More on Chorded Cycle

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March 22-23, 2014
Over the years there have been many results that find conditions sufficient for cycles (often with various properties like containing a set of vertices, or a set of edges, etc.).

But the one property that was greatly ignored was the following:
Question

What conditions imply a graph contains a cycle with a chord?

Here a **chord** is an edge between two vertices on the cycle that is not an edge of the cycle.
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Theorem

If $G$ has minimum degree at least 3, then $G$ contains a chorded cycle.

longest path in $G$
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Other conditions for a chorded cycle.
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- Some specified number of disjoint chorded cycles.
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- Some specified number of disjoint chorded cycles.
- Some specified number of disjoint doubly chorded cycles.
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- Other conditions for a chorded cycle.
- Some specified number of disjoint chorded cycles.
- Some specified number of disjoint doubly chorded cycles.
- Cycles with a designated minimum number of chords.
Theorem

If $G$ is a graph on $n \geq 4k$ vertices with minimum degree $\delta(G) \geq 3k$, then $G$ contains at least $k$ independent chorded cycles.

Note: This can be viewed as a generalization of the Corradi-Hajnal Theorem.
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Theorem

Let $G$ be a graph of order $n \geq 3k$ with minimum degree $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.
Clearly, $n \geq 4k$ is needed as the cycles need at least 4 vertices each.

For $n \geq 6k$, the graph $K_{3k-1, n-3k+1}$ has $\delta = 3k - 1$ and no collection of $k$ independent chorded cycles, as chorded cycles here require 3 vertices from each partite set.

$$K_{3k-1, n-3k+1}$$
Theorem

If \( G \) is a graph on \( n \geq 4k \) vertices such that for any pair of non-adjacent vertices \( x, y \),

\[
|N(x, y)| \geq 4k + 1,
\]

then \( H \) contains at least \( k \) independent chorded cycles.
Theorem

If $G$ is a graph on $n \geq 6k$ vertices with

$$\sigma_2(G) \geq 6k - 1,$$

then $G$ contains $k$ vertex disjoint doubly chorded cycles.
Sharpness

\[ a = 3k - 1 \]

\[ \bullet \bullet \ldots \bullet \]

\[ + \]

\[ K_{a,b} \]

\[ \bullet \bullet \bullet \ldots \bullet \]

\[ b = n - 3k + 1 \]
Other Natural Questions for Chorded Cycles

- Make a set of edges the chords.
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- Place \( k \) vertices on \( k \) vertex disjoint chorded cycles.
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Other Natural Questions for Chorded Cycles

- Make a set of edges the chords.
- Place $k$ vertices on $k$ vertex disjoint chorded cycles.
- Place $k$ indep. edges on $k$ vertex disjoint chorded cycles.
- Place $k$-path linear forest on $k$ disjoint chorded cycles.
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- Control the order of the chorded cycles.
Other Natural Questions for Chorded Cycles

- Make a set of edges the chords.
- Place $k$ vertices on $k$ vertex disjoint chorded cycles.
- Place $k$ indep. edges on $k$ vertex disjoint chorded cycles.
- Place $k$-path linear forest on $k$ disjoint chorded cycles.
- Control the order of the chorded cycles.
- Expand our chorded cycle system to span $V(G)$. 
Specifying certain edges to be chords

Question
Can we make an independent set of $k$ edges the chords of $k$ vertex disjoint chorded cycles?
Theorem

Let $k \geq 1$ be an integer and $G$ be a graph of order $n \geq 14k$. If $\sigma_2(G) \geq n + 3k - 2$, then for any $k$ independent edges $e_1, e_2, \ldots, e_k$ of $G$, the graph $G$ contains $k$ vertex disjoint cycles $C_1, C_2, \ldots, C_k$ such that $e_i$ is a chord of $C_i$ for all $1 \leq i \leq k$. Furthermore, $4 \leq |V(C_i)| \leq 5$ for each $i$. 
Sharpness Example

\begin{align*}
K_{2k} & \quad e_1 \\
K_{2k-1} & \quad e_2 \\
K_{n-4k+1} & \quad e_k \\
x & \quad \ldots \\
& \quad + \\
y & \quad + \\
\end{align*}
Question- Placing vertices on chorded cycles

Question

When can we distribute \( k \) vertices on \( k \) disjoint chorded cycles?
Theorem

Let \( k \geq 1 \) be an integer and let \( G \) be a graph of order \( n \geq 16k - 12 \). If \( \delta(G) \geq n/2 \) then for any set of \( k \) vertices \( \{v_1, v_2, \ldots, v_k\} \) there exists a collection of \( k \) vertex disjoint chorded cycles \( \{C_1, \ldots, C_k\} \) such that \( v_i \in V(C_i) \) and \( |V(C_i)| \leq 6 \) for each \( i = 1, 2, \ldots, k \).
Question

When can we distribute $k$ independent edges on $k$ disjoint chorded cycles?
Theorem

Let $G$ be a graph of order $n \geq 18k - 2$ and let $e_1, e_2, \ldots, e_k$ be a set of $k$ independent edges in $G$. If

$$\delta(G) \geq \frac{n + 2k - 2}{2}$$

then there exists a system of $k$ chorded cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$. 

[with M. Cream, R. Faudree and K. Hirohata]
As a Corollary to the proof we obtain the fact the edges $e_1, e_2, \ldots, e_k$ can be a mix of either chords or edges of the cycles (again one edge per cycle).

Further, we can show that the cycle system can also be extended to span $V(G)$. 

[with M. Cream, R. Faudree and K. Hirohata]
Theorem

Let $G$ be a graph of order $n \geq 22k - 2$ and let $e_1, \ldots, e_k$ be $k$ independent edges in $G$. Then if

$$\delta(G) \geq \frac{n + 2k - 2}{2}$$

then there exists a system of $k$ vertex disjoint doubly chorded cycles $C_i, \ldots, C_k$ such that $e_i \in E(C_i)$ and $|V(C_i)| \leq 6$ for each $i = 1, 2, \ldots, k$.

Corollary

The above system can be extended to span $V(G)$. 
Question

When can we distribute a $k$ path linear forest on $k$ disjoint chorded cycles?
Fact

Given independent path $P_{r_1}, P_{r_2}, \ldots, P_{r_k}$ with each $r_i \geq 2$ let $r = \sum r_i$. Then the number of interior vertices in this path system is $r - 2k$.

Theorem

Let $P_{r_1}, P_{r_2}, \ldots, P_{r_k}$ be a linear forest in a graph $G$ of order $16k + r - 2$ with

$$\delta(G) \geq n/2 + r - 1 - k.$$ 

Then there exists a system of $k$ chorded cycles $C_1, \ldots, C_k$ such that the path $P_{r_i}$ lies on the cycle $C_i$ and $|V(C_i)| \leq r_i + 4$. 

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