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# New Conditions for $k$ -ordered Hamiltonian Graphs

Guantao Chen\*  
Georgia State University  
Atlanta GA 30303

Ronald J. Gould  
Emory University  
Atlanta GA 30322

Florian Pfender  
Emory University  
Atlanta GA 30322  
fpfende@mathcs.emory.edu

## Abstract

We show that in any graph  $G$  on  $n$  vertices with  $d(x) + d(y) \geq n$  for any two nonadjacent vertices  $x$  and  $y$ , we can fix the order of  $k$  vertices on a given cycle and find a hamiltonian cycle encountering these vertices in the same order, as long as  $k < n/12$  and  $G$  is  $\lceil (k+1)/2 \rceil$ -connected. Further we show that every  $\lfloor 3k/2 \rfloor$ -connected graph on  $n$  vertices with  $d(x) + d(y) \geq n$  for any two nonadjacent vertices  $x$  and  $y$  is  $k$ -ordered hamiltonian, i.e. for every ordered set of  $k$  vertices we can find a hamiltonian cycle encountering these vertices in the given order. Both connectivity bounds are best possible.

## 1 Introduction

One of the most widely studied classes of graphs are hamiltonian graphs. In this paper we are interested in the following question: When can we guarantee a certain set  $S$  of vertices to appear on a hamiltonian cycle in a given order? In [6], Ng and Schultz first explored the following related concept introduced by Chartrand. A graph is called  *$k$ -ordered hamiltonian*, if for every vertex set  $S$  of size  $k$  there is a hamiltonian cycle encountering the vertices in  $S$  in a given order. Clearly, every hamiltonian graph is 3-ordered hamiltonian. Ng and Schultz [6] showed that  $k$ -ordered hamiltonian graphs must be  $(k-1)$ -connected. Further, they showed the following theorem.

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**Theorem 1** [6] Let  $G$  be a graph of order  $n$  and let  $k$  be an integer with  $3 \leq k \leq n$ . If  $d(u) + d(v) \geq n + 2k - 6$  for every pair  $u, v$  of nonadjacent vertices of  $G$ , then  $G$  is  $k$ -ordered hamiltonian.

This bound was later improved in [3] and [4] by Faudree et al. for small values of  $k$ .

**Theorem 2** [4] Let  $G$  be a graph of order  $n$  and let  $k$  be an integer with  $3 \leq k \leq n/2$ . If  $d(u) + d(v) \geq n + (3k - 9)/2$  for every pair  $u, v$  of nonadjacent vertices of  $G$ , then  $G$  is  $k$ -ordered hamiltonian.

Instead of increasing the bound on the degree sum from the Ore-bound for hamiltonicity as in these papers, we choose to ask for a higher connectivity with the resultant effect of being able to lower the degree sum condition. We will first prove the following theorem.

**Theorem 3** Let  $G$  be a graph on  $n$  vertices with  $d(x) + d(y) \geq n$  for any two nonadjacent vertices  $x$  and  $y$ . Let  $k < n/12$  be an integer, and let  $C$  be a cycle encountering a vertex sequence  $S = \{x_1, \dots, x_k\}$  in the given order. If  $G$  is  $\lceil (k+1)/2 \rceil$ -connected, then  $G$  has a hamiltonian cycle encountering  $S$  in the given order.

**Corollary 4** Let  $G$  be a graph on  $n$  vertices with minimum degree  $\delta(G) \geq n/2$ . Let  $k < n/12$  be an integer, and let  $C$  be a cycle encountering a vertex sequence  $S = \{x_1, \dots, x_k\}$  in the given order. If  $G$  is  $\lceil (k+1)/2 \rceil$ -connected, then  $G$  has a hamiltonian cycle encountering  $S$  in the given order.

The connectivity bound is best possible, as illustrated by the following graph  $G_1$ . Let  $L, K, R$  be complete graphs with  $|R| = \lceil (2n - k)/4 \rceil$ ,  $|K| = \lfloor k/2 \rfloor$ ,  $|L| = n - |K| - |R|$ . Let  $G_1$  be the union of the three graphs, adding all possible edges containing vertices of  $K$ . Clearly,  $\delta(G_1) > n/2$ , and  $G_1$  is  $\lfloor k/2 \rfloor$ -connected. Let  $S = \{x_1, \dots, x_k\}$  with  $x_i \in K$  if  $i$  is even and  $x_i \in R$  otherwise. The cycle  $C = x_1 x_2 \dots x_k x_1$  contains  $S$  in the right order, but no cycle containing  $S$  in the right order can contain any vertices of  $L$ .

