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Generalizing Pancyclic and k-Ordered Graphs

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Abstract. Given positive integers $k \leq m \leq n$, a graph G of order n is (k, m)-*pancyclic* if for any set of k vertices of G and any integer r with $m \leq r \leq n$, there is a cycle of length r containing the k vertices. Minimum degree conditions and minimum sum of degree conditions of nonadjacent vertices that imply a graph is (k, m)-pancylic are proved. If the additional property that the k vertices must appear on the cycle in a specified order is required, then the graph is said to be (k, m)-*pancyclic ordered.* Minimum degree conditions and minimum sum of degree conditions for nonadjacent vertices that imply a graph is (k, m)-pancylic ordered are also proved. Examples showing that these constraints are best possible are provided.

1. Introduction

In this paper we will deal only with finite graphs without loops or multiple edges. Notation will be standard, and we will generally follow the notation of Chartrand and Lesniak in [CL96]. Given a vertex x on a cycle C with an orientation, then the successor of x on C will be denoted by x^+ and the predecessor by x^-. For a graph G we will use G to represent the vertex set $V(G)$ and the edge set $E(G)$ when the meaning is clear. Given a subset (or subgraph) H of a graph G and a vertex v, let $d_H(v)$ denote the degree of v relative to H. Given a subset H of vertices of a graph G, the subgraph induced by H will also be denoted by H when it does not lead to confusion. Thus, for example, $G - H$ will denote a set of vertices as well as a subgraph, depending on the context.

Various degree conditions have been investigated which imply that a graph has hamiltonian type properties. The most common degree condition is the minimum degree of a graph G, which will be denoted by $\delta(G)$. Another common degree condition studied is the sum of degrees of nonadjacent vertices. For a graph G, let

$\sigma_2(G) \geq s$ mean that $d(u) + d(v) \geq s$ for each pair of nonadjacent vertices in G. A graph G is called *pancyclic* whenever G of order n contains a cycle of each length r for $3 \leq r \leq n$. A stronger related property is *vertex pancyclic* which requires for any specified vertex v of G there are cycles of length 3 through n containing v. The following definition generalized these ideas:

Definition 1. Let $0 \leq k \leq m$ be fixed integers and G be a graph of order n. The graph G is (k, m)-pancyclic if $n \geq m$ and for any set S_k of k vertices there is a cycle C_r of G containing S_k for each $m \leq r \leq n$.

Note, $(0, 3)$-pancyclic and $(1, 3)$-pancyclic graphs are pancyclic and vertex pancyclic graphs respectively.

The following was introduced by Gary Chartrand [private communication] but first used by Ng and Schultz [7]. A graph G is *k-ordered (hamiltonian)* if given any ordered set S of k vertices, there is a (hamiltonian) cycle that contains S and the vertices of S are encountered on the cycle in the specified order. Results on $\delta(G)$ and $\sigma_2(G)$ that imply a graph G is k-ordered or k-ordered hamiltonian can be found in [6] and [4]. Here, we investigate a generalization of both k-ordered and pancyclic graphs given in the following:

Definition 2. Let $0 \leq k \leq m$ be fixed integers and G be a graph of order n. The graph G is (k, m)-*pancyclic ordered* if $n \geq m$ and for any ordered set S_k of k vertices there is a cycle C_r containing S_k and encountering the vertices of S_k in the specified order for each $m \leq r \leq n$.

Dirac [3] proved that any graph G of order n with $\delta(G) \geq n/2$ is hamiltonian, and Ore in [O60] proved that if $\sigma_2(G) \geq n$ the graph is also hamiltonian. Bondy [1] proved that if $\sigma_2(G) \geq n + 1$, then G is pancyclic. Kierstead, Sárkőzy, and Selkow proved the following result on minimum degree conditions for a graph to be k-ordered hamiltonian.

