Minimum Degree and Dominating Paths

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Joint work with Ralph Faudree, Mike Jacobson and Doug West
Dedicated to Ralph J. Faudree

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Figure: Ralph Faudree - Outstanding Mathematician and Great Friend
Figure: Ralph Faudree - Powerful Administrator
Figure: From experience he did not believe all he heard!
Figure: Ralph and Pat on the 3-gorges river cruise.
Figure: The Gang of 7: Back: Burr, Jacobson, Rousseau, Schelp
Front: RG, Uncle Paul, Ralph: March 1984
Figure: Dinner in Budapest with Miki Simonovits.
A set $S \subseteq V(G)$ is a connected dominating set provided $G[S]$ is connected and each vertex of $V(G) - S$ has a neighbor in $S$.

Fact: large minimum degree implies small connected dominating set.
A set $S \subset V(G)$ is a connected dominating set provided $G[S]$ is connected and each vertex of $V(G) - S$ has a neighbor in $S$.

Fact: large minimum degree implies small connected dominating set.

Theorem (Caro – West – Yuster, 2000)
For large fixed $k$, every $n$-vertex graph $G$ with $\delta(G) \geq k$ has a connected dominating set with size at most

$$\frac{(1+o(1))\ln k}{k}n$$
Question

What do we known about $G[S]$?
**Question**

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Spanning trees are connected dominating sets, with the leaves as the dominated vertices.
But what can we say about the structure of $G[S]$, even if it is a tree of some sort?
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One of our goals will be to try and say something about one special case of a connected dominating set that is a tree - namely a path.
More background - Maybe the set is not so small

**Theorem**


Some k-regular graphs have no dominating set of size less than

$$\frac{1 + \ln (k+1)}{k+1} n$$

A probabilistic argument for this.
Definition

A path $P$ such that every vertex of $G$ is on $P$, or adjacent to a vertex of $P$ is called a dominating path.
Control $G[S]$ - dominating paths

**Definition**

A path $P$ such that every vertex of $G$ is on $P$, or adjacent to a vertex of $P$ is called a **dominating path**.

**Theorem**

(*Dirac*, 1952)

Every $n$-vertex graph with $\delta(G) \geq (n - 1)/2$ has a spanning path, hence a dominating path with $n - 2$ vertices.
Definition

Vertices $u$ and $v$ are $\lambda$-distant provided $\text{dist}(u, v) \geq \lambda$.

Theorem

(Broersma, 1988)

Let $G$ be a $k$-connected graph ($k \geq 1$) and let $\lambda \geq 2$. If the degree sum of any $k + 2$ mutually $(2\lambda - 1)$-distant vertices is at least $n - 2k - 1 - (\lambda - 2)k(k + 2)$, then $G$ has a path where every vertex is at distance less than $\lambda$ of this path.

Corollary

Let $G$ be a $k$-connected graph. If the degree sum of any $k + 2$ mutually 3-distant vertices is at least $n - 2k - 1$

then $G$ has a dominating path.
Dominating cycles were usually long cycles

**Theorem**

(Yoshimoto, 2008)

If

\[ \text{deg } e_1 + \text{deg } e_2 > |V(G)| - 4 \]

for any two remote edges \( e_1, e_2 \), then all longest cycles in \( G \) are dominating and this bound is best possible.
More long cycles

Theorem
(Bondy, 1980)
If $G$ is 2-connected of order $n$ and

$$\sigma_3(G) \geq n + 2,$$

then each longest cycle of $G$ is dominating.

Theorem
(Yamashita, 2008)
If $G$ is 3-connected of order $n$ and

$$\sigma_4(G) \geq n + \kappa(G) + 3,$$

then $G$ contains a longest cycle which is dominating.
Question

What minimum degree guarantees a “small” dominating path?
Our driving question

Question
What minimum degree guarantees a “small” dominating path?

Question
How small is small?
Our Results

Theorem

Every $n$-vertex connected graph $G$ with

$$\delta(G) \geq \frac{n}{3} - 1$$

contains a dominating path, and the inequality is sharp.
Sharpness Example

\[ \text{Minimum Degree and Dominating Paths} \]
Theorem

If $G$ is an $n$-vertex 2-connected graph with

$$\delta(G) \geq (n + 1)/4,$$

then $G$ contains a dominating path.

(This is almost sharp in the sense there is an example with $\delta(G) = (n - 6)/4$ that fails.)
(minus edges to one vertex per clique)

$K_k + (k+2)K_k$
More control of the path length

**Theorem**

If $\delta(G) \geq n/3$, then $G$ has a dominating $k$ vertex path for every $k$ from the least value to at least

$$\min \{ n, 2\delta(G) + 1 \}$$

and this is sharp.
Better yet!