A graph is called  $k$ -ordered, if for every vertex sequence  $S$  of size  $k$  there is a cycle encountering the vertices in  $S$  in the given order. Now observe that every  $k$ -ordered graph is  $(k-1)$ -connected. Thus, we get the following corollaries (these are very similar to theorems used in [6] and [4]).

**Corollary 5** Let  $G$  be a graph on  $n$  vertices with  $d(x) + d(y) \geq n$  for any two nonadjacent vertices  $x$  and  $y$ . Let  $k < n/12$  be an integer, and suppose that  $G$  is  $k$ -ordered. Then  $G$  is  $k$ -ordered hamiltonian.

**Corollary 6** Let  $G$  be a graph on  $n$  vertices with minimum degree  $\delta(G) \geq n/2$ . Let  $k < n/12$  be an integer, and suppose that  $G$  is  $k$ -ordered. Then  $G$  is  $k$ -ordered hamiltonian.

$n$  and let  $k$  be an integer with every pair  $u, v$  of nonadjacent vertices.

[4] by Faudree et al. for small

$n$  and let  $k$  be an integer with  $(n-9)/2$  for every pair  $u, v$  of red hamiltonian.

the sum from the Ore-bound for to ask for a higher connectivity weaker the degree sum condition.

with  $d(x) + d(y) \geq n$  for any  $n/12$  be an integer, and let  $C$  be  $\{x_1, \dots, x_k\}$  in the given order. hamiltonian cycle encountering

with minimum degree  $\delta(G) \geq n/12$  a cycle encountering a vertex  $v$ . If  $G$  is  $\lceil (k+1)/2 \rceil$ -connected,  $S$  in the given order.

illustrated by the following graph  $G_1$  is the union of the three graphs,  $L_1, K_1, R_1$  of  $K$ . Clearly,  $\delta(G_1) > n/2$ ,  $\{x_k\}$  with  $x_i \in K$  if  $i$  is even  $x_k x_1$  contains  $S$  in the right order can contain any vertices

sequence  $S$  of size  $k$  there is given order. Now observe  $C$ . Thus, we get the following result used in [6] and [4].

with  $d(x) + d(y) \geq n$  for any  $n/2$  be an integer, and suppose hamiltonian.

with minimum degree  $\delta(G) \geq n/2$  that  $G$  is  $k$ -ordered. Then  $G$

We further prove the following theorem.

**Theorem 7** Let  $G$  be a graph on  $n$  vertices with  $d(x) + d(y) \geq n$  for any two nonadjacent vertices  $x$  and  $y$ . Let  $k \leq n/176$  be an integer. If  $G$  is  $\lfloor 3k/2 \rfloor$ -connected, then  $G$  is  $k$ -ordered hamiltonian.

The connectivity bound is best possible, as illustrated by the following graph  $G_2$ . Let  $L_2, K_2, R_2$  be complete graphs with  $|R_2| = \lfloor k/2 \rfloor$ ,  $|K_2| = 2\lfloor k/2 \rfloor - 1$ ,  $|L_2| = n - |K_2| - |R_2|$ . Let  $G'_2$  be the union of the three graphs, adding all possible edges containing vertices of  $K_2$ . Let  $x_i \in L_2$  if  $i$  is odd, and let  $x_i \in R_2$  otherwise. Add all edges  $x_i x_j$  whenever  $|i-j| \notin \{0, 1, k-1\}$ , and the resulting graph is  $G_2$ . The degree sum condition is satisfied and  $G_2$  is  $(\lfloor 3k/2 \rfloor - 1)$ -connected. But there is no cycle containing the  $x_i$  in the right order, since such a cycle would contain  $2\lfloor k/2 \rfloor$  paths through  $K_2$ .

For the analogous theorem with a bound on the minimum degree we get a slight improvement on the connectivity bound for odd  $k$ .

**Theorem 8** Let  $G$  be a graph on  $n$  vertices with minimum degree  $\delta(G) \geq n/2$ . Let  $k \leq n/176$  be an integer. If  $G$  is  $3\lfloor k/2 \rfloor$ -connected, then  $G$  is  $k$ -ordered hamiltonian.

Again, the connectivity bound is best possible, as illustrated by the following graph  $G_3$ . Let  $L_3, K_3, R_3$  be complete graphs with  $|R_3| = \lceil (n-k)/2 \rceil$ ,  $|K_3| = 2\lfloor k/2 \rfloor - 1$ ,  $|L_3| = n - |K_3| - |R_3|$ . Let  $G'_3$  be the union of the three graphs, adding all possible edges containing vertices of  $K_3$ . Let  $x_i \in L_3$  if  $i$  is odd, and let  $x_i \in R_3$  otherwise. Add all edges  $x_i x_j$  whenever  $|i-j| \notin \{0, 1, k-1\}$ , and the resulting graph is  $G_3$ . The degree condition is satisfied, and  $G_3$  is  $(3\lfloor k/2 \rfloor - 1)$ -connected. But there is no cycle containing the  $x_i$  in the right order, since such a cycle would contain  $2\lfloor k/2 \rfloor$  paths through  $K_3$ .