Theorem 1 [6]. *Let $k \geq 2$and G a graph of order $n \geq 11k - 3$. If*

$$\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1,$$

then G is k -ordered hamiltonian.

The graph F_1 in Fig. 1, which is $K_{2\lfloor k/2 \rfloor - 1} + (K_{\lceil (n-2\lfloor k/2 \rfloor + 1)/2 \rceil} \cup K_{\lfloor (n-2\lfloor k/2 \rfloor + 1)/2 \rfloor})$, verifies that Theorem 1 is sharp. The graph F_1 is not k-ordered and $\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 2$. By having consecutive vertices of the set $S = \{x_1, x_2, \cdots, x_k\}$ in alternate components of $F_1 - A$, then a cycle encountering S in the appropriate order cannot exist.

Theorem 2 [4]. *Let k be an integer with $3 \leq k \leq n/2$. If*

$$\sigma_2(G) \geq n + (3k - 9)/2,$$

then G is k-ordered hamiltonian.

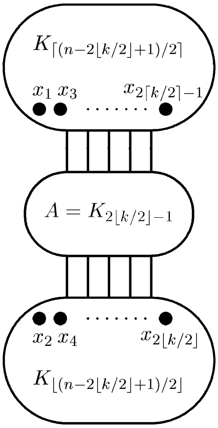

Fig. 1. F_1

The graph F_2 in Fig. 2, which was given by Ng and Schultz [7] verifies that the degree condition in Theorem 2 cannot be reduced. The graph F_2 is obtained by first considering the graph $K_{k-1} + (K_{n-2k+1} + (K_k - E(C_k)))$, and then removing all edges between odd labeled vertices on the cycle C_k and the complete graph K_{n-2k+1}. There is no cycle containing the vertices $S = \{x_1, x_2, \cdots, x_k\}$ of F_2 in the correct order, since the vertices in the subgraph K_{k-1} separates the odd indexed vertices of S from the remainder of the graph, except for edges in $K_k - C_k$ which canot be used. Also, $\sigma_2(F_2) \geq n + (3k-10)/2$ when k is even.

In Section 2 conditions on $\delta(G)$ and $\sigma_2(G)$ for a graph to be (k,m)-pancyclic will be given. Examples to show that the conditions are optimal are also provided. The same will be done in Section 3 for (k,m)-pancyclic ordered graphs.

2. Pancyclic Graphs

For $1 \leq k \leq m \leq n$ the following result provides the sharpest $\sigma_2(G)$ conditions which imply that a graph G of order n is (k,m)-pancyclic.

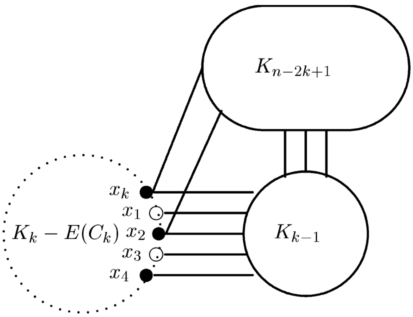

Fig. 2. F_2

Theorem 3. *Let $1 \leq k \leq m \leq n$ be integers, and Gbe a graph of order n. The graph Gis (k, m)-pancyclic if $\sigma_2(G)$ satisfies any of the following conditions:*

(i)	$\sigma_2(G) \geq n$	*when* $m = n$,
(ii)	$\sigma_2(G) \geq \lfloor (4n-1)/3 \rfloor$	*when* $k = 1$ and $m = 3$,
(iii)	$\sigma_2(G) \geq 2n-3$	*when* $k = 2$ or 3 and $m = 3$,
(iv)	$\sigma_2(G) \geq 2n-m$	*when* $k = 3$ and $m = 4$ or 5,
(v)	$\sigma_2(G) \geq 2n - 2\lceil (m-1)/2 \rceil - 1$	*when* $4 \leq k \leq m < 2k$, $n > m$,
(vi)	$\sigma_2(G) \geq 2\lfloor n/2 \rfloor + 1$	*when* $k \geq 2$, $m \geq 2k$, and $n > m$.

