges

$$e_1, e_2, \ldots, e_k$$

can be a mix of either chords or edges of the cycles (again one edge per cycle).

Further, we can show that the cycle system can also be extended to span $V(G)$.

Doubly Chorded Cycles

[with M. Cream, R. Faudree and K. Hirohata]

Theorem

Let G be a graph of order $n \geq 22k - 2$ and let $e_1, \ldots, e_k$ be k independent edges in G. Then if

$$\delta(G) \geq \frac{n + 2k - 2}{2}$$

then there exists a system of k vertex disjoint doubly chorded cycles $C_i, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

Corollary

The above system can be extended to span $V(G)$.

Question

When can we distribute a k path linear forest on k disjoint chorded cycles?

Containing Linear Forests

Fact

Given independent path $P_{r_1}, P_{r_2}, \ldots, P_{r_k}$ with each $r_i \geq 2$ let $r = \sum r_i$. Then the number of interior vertices in this path system is $r - 2k$.

Theorem

Let $P_{r_1}, P_{r_2}, \ldots, P_{r_k}$ be a linear forest in a graph G of order $16k + r - 2$ with

$$\delta(G) \geq n/2 + r - 1 - k.$$

Then there exists a system of k chorded cycles $C_1, \ldots C_k$ such that the path P_{r_i} lies on the cycle C_i and $|V(C_i)| \leq r_i + 4$.