## 2 Proof of Theorem 3

Assume that  $C$  is a maximal cycle encountering  $S$  in the given order. If  $C$  is hamiltonian, we are done. So, assume  $|C| < n$ , and let  $H$  be a component of  $G - C$ , say  $|H| = r$ . The sequence  $S$  splits  $C$  into  $k$  segments  $[x_1 C x_2], \dots, [x_k C x_1]$ .

**Claim 1** There is at most one adjacency of  $H$  in each segment  $[x_i C x_{i+1}]$ .

Suppose the contrary. Let  $x, y$  be two adjacencies of  $H$  inside  $[x_i C x_{i+1}]$  with no other adjacencies of  $H$  in  $(x C y)$ . Let  $v \in H \cap N(x)$ . Let  $|x C y| = s$ . Since  $v$  is not insertible in  $C$  we get

$$d(v) \leq r - 1 + \frac{n - r - s + 1}{2}.$$

Insert the vertices of  $(xCy)$  one by one into  $[yCx]$ . If all of them can be inserted, we can extend  $C$  through  $v$ , so there is a vertex  $w$  that can not be inserted. We get

$$d(w) \leq s - 1 + \frac{n - r - s + 1}{2},$$

so

$$d(v) + d(w) \leq n - 1,$$

a contradiction. This proves the claim.  $\square$

By claim 1,  $C$  has at most  $k$  adjacencies to  $H$ . Let  $v \in H$ , and  $w \in C$  be a vertex not adjacent to  $H$ . Then

$$n \leq d(v) + d(w) \leq (r - 1 + k) + (n - r - 1) = n + k - 2.$$

Thus,  $w$  is adjacent to all but at most  $k - 2$  vertices of  $G - H$ . Further,  $v$  is adjacent to all but at most  $k - 2$  vertices in  $H$ . We claim that  $H$  is hamiltonian connected as follows: Either  $H$  is complete and we are done, or two vertices  $v, u \in H$  are not adjacent. Then  $|H| \geq \frac{d(v)+d(u)}{2} - k \geq \frac{n}{2} - k$ , using Claim 1 and the degree sum condition. Now  $\delta_H(H) \geq |H| - k + 2 > |H|/2 + 1$ , which implies hamiltonian connectedness.

**Claim 2**  $G - C$  has at most one component.

Suppose the contrary, let  $H'$  be another component with  $|H'| = r'$ . Let  $v \in H, v' \in H'$ . Since  $G$  is  $\lceil (k+1)/2 \rceil$ -connected,  $H$  can be adjacent to at most  $\lfloor (k-1)/2 \rfloor$  vertices from  $S$ , else there is a contradiction with Claim 1. The same is true for  $H'$ . Thus, for some  $i, x_i \notin N(H) \cup N(H')$ . But now,

$$\begin{aligned} 3n &\leq 2(d(x_i) + d(v) + d(v')) \leq \\ &2((n - r - r' - 1) + (r - 1 + k) + (r' - 1 + k)) = \\ &2n + 4k - 6, \end{aligned}$$

a contradiction that proves the claim.  $\square$

Since  $G$  is  $\lceil (k+1)/2 \rceil$ -connected, there is a segment  $[x_j C x_{j+2}]$  with two adjacencies  $y, z$  of  $H$ . By claim 1, we may assume that  $y \in [x_j C x_{j+1}]$ , and  $z \in [x_{j+1} C x_{j+2}]$ . If  $|H| \geq k$  we can even guarantee that  $|(N(y) \cup N(z)) \cap H| \geq 2$ .