Also, all of the conditions on $\sigma_2(G)$are sharp.

Before proving this Theorem we will provide several results that will be used in the proof.

Lemma 1. *Let $k \geq 2$ be a fixed integer, $S_k \subseteq V(G)$ a set of k vertices, and G a graph of order n with $\sigma_2(G) \geq n$.*

(i) If S_k is hamiltonian or an edge, then there is a cycle of length either $k+1$ or $k+2$ in G containing S_k.
(ii) Otherwise, if p is the minimum integer $(1 \leq p \leq k)$ such that S_k has a spanning linear forest with $k-p$ edges, then there is a cycle of order $k+p$ in G containing S_k.

Proof. Case (i). Let C be a hamiltonian cycle of S_k or $C = S_k$ is an edge, $H = G - C$, and assume there is no cycle C_{k+1} containing S_k. Let $y \in H$ with adjacency $x \in C$. Then, $y\,x^+ \notin E(G)$. Moreover, if $y\,z \in E(G)$ for $z \in C$, then $x^+z^+ \notin E(G)$, since this would result in a C_{k+1}. Hence, $d_C(x^+) + d_C(y) \leq k$, and so $d_H(y) + d_H(x^+) \geq n-k$. This implies there is a $y' \in H$ that is commonly adjacent to y and x^+, since y and x^+ are not adjacent, which results in a C_{k+2}. This completes the proof of Case (i).

Case (ii). First select a spanning linear forest in S_k with $k-p$ edges, where p is minimum. Then there are p paths (including paths with just one vertex) in the system. Denote the endpoints of these p paths by $\{x_1, y_1\}, \{x_2, y_2\}, \cdots, \{x_p, y_p\}$ respectively, with, of course, $x_i = y_i$ when the i^{th} path has just one vertex. Because of the minimality of p, the only possible edges between vertices in the set $\{x_1, y_1, \cdots, x_p, y_p\}$ are $x_i y_i$ for $1 \leq i \leq p$. Therefore, for each i, the independent vertices y_i and x_{i+1} (taken modulo p) are part of an independent set with p vertices in S_k. Also, if x_{i+1} is adjacent to any vertex in one of the paths in this spanning linear forest, then y_i is not adjacent to the predecessor. Thus, the sum of the degrees of x_{i+1} and y_i in S_k is at most $k-p$. Let $H = G - S_k$. Then there are p vertices in H commonly adjacent to y_i and x_{i+1}. Hence, there exist p vertices $\{z_1, z_2, \cdots, z_p\}$ in H such that (y_i, z_i, x_{i+1}) is a path with three vertices. This gives a cycle with $k+p$ vertices containing S_k, and completes the proof of Case (ii) and Lemma 1. □

Lemma 2. *Let $k \geq 0$ be a fixed integer, G a graph of order n, and S_k a subset of $V(G)$ of order k. If $\sigma_2(G) \geq 2\lfloor n/2 \rfloor + 1$, and C_r is any cycle of G containing S_k with $k < r < n$, then there is a C_{r+1} in G containing S_k.*

Proof. Let $C_r = (u_1, u_2, \cdots, u_r, u_1)$ be a cycle of length r in G containing $S = S_k$. Assume there is no cycle of length $r+1$ containing S. By a result of Hendry [5] the subgraph C induced by the vertices of C_r is either complete or a regular complete bipartite graph. Since G must be connected, there are edges between C and $G - C = H$.

First suppose that C is complete. Select a vertex u in C, also in S if possible, having an adjacency v in H. By assumption, $N_C(v) = \{u\}$. Let u' be a vertex in $C - \{u\}$, also in S if possible. By the choice of u and u', there is a vertex w in $C - (S \cup \{u, u'\})$. The degree condition implies that the nonadjacent vertices u' and v have two common adjacencies. Hence, they have a common adjacency v' in H. The required cycle C_{r+1} is obtained by adjoining v and v' to C and deleting w. This gives a contradiction.