Theorem

If \( \delta(G) \geq cn \) with \( c > 1/3 \), then \( G \) has a dominating path with length logarithmic in \( n \) (base depends only on \( c \)).
Theorem

If $\delta(G) \geq cn$ with $c > 1/3$, then $G$ has a dominating path with length logarithmic in $n$ (base depends only on $c$).

Remark: The Alon - Wormald result implies min deg $k$ guaranteeing an $s$-vertex dominating path requires $s > \frac{\ln k}{k} n$. But when $s$ is constant a direct argument gives more.
Theorem

Fix \( s \in \mathbb{N} \) and \( c \in \mathbb{R} \) with \( c < 1 \). For \( n \) sufficiently large, with

\[
\delta(G) \geq n - 1 - cn^{1-1/s}
\]

the graph contains an \( s \)-vertex dominating path.
Theorem

Given $s \in \mathbb{N}$ and $c > 1$, for $n$ suff. large, some $n$-vertex graph with

$$\delta \geq n - c(s \ln n)^{1/2} \ n^{1-1/s}$$

has no dominating set of size at most $s$. 

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Minimum Degree and Dominating Paths
**Theorem**

Given \( s \in N \) and \( c > 1 \), for \( n \) suff. large, some \( n \)-vertex graph with \( \delta \geq n - c(s/n)^{1/2} n^{1-1/s} \) has no dominating set of size at most \( s \).

**Corollary**

*In particular, no \( s \)-vertex dominating path.*
Theorem

Let $G$ be a connected $n$-vertex graph with

$$\delta(G) \geq an + \log_{a/(1-a)} n$$

where $a > 1/2$. For $n$ suff. large, $G$ has an $s$-vertex dominating path whenever

$$\log_{a/(1-a)} n \leq s \leq n$$

starting from any vertex.
For $1/3 < a < 1$, there is a constant $c = c(a)$ such that if $n$ is sufficiently large and $\delta(G) \geq an$, then $G$ contains a dominating path with at most $c \log_{1/(1-a)} n$ vertices.
Theorem

Fix $s \in \mathbb{N}$ and $c \in \mathbb{R}$ with $c < 1$. For suff. large $n$,

$$\delta(G) \geq n - 1 - cn^{1-1/s}$$

ensures an $s$-vertex dominating path.

Near Sharpness.

Theorem

Given $s \in \mathbb{N}$ and $c > 1$, for suff. large $n$, some $n$-vertex graph with $\delta(G) \geq n - c(sln n)^{1/s}n^{1-1/s}$ has no dominating set of size at most $s$. 
Balanced Caterpillars

Definition

spanning caterpillar = spanning tree consisting of a single path (spine) plus leaves.

Definition

balanced = if the vertices of the spine all have the same number of neighbors; nearly balanced = the numbers differ by at most 1.
Theorem

Fix $p \in \mathbb{N}$. For $n$ suff. large, with $(p + 1)$ dividing $n$ and

$$\delta(G) \geq (1 - \frac{p}{(p + 1)^2})n$$

then $G$ contains a balanced spanning caterpillar with $\frac{n}{p+1}$ spine vertices.
Case: $p = 1$ says $\delta \geq 3n/4$ and $n$ even. This implies...
Theorem

Fix a positive integer \( s \) and a real constant \( c \) less than 1. Let \( G \) be an \( n \)-vertex graph such that \( \delta(G) \geq n - cn^{1-1/s} \). If \( n \) is suff. large, then \( G \) contains a nearly balanced spanning caterpillar with \( k \) spine vertices for each \( k \) such that

\[
s \leq k \leq 0.5 \frac{\log n}{\log \log n}.
\]
Question: For \( k \in \mathbb{N} \), when \( G \) is \( k \)-connected, what threshold on \( \delta(G) \) guarantees a dominating path?
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Conjectured to be \( \delta(G) \geq \frac{n-2k-1}{k+2} \); that much is needed - known to be about right when \( k \leq 2 \).
Open Problems

Question: For $k \in \mathbb{N}$, when $G$ is $k$-connected, what threshold on $\delta(G)$ guarantees a dominating path?

Conjectured to be $\delta(G) \geq \frac{n-2k-1}{k+2}$; that much is needed - known to be about right when $k \leq 2$

Question: For $\delta \geq an$ with $a > 1/3$, how short a dominating path can we get (for $n$ large)?
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Question: For $\delta \geq \alpha n$ with $\alpha > 1/3$, how short a dominating path can we get (for $n$ large)?

Known to be at most $c \log_{1/(1-a)} n$. 

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Question: For $s \in \mathbb{N}$ and $n$ large, what threshold on $\delta(G)$ ensures an $s$-vertex dominating path?
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Known to be at most $c\log_{1/(1−a)} n$.

Question: For $s \in \mathbb{N}$ and $n$ large, what threshold on $\delta(G)$ ensures an $s$-vertex dominating path?

at most $n – \Omega(n^{1−1/s}$ and at least $n – O(sln n)^{1/s} n^{1−1/s}$.