**Claim 3**  $|C| \geq n/2$ .

Suppose  $|C| < n/2$ . Then  $|H| \geq n/2$ , and  $y, z$  could be picked such that  $uy, vz \in E(G)$  for two vertices  $u, v \in H$ . Find a hamiltonian path  $P$  in  $H$

o  $[yCx]$ . If all of them can be  
ere is a vertex  $w$  that can not

$$\frac{-s+1}{2},$$

- 1,

□

to  $H$ . Let  $v \in H$ , and  $w \in C$

$$-r-1) = n+k-2.$$

2 vertices of  $G-H$ . Further  
es in  $H$ . We claim that  $H$  is  
3 complete and we are done, or

$$|H| \geq \frac{d(v)+d(w)}{2} - k \geq \frac{n}{2} - k,$$

Now  $\delta_H(H) \geq |H| - k + 2 >$   
tedness.

ponent with  $|H'| = r'$ . Let  
cted,  $H$  can be adjacent to at  
is a contradiction with Claim  
 $i, x_i \notin N(H) \cup N(H')$ . But

$$-(r'-1+k) =$$

$$2n+4k-6,$$

□

segment  $[x_jCx_{j+2}]$  with two  
ume that  $y \in [x_jCx_{j+1}]$ , and  
rantee that  $|(N(y) \cup N(z)) \cap$

$z$  could be picked such that  
a hamiltonian path  $P$  in  $H$

from  $u$  to  $v$ . Observe that  $N(x_{j+1}) \cup N(x_{j+2}) \subseteq C$ . If  $x_{j+1}x_{j+2} \in E(G)$ ,  
then the cycle  $uPvzC^{-}x_{j+1}x_{j+2}Cx_ju$  is longer than  $C$ , a contradiction.  
Thus,  $x_{j+1}x_{j+2} \notin E(G)$ . But now

$$|C| \geq \frac{d(x_{j+1}) + d(x_{j+2})}{2} + 2 > \frac{n}{2},$$

the contradiction proving the claim. □

For the final contradiction we differentiate two cases.

**Case 1** *There exists a vertex  $w \in (yCx_{j+1}) \cup (zCx_{j+2})$ .*

Let  $N = N(x_{j+1}) \cap N(x_{j+2}) \cap N(w)$ . Since none of the vertices  $x_{j+1}, x_{j+2}, w$   
is adjacent to  $H$ , each is adjacent to all but at most  $k-2$  vertices of the  
cycle. Thus,  $|N| \geq |C| - 3k + 6$ .

**Claim 4** *For some  $i$ ,  $|N \cap [x_iCx_{i+1}]| \geq 4$ .*

Suppose not, then

$$n/2 \leq |C| \leq 3k + |C| - |N| \leq 6k - 6,$$

a contradiction for  $n \geq 12k$ . □

Let  $i$  be as in the last claim, and let  $v_1, v_2, v_3, v_4 \in N \cap [x_iCx_{i+1}]$  be  
the first four of these vertices in that order.

If  $v_4 \in (yCx_{j+1})$ , define a new cycle as follows:  $C' = zC^{-}v_4x_{j+2}CyuPvz$   
(see Figure 1).

If  $v_4 \in (zCx_{j+2})$ , let  $C' = zC^{-}x_{j+2}v_4CyuPvz$ .

Otherwise observe that by claim 1, there is at most one adjacency  $x$  of  $H$   
in  $[v_1Cv_4]$ .

For  $i \neq j+1$ , define the new cycle  $C'$  as follows:

If  $x \in [v_1Cv_2]$ , let  $C' = zC^{-}x_{j+1}v_3x_{j+2}Cv_2wv_4CyuPvz$  (see Figure 2).

If  $x \in [v_3Cv_4]$ , let  $C' = zC^{-}x_{j+1}v_2x_{j+2}Cv_1wv_3CyuPvz$ .

Otherwise, let  $C' = zC^{-}x_{j+1}v_2Cv_3x_{j+2}Cv_1wv_4CyuPvz$ .

For  $i = j+1$ , a very similar construction works:

let  $C' = zC^{-}v_4wv_1C^{-}x_{j+1}v_2Cv_3x_{j+2}CyuPvz$ .

In any case, no vertex in  $C - C'$  is adjacent to  $H$ , so all of them have high  
degree to  $C$  and thus high degree to  $C \cap C'$ . Therefore, we can insert them  
one by one into  $C'$  creating a longer cycle, a contradiction.