Now assume that C is a regular complete bipartite graph with parts A and B. Note that $|A| = |B| = t, t \geq 2$. If there is a vertex $z \in N(A) \cap N(B)$, then the required cycle results. Therefore $N(A) \cap N(B) = \emptyset$. With no loss of generality we can assume that $|N(A)| \geq |N(B)|$, and so $|N(B)| \leq \lfloor (n - 2t)/2 \rfloor$. But this implies for $x, y \in B$ that

$$d(x) + d(y) \leq 2t + 2\lfloor (n - 2t)/2 \rfloor = 2\lfloor n/2 \rfloor,$$

which contradicts the hypothesis. This completes the proof of Lemma 2. □

Theorem 4. *Let $k \geq 0$ be a fixed integer, and G a graph of order $n \geq \max\{4, 2k\}$. If $\sigma_2(G) \geq 2\lfloor n/2 \rfloor + 1$, then G is $(0,3)$-pancyclic when $k = 0$ and G is $(k, \max\{4, 2k\})$-pancyclic when $k \geq 1$.*

Proof. When $k = 0$ it is straightforward to prove that G contains a cycle of length 3. When $k = 1$ it is also easy to show that each vertex of G is on a cycle of length 3 or 4. If $k \geq 2$, then for any set S_k of k vertices there is a cycle C_r of G containing S_k with $k < r \leq 2k$ by Lemma 1. Theorem 4 follows from Lemma 2. □

The proof of Theorem 4 implies the existence of additional cycles depending on the structure of the set S_k of k vertices. There is additional strength as described in the following corollary.

Corollary 1. *Let $k \geq 0$ be a fixed integer, G a graph of order $n \geq \max\{4, 2k\}$, and S_k a fixed set of k vertices of G. Let $p = p(S_k)$ be the integer described in Lemma 1(ii) and if S_k is hamiltonian, let $p = 2$. If $\sigma_2(G) \geq 2\lfloor n/2 \rfloor + 1$, then G has cycles containing S_k of each length from $k + p$ to n.*

Theorem 5. *For and $n > m$, let G be a graph of order n. If $\sigma_2(G) \geq 2n - m$ when m is odd and $\sigma_2(G) \geq 2n - m - 1$ when m is even, then G is a (k, m)-pancyclic graph.*

Proof. Let $S = S_k$ be a fixed set of k vertices of G. Given an integer t with $m \leq t \leq n$, select an induced subgraph T with t vertices and $S \subset T$. Since each pair of nonadjacent vertices of G has a most m nonadjacencies including the pair, $\sigma_2(T) \geq 2t - m$ if m is odd and $\sigma_2(T) \geq 2t - m - 1$ if m is even. Since $2t - m \geq t$ and $2t - m - 1 \geq t$ except when $t = m$, the graph T is hamiltonian by [O60]. Also, even in the remaining case when $|T| = m$ and $\sigma_2(T) \geq m - 1$ with m even, we have that T is hamiltonian by the argument of Ore in [8]. Thus, there is a cycle of each order $m \leq t \leq n$ containing the set S. This completes the proof of Theorem 5. □

The next result can be found in [9], but the proof is very short, so to make the paper more self contained, we include it.

Theorem 6. *If G is a graph of order $n \geq 3$, and $\sigma_2(G) \geq \lfloor (4n-1)/3 \rfloor$, then G is a $(1,3)$-pancyclic graph.*

Proof. Since $\lfloor (4n-1)/3 \rfloor \geq n + 1$, we have from Theorem 4 that G is $(1,4)$-pancyclic. Hence, we must show that each vertex of G is in a triangle. Since $\sigma_2(G) \geq \lfloor (4n-1)/3 \rfloor$, $\delta(G) \geq n/3 + 1$. If there is a vertex of G with no edge in its neighborhood, then there are two nonadjacent vertices each with degrees at most $n - (n/3 + 1)$, and so $\sigma_2(G) \leq 2n - 2(n/3 + 1) < \lfloor (4n-1)/3 \rfloor$. This gives a contradiction that completes the proof of Theorem 6. □

Before proving Theorem 3, we will state a corollary that follows with just a few additional observations that will be made in the proof of Theorem 3.