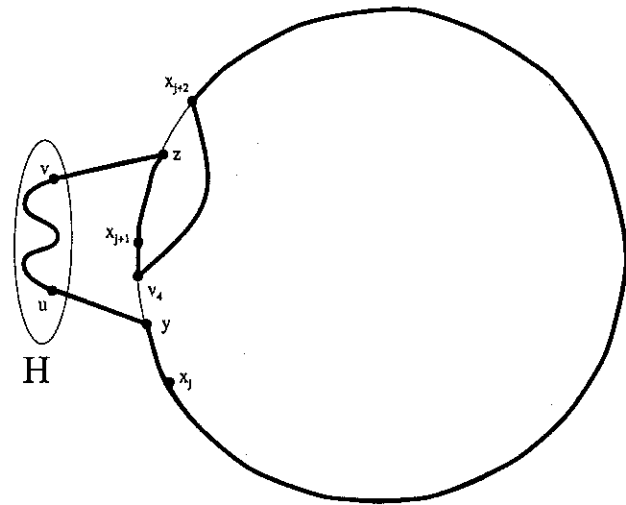


Figure 1: a possible  $C'$

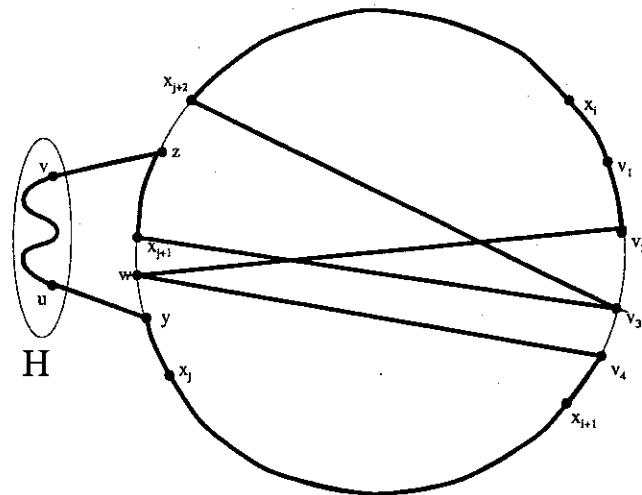


Figure 2: a possible  $C'$

**Case 2** Suppose  $(yCx_{j+1}) \cup (zCx_{j+2}) = \emptyset$ .

Let  $N' = N(x_{j+1}) \cap N(x_{j+2})$ . Then  $|N'| \geq |C| - 2k + 4$ .

**Claim 5** For some  $l$ ,  $|N' \cap [x_lCx_{l+1}]| \geq 5$ .

Suppose not. Then

$$n/2 \leq |C| \leq 4k + |C| - |N'| \leq 6k - 4,$$

a contradiction for  $n \geq 12k$ . □

Let  $l$  be as in the last claim, and let  $z_1, z_2, z_3, z_4, z_5 \in N' \cap [x_lCx_{l+1}]$  be the first five of these vertices in that order. At most one of them is adjacent to  $H$ , say  $z_2$ . Now a very similar argument as in the last case gives the desired contradiction, just replace  $x_{j+1}$  by  $z_1$ ,  $x_{j+2}$  by  $z_5$ , and  $w$  by  $z_4$ . One possible cycle would then be (for  $l < i < j$ ):  $C' = zCx_{j+1}z_2Cx_3x_{j+2}Cz_1v_2Cv_3z_5Cv_1z_4v_4CyuPuz$  (see Figure 3). □

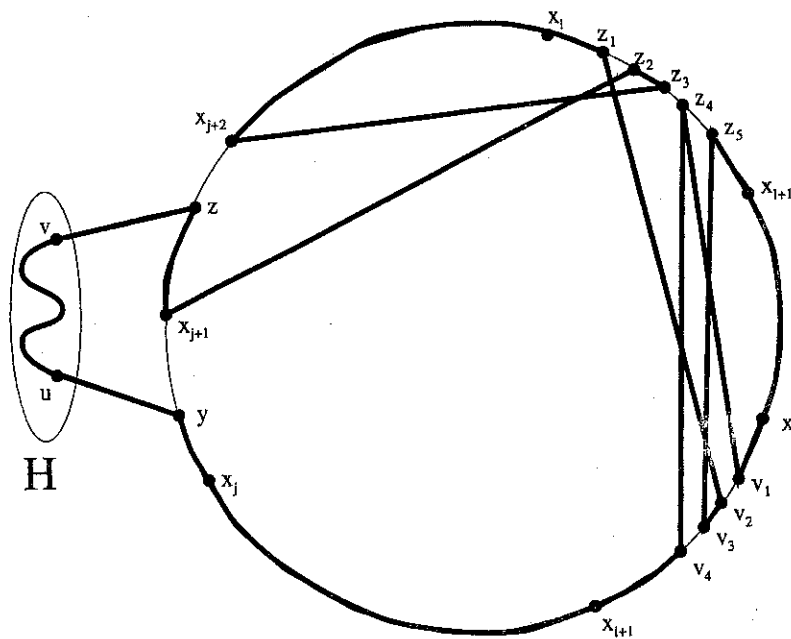


Figure 3: a possible  $C'$

### 3 Proof of Theorems 7 and 8

By Corollary 5, all we need to show is that  $G$  is  $k$ -ordered. For this purpose, we will use a slightly stronger concept.

We will say that a graph  $G$  on at least  $2k$  vertices is  $k$ -linked, if for every vertex set  $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$  of  $2k$  vertices, there are  $k$  disjoint  $x_i y_i$ -paths. The property remains the same if we allow repetition in  $T$ , and ask for  $k$  internally disjoint  $x_i y_i$ -paths. Thus, as an easy consequence, every  $k$ -linked graph is  $k$ -ordered.

An important theorem about  $k$ -linked graphs is the following theorem of Bollobás and Thomason:

**Theorem 9** [1] *Every  $22k$ -connected graph is  $k$ -linked.*

The following lemmas will be used later.

**Lemma 10** *If a  $2k$ -connected graph  $G$  has a  $k$ -linked subgraph  $H$ , then  $G$  is  $k$ -linked.*

**Proof:** Let  $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$  be a set of  $2k$  vertices in  $V(G)$ . Since  $G$  is  $2k$ -connected, there are  $2k$  disjoint paths from  $T$  to  $V(H)$  (trivial paths for vertices in  $T \cap H$ ). Now we can connect these paths in the desired way inside  $H$ , since  $H$  is  $k$ -linked.  $\square$

**Lemma 11** *If  $G$  is a graph,  $v \in V(G)$  with  $d(v) \geq 2k - 1$ , and if  $G - v$  is  $k$ -linked, then  $G$  is  $k$ -linked.*

**Proof:** Let  $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$  be a set of  $2k$  vertices in  $V(G)$ . If  $v \notin T$ , we can find disjoint  $x_i y_i$ -paths inside  $G - v$ . Thus assume that  $v \in T$ , without loss of generality we may assume that  $v = x_1$ . If  $y_1 \in N(v)$ , we can find disjoint  $x_i y_i$ -paths for all  $i \geq 2$  in  $G - v - y_1$ , since  $G - v - y_1$  is  $(k - 1)$ -linked. Adding the path  $vy_1$  completes the desired set of paths in  $G$ . If  $y_1 \notin N(v)$ , then there exists a vertex  $x'_1 \in N(v) - T$ , since  $d(v) \geq 2k - 1$ . We can find disjoint  $x_i y_i$ -paths for  $i \geq 2$  and a  $x'_1 y_1$ -path in  $G - v$ , which we can then extend to an  $x_1 y_1$ -path in  $G$ .  $\square$

Further, we will use a theorem of Mader about dense graphs:

**Theorem 12** [5] *Every graph  $G$  with  $|V(G)| = n \geq 2k - 1$ , and  $|E(G)| \geq (2k - 3)(n - k + 1) + 1$  has a  $k$ -connected subgraph.*

**Corollary 13** [5] *Every graph  $G$  with  $|V(G)| = n \geq 2k - 1$ , and  $|E(G)| \geq 2kn$  has a  $k$ -connected subgraph.*

**Proof of Theorem 7.** Let  $G$  be a graph fulfilling the stated conditions. Let  $S = \{x_1, \dots, x_k\}$  be a set of  $k$  vertices. To show that  $G$  is  $k$ -ordered we need to find a cycle  $C$  including the vertices of  $S$  in the given order. Corollary 5 will then provide Theorem 7. Let  $K$  be a minimal cutset of  $G$ . Let  $L$  and  $R$  be two components of  $G - K$  with  $|L| \leq |R|$ .



nd 8

$G$  is  $k$ -ordered. For this purpose,

st  $2k$  vertices is  $k$ -linked, if for  $\{x_1, \dots, x_k, y_1, \dots, y_k\}$  of  $2k$  vertices, there exists a set of  $k$  disjoint paths  $P_i$  from  $x_i$  to  $y_i$ . It remains the same if we allow repeated vertices. Thus, as an easy exercise,

graphs is the following theorem:

$G$  is  $k$ -linked.

is a  $k$ -linked subgraph  $H$ , then  $G$

$\{x_1, \dots, x_k, y_1, \dots, y_k\}$  be a set of  $2k$  vertices in  $H$ . We can connect these paths in  $H$ .  $\square$

$d(v) \geq 2k - 1$ , and if  $G - v$  is

$\{x_1, \dots, x_k, y_1, \dots, y_k\}$  be a set of  $2k$  vertices in  $G - v$ . Thus assume  $v = x_1$ . If  $v = y_1$ , we may assume that  $v = x_1$ . If  $v = y_i$  for all  $i \geq 2$  in  $G - v$ , since  $v y_1$  completes the desired set of paths, we are done. Otherwise, let  $x'_1 \in N(v) - T$ , since  $v x'_1$  is a path in  $G$ .  $\square$

about dense graphs:

$n \geq 2k - 1$ , and  $|E(G)| \geq \frac{n^2}{4}$ .