Corollary 2. *Let $(1 \leq k \leq m \leq n)$ be positive integers, and let G be a graph of order n. The graph G is (k,m)-pancyclic if $\delta(G)$ satisfies any of the following conditions:*

(i) $\delta(G) \geq n/2$ *when* $m = n$,
(ii) $\delta(G) \geq (n+1)/2$ *when* $k = 1$ and $m = 3$,
(iii) $\delta(G) \geq n - 1$ *when* $k = 2$ or 3 and $m = 3$,
(iv) $\delta(G) \geq n - 2$ *when* $k = 3$ and $m = 4$ or 5,
(v) $\delta(G) \geq n - (m/2)$ *when* $4 \leq k \leq m < 2k$, $n > m$,
(vi) $\delta(G) \geq \lfloor (n+2)/2 \rfloor$ *when* $k \geq 2$, $m \geq 2k$, and $n > m$.

Also, all of the conditions on $\delta(G)$ are sharp.

Proof. (of Theorem 3 and Corollary 2). The proof will be broken into 6 cases.

Case (i). Let $m = n$. Then all that is required is a hamiltonian cycle. By a classic result of Ore [8] (Dirac [3]), a graph G of order n is hamiltonian if $\sigma_2(G) \geq n$ ($\delta(G) \geq n/2$). Also, the unbalanced complete bipartite graph $G = K_{(n+1)/2,(n-1)/2}$ is not hamiltonian for odd n and $\sigma_2(G) = n - 1$ and $\delta(G) = (n-1)/2$. □

Case (ii). Observe that the complete bipartite graph $G = K_{n/2,n/2}$ is not $(1,3)$-pancyclic and $\delta(G) = n/2$, however if $\delta(G) \geq (n+1)/2$ in any graph G of order n, then each vertex of G is in a triangle and G is vertex pancyclic (see [1]). Consider the graph G of order n obtained from the graph $K_{n-1} - E(K_{\lfloor (n+1)/3 \rfloor})$ by adding a vertex v that is adjacent to the set of $\lfloor (n+1)/3 \rfloor$ independent vertices. Then,

$\sigma_2(G) = \lfloor(4n-2)/3\rfloor$, but the graph G is not $(1,3)$-pancyclic, since there is no triangle containing the vertex v. This, along with Theorem 6, completes the proof of Case (ii). □

Case (iii). The graph must be complete for each pair or triple of vertices of a graph G to be contained in a triangle. Thus, $\sigma_2(G) \geq 2n-3$ and $\delta(G) \geq n-1$ is required and nothing less is sufficient. □

Case (iv). The graph $G = K_n - P_3$ has $\sigma_2(G) = 2n-5$ and $\delta(G) = n-3$, and there is no cycle of length 4 containing the 3 vertices of the missing P_3. It is also easily seen that if $\sigma_2(G) \geq 2n-4$ or $\delta(G) \geq n-2$, then $\overline{G}$ is just a matching and so G is $(3,4)$-pancyclic. The graph $G = K_n - E(K_3)$ has $\sigma_2(G) = 2n-6$ and $\delta(G) = n-3$, and there is no cycle of length 5 containing the 3 vertices of the missing K_3. It is also easily seen that if $\sigma_2(G) \geq 2n-5$ (or if $\delta(G) \geq n-2$), then $\overline{G}$ is at most a disjoint union of P_3's, and so G is $(3,5)$-pancyclic. These observations verify Case (iv). □