$n \geq 2k - 1$ , and  $|E(G)| \geq \frac{n^2}{4}$ .

filling the stated conditions. To show that  $G$  is  $k$ -ordered, let  $S$  be a minimal cutset of  $G$ . Then  $|L| \leq |R|$ .

Case 1 Suppose  $|K| \geq 2k$ .

The degree sum condition forces  $|E(G)| \geq n^2/4 \geq 44kn$ . By Corollary 13,  $G$  has a  $22k$ -connected subgraph  $H$ , which is  $k$ -linked by Theorem 9. By Lemma 10,  $G$  is  $k$ -linked and thus  $k$ -ordered.

Case 2 Suppose  $3\lfloor k/2 \rfloor \leq |K| \leq 2k - 1$ .

First note that  $L$  and  $R$  are the only components of  $G - K$ . Otherwise, let  $x \in L, y \in R, z \in G - (K \cup L \cup R)$ , then

$$\begin{aligned} 3n &\leq 2d(x) + 2d(y) + 2d(z) \\ &\leq 2|L| + 2|K| + 2|R| + 2|K| + 2(n - |L| - |R|) \\ &\leq 2n + 4|K| < 2n + 8k, \end{aligned}$$

a contradiction.

Claim 1  $R$  is  $k$ -linked, and  $L$  is  $k$ -linked or complete.

Let  $v \in L, w \in R$ . Then

$$n \leq d(v) + d(w) \leq |L| - 1 + |K| + |R| - 1 + |K| \leq n + 2k - 3.$$

Thus  $w$  is connected to all but at most  $2k - 3$  vertices in  $R$ . Therefore,  $R$  is  $2k$ -connected. Again,

$$|E(R)| \geq |R|(|R| - 2k + 2) \geq |R|(n/2 - 3k + 2) \geq 44k|R|.$$

Thus,  $R$  has a  $22k$ -connected and therefore  $k$ -linked subgraph, and so  $R$  is  $k$ -linked by Corollary 13, Theorem 9 and Lemma 10.

If  $L$  is complete we are done. Otherwise, let  $x, y \in L$  with  $xy \notin E$ , then

$$|L| \geq \frac{d(x) + d(y)}{2} - |K| \geq \frac{n}{2} - 2k + 1.$$

Every vertex in  $L$  is connected to all but at most  $2k - 3$  vertices in  $L$ , therefore  $L$  is  $2k$ -connected. By a similar argument as before,  $L$  is  $k$ -linked, establishing the claim.  $\square$

Claim 2 For every vertex  $v \in K$ , at least one of the following holds:

1.  $d_R(v) \geq 2k$ ,
2.  $d_L(v) \geq 2k$ ,
3.  $d_L(v) = |L|$ .

Suppose the claim is false for some vertex  $v \in K$ . Let  $x \in L - N(v)$ ,  $y \in R - N(v)$ . Then

$$\begin{aligned} 2n &\leq d(x) + 2d(v) + d(y) \\ &< |L| + |K| + 2(|K| + 4k) + |R| + |K| \\ &\leq n + 3|K| + 4k < n + 10k, \end{aligned}$$

a contradiction. □

The last claim yields a partition of  $K$  as follows:

$$\begin{aligned} K_R &= \{v \in K \mid d_R(v) \geq 2k\}, \\ K_{L1} &= \{v \in K \mid d_L(v) \geq 2k\} - K_R, \\ K_{L2} &= \{v \in K \mid d_L(v) = |L|\} - (K_R \cup K_{L1}). \end{aligned}$$

Note that either  $K_{L1} = \emptyset$  or  $K_{L2} = \emptyset$ , and that the graph induced on  $K_{L2}$  is complete, since all vertices in  $K_{L2}$  have degree less than  $4k$ .

Now let  $R' = \langle R \cup K_R \rangle$ ,  $L' = \langle L \cup K_{L1} \cup K_{L2} \rangle$ . By Claim 1, Claim 2 and Lemma 11,  $R'$  is  $k$ -linked and  $L'$  is  $k$ -linked or complete.

For the last part of the proof, let  $S_L = L' \cap S$ ,  $S_R = R' \cap S$ . Create a new graph  $G'$  as follows: For every  $i$  with  $x_i \in S_L$  and  $x_{i-1}, x_{i+1} \in S_R$ , add a vertex  $x'_i$  with  $N(x'_i) = N(x_i) \cup \{x_i\}$ . It is easy to see that  $G'$  is  $\lfloor 3k/2 \rfloor$ -connected. Therefore,  $G' - S_R$  is  $(\lfloor 3k/2 \rfloor - |S_R|)$ -connected. Using this fact, we can find independent paths in  $G' - S_R$  from each of the vertices in  $S_L \cup \bigcup x'_i$  into  $R' - S_R$ , since  $|S_L \cup \bigcup x'_i| \leq \min\{k, 2|S_L|\} \leq 3k/2 - |S_R|$ . Denote the set of last edges of these paths by  $M$ . Now contract the edges  $x_i x'_i$  to get back to  $G$ .

The existence of the cycle  $C$  is now guaranteed, since we can pick appropriate vertices in  $S_L \cup (M \cap L')$  and in  $S_R \cup (M \cap R')$ , and then use the fact that  $R'$  is  $k$ -linked and  $L'$  is  $k$ -linked or complete to find the necessary connections. This completes the proof of Theorem 7. □

**Proof of Theorem 8.** Observe that the connectivity only played a role in the last part of the previous proof. Let  $G$  be a graph as in Theorem 8. If  $G$  is  $\lfloor 3k/2 \rfloor$ -connected, we are done by Theorem 7. Thus, we may assume that  $k$  is odd and  $G$  has a minimal cut set of size  $3\lfloor k/2 \rfloor$ . Further, we know that  $G$  splits in two parts  $L'$  and  $R'$ , each of which is  $k$ -linked (observe that the degree condition forces  $|L'| > 2k$ ) by the proof of Theorem 7.

Since  $k$  is odd, there are two consecutive vertices in  $S$  on the same side, we may assume  $x_1$  and  $x_k$  is such a pair. Since  $G$  is  $(3(k-1)/2)$ -connected, there exists a matching  $M = \{e_1, \dots, e_{3(k-1)/2}\}$  of edges between  $R'$  and

$x, v \in K$ . Let  $x \in L - N(v)$ .

$$\begin{aligned} & \leq |R| + |K| \\ & \leq n + 3|K| + 4k < n + 10k. \end{aligned}$$

It follows:

- $k$ },
- $k$ } -  $K_R$ ,
- $k$ } -  $(K_R \cup K_{L1})$ .

that the graph induced on  $K_{L2}$  has degree less than  $4k$ .  $\cup K_{L2}$ . By Claim 1, Claim 2 is linked or complete.

$L' \cap S, S_R = R' \cap S$ . Create  $x_i \in S_L$  and  $x_{i-1}, x_{i+1} \in S_H$ . It is easy to see that  $G'$  is  $\lfloor k/2 \rfloor - |S_R|$ -connected. Using  $|S_R| - S_R$  from each of the vertices  $x_i \in S_L$ ,  $\lfloor k/2 \rfloor - |S_R| \leq 3k/2 - |S_R|$ . By  $M$ . Now contract the edges

guaranteed, since we can pick appropriate  $x_i \in S_L$  and  $x_{i-1}, x_{i+1} \in S_H$ . By Claim 1, Claim 2 is linked or complete.  $\square$

connectivity only played a role in the proof of Theorem 8. If  $G$  is a graph as in Theorem 8. If  $G$  is a graph as in Theorem 7. Thus, we may assume  $|S| \geq 3\lfloor k/2 \rfloor$ . Further, we know  $G$  is  $\lfloor k/2 \rfloor$ -connected which is  $k$ -linked (observe that  $G$  is  $\lfloor k/2 \rfloor$ -connected by the proof of Theorem 7. vertices in  $S$  on the same side,  $G$  is  $(3(k-1)/2)$ -connected,  $\lfloor k/2 \rfloor$  of edges between  $R'$  and

$L'$ . We can renumber the edges of  $M$  such that  $e_i \cap S \subseteq \{x_i\}$  for all  $i \leq k-2$ , and  $e_{k-1} \cap S \subseteq \{x_{k-1}, x_k\}$ . Let  $x_{k+1} = x_1$ . To construct the cycle  $C$ , we need to find  $x_i x_{i+1}$ -paths for all  $i \leq k$ . If  $x_i \in L'$  and  $x_{i+1} \in R'$ , or if  $x_i \in R'$  and  $x_{i+1} \in L'$ , we want to find a path from  $x_i$  to  $x_{i+1}$  through  $L'(R')$  and a path from  $e_i$  to  $x_{i+1}$  through  $R'(L')$ . Note that this case can only occur if  $i \leq k-1$ . If  $x_i, x_{i+1} \in L'(R')$ , we want to find a  $x_i x_{i+1}$ -path in  $L'(R')$ . The simultaneous existence of all these paths is guaranteed since  $R'$  and  $L'$  are  $k$ -linked. This completes the proof of Theorem 8.  $\square$

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