Case (v). For integers $4 \leq k \leq m < 2k$ with $n > m$, consider the graph $G = K_n - E(K_{\lceil(m+1)/2\rceil})$. Let S be a set of k vertices that contains the independent set of $\lceil(m+1)/2\rceil$ vertices of G. Any cycle of G that contains the independent set of $\lceil(m+1)/2\rceil$ vertices would necessarily have at least $2\lceil(m+1)/2\rceil$ vertices. Thus, we have a graph G with $\sigma_2(G) = 2n - 2\lceil(m+1)/2\rceil$ and $\delta(G) = n - \lceil(m+1)/2\rceil$ such that G is not a (k,m)-pancyclic graph when m is odd and is not a (k,m)-pancyclic or a $(k,m+1)$-pancyclic graph when m is even. This, along with Theorem 5, completes the proof of Case (v). □

Case (vi). This most interesting case deals with $m \geq 2k \geq 4$. The complete bipartite graph $G = K_{n/2,n/2}$ for n even has no odd cycles, but $\sigma_2(G) = n$ ($\delta(G) = n/2$), and so the condition $\sigma_2(G) \geq 2\lfloor n/2\rfloor + 1$ or the condition $\delta(G) \geq \lfloor(n+2)/2\rfloor$ is needed to imply a graph of order $n > m$ is (k,m)-pancyclic for $m \geq 2k$. For odd n the complete bipartite graph $K_{(n-1)/2,(n+1)/2}$ plays the same role. Also, the condition $m \geq 2k$ is necessary as a lower bound on the cycle lengths, since if the set S_k is independent, then the smallest cycle containing S_k will contain at least $2k$ vertices. The graph $G = K_n - E(K_k)$ for $n > m \geq 2k$ has an independent set with k vertices and satisfies the corresponding sum of degree and minimum degree conditions for both n odd and even. This along with Theorem 4, completes the proof of Case (vi). □

This completes the proof of Theorem 3 and also Corollary 2. □

3. Pancyclic Ordered Graphs

For $k = 1, 2$ or 3 a graph that is (k,m)-pancyclic is also (k,m)-pancyclic ordered, since a proper orientation and starting point on a cycle will give any order of a set of 3 vertices. Thus, Theorem 3 from the previous section immediately implies the following.

Theorem 7. *Let $1 \leq k \leq 3 \leq m \leq n$ be positive integers, and let G be a graph of order n. The graph G is (k, m)-pancyclic ordered if $\sigma_2(G)$ satisfies any of the following conditions:*

(*i*) $\sigma_2(G) \geq \lfloor (4n+1)/3 \rfloor$ *when* $k = 1$ and $m = 3$,
(*ii*) $\sigma_2(G) \geq 2n - m$ *when* $k = 2$ and $m = 3$ or $k = 3$ and $m = 3, 4, 5$,
(*iii*) $\sigma_2(G) \geq 2\lfloor n/2 \rfloor + 1$ *when* $k = 1, 2, 3$ and $m \geq \max\{4, 2k\}$, and $n > m$,
(*iv*) $\sigma_2(G) \geq n$ *when* $m = n$.

Also, all of the conditions on $\sigma_2(G)$are sharp.

The $\sigma_2(G)$ conditions of Theorem 7 can be replaced by $\delta(G) = \sigma_2(G)/2$ yielding a sharp result. We are left with only the cases $k \geq 4$. We will first prove the following theorem, which deals with the cases when $m < 2k$.

Theorem 8. *Let $4 \leq k \leq m \leq n$ be positive integers, and let G be a graph of order n. The graph G is (k, m)-pancyclic ordered if $\sigma_2(G)$ satisfies any of the following conditions:*

(i) $\sigma_2(G) \geq 2n - 3$ *when* $k \leq m < \lfloor 3k/2 \rfloor$,
(ii) $\sigma_2(G) \geq 2n - 4$ *when* $\lfloor 3k/2 \rfloor \leq m < \lceil (5k-2)/3 \rceil$,
(iii) $\sigma_2(G) \geq 2n - 5$ *when* $\lceil (5k-2)/3 \rceil \leq m < 2k$,

Also, all of the conditions on $\sigma_2(G)$ are sharp.

Proof. Case (i). The graph $G = K_n - E(\lfloor n/2 \rfloor K_2)$ has $\sigma_2(G) = 2n - 4$. If a set S of k vertices are ordered such that $\lfloor k/2 \rfloor$ of the consecutive pairs in the set are not adjacent, then the smallest cycle containing S in the appropriate order is $\lfloor 3k/2 \rfloor$. Thus G is not (k, m)-pancyclic ordered if $m < \lfloor 3k/2 \rfloor$. On the other hand, if $\sigma_2(G) \geq 2n - 3$ for a graph G of order n, then $G = K_n$ and so is (k, m)-pancyclic ordered for all $m \geq k$. This completes the proof of Case (i).

Case (ii). The graph $G = K_n - E(\lfloor n/3 \rfloor P_3)$ has $\sigma_2(G) = 2n - 5$. For any set S of k vertices that are ordered such that there are a maximum number of consecutive pairs in the order that are not adjacent ($\lfloor k/3 \rfloor$ missing P_3's and possibly an additional missing edge), then the smallest cycle containing S in the appropriate order is $\lceil (5k-2)/3 \rceil$. Thus G is not (k, m)-pancyclic ordered if $m < \lceil (5k-2)/3 \rceil$. On the other hand, if $\sigma_2(G) \geq 2n - 4$ for a graph G of order n, then $(K_n - \lfloor n/2 \rfloor K_2) \subseteq G$ and so G is easily seen to be (k, m)-pancycylic ordered for all $m \geq \lfloor 3k/2 \rfloor$, completing the proof of Case (ii).

Case (iii). The graph $G = K_n - E(C_k)$ has $\sigma_2(G) = 2n - 6$. For the set S of k vertices that contains the cycle C_k in the natural order, the smallest cycle containing the set S in the appropriate order is $2k$. Thus G is not (k, m)-pancyclic ordered if $m < 2k$. On the other hand, if $\sigma_2(G) \geq 2n - 5$ for a graph G of order n, then $\overline{G}$ is a disjoint union of paths of length at most 2. Thus, clearly G is (k, m)-pancyclic ordered for all $m \geq \lceil (5k-2)/3 \rceil$, which completes the proof of Case (iii) and Theorem 8. □

By considering set $S = \{x_1, x_2, \cdots, x_k\}$, the graph F_3 in Fig. 3 and the graph F_2 in Fig. 2 provide examples which are not (k, m)-pancyclic ordered, giving the lower bounds for the required minimum degree sum. This verifies the following proposition.

Proposition 1. *Let $k \geq 4$ be a positive integer and G is a graph of order n.*

If $2k \leq m \leq (5k-3)/2$, the graph G is not (k, m)-pancyclic ordered unless $\sigma_2(G) \geq n + 4k - m - 6$.

If $m > (5k-3)/2$, the graph G is not (k, m)-pancyclic ordered unless $\sigma_2(G) \geq n + (3k-9)/2$.

Proof. In the case $2k \leq m \leq (5k-3)/2$, $\sigma_2(F_3) = n + 4k - m - 7$, and there is no cycle of length m containing S in the appropriate order. □

In the remainder of the paper we provide the following result that shows that these bounds are also sufficient when $m \geq 2k$.

Theorem 9. *If $k \geq 4$ and G is a graph of order n with $\sigma_2(G) \geq n + 4k - m - 6$, then G is (k, m)-pancyclic ordered for $2k \leq m \leq (5k-3)/2$.*

Proof. Suppose that G satisfies the above conditions, and to the contrary $S = \{v_1, v_2, \cdots, v_k\}$ is an ordered set of k vertices of G that implies that G is not (k, m)-pancyclic ordered.

Claim 1. There is a cycle D with at most m vertices that contains S and encounters the vertices of S in the required order.

Note that if u and v are nonadjacent vertices, then they have $n + 4k - m - 6 - (n-2) = 4k - m - 4$ common adjacencies. If, in addition, u and v are in S, then they have $3k - m - 2$ common adjacencies in $G - S$. Let k_1 be the

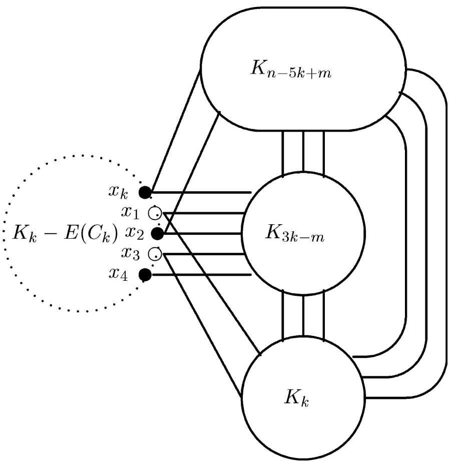

Fig. 3. F_3

number of consecutive pairs with $v_iv_{i+1} \in E(G)$ taking the indices modulo k. All of these k_1 edges will be placed in the cycle D. First consider the case when $m = 2k$. If $k_1 \geq 2$, then a cycle of length $2k - k_1$ can be constructed by choosing a path of length 2 between v_j and v_{j+1} using a vertex u in $G - S$ when $v_jv_{j+1} \notin E(G)$. If $k_1 = 0$ or 1, then every pair of consecutive vertices v_i and v_{i+1} with $v_iv_{i+1} \notin E(G)$ has at least $k - k_1$ common adjacencies in $G - S$, by noting the possible common adjacencies of v_i and v_{i+1} in S. Thus, by placing paths of length 2 between non-adjacent vertices, there is a cycle D with $2k - k_1$ vertices that contains S and encounters the vertices in the required order, completing the proof of Claim 1 when $m = 2k$.

A similar approach will be taken when $m > 2k$, except that some of the paths between nonadjacent consecutive vertices of S will possibly have length 3. We will build the cycle as follows: after including the k_1 edges between consecutive pairs of vertices of S, choose a maximum number k_2, of 2-paths joining nonadjacent consecutive pairs of S. Join any remaining pairs with 3-paths, if possible, until all consecutive pairs are joined, or no others can be joined by 1-, 2- or 3-paths. Let k_3 be the number of paths of length 3 between consecutive vertices of S. If all pairs are joined then the cycle D results. Thus, $k_1 + k_2 + k_3 \leq k - 1$, and the length of the path system is $k_1 + 2k_2 + 3k_3$. Note that $k_2 \geq 3k - m - 2$ in general, and if $k_1 \leq 1$, then $k_2 \geq 3k - m - k_1$.

Let S_2 be the k_2 central vertices in the paths of length 2 and S_3 be the $2k_3 \leq 2(k - 1 - k_2 - k_1)$ central vertices of the paths of length 3. Assume that S contains a consecutive pair v_i and v_{i+1} not joined by a 1-, 2- or 3-path. Also, let S_{v_i} and $S_{v_{i+1}}$ be the neighborhoods of v_i and v_{i+1} respectively in $G - (S \cup S_2 \cup S_3)$. We can assume that $S_{v_i} \cap S_{v_{i+1}} = \emptyset$ and there are no edges between S_{v_i} and $S_{v_{i+1}}$, since this contradicts the way the path system was constructed. We can also assume that v_i and v_{i+1} have at most $k - 2 - \epsilon$ adjacencies in S, where $\epsilon = 1$ if $k_1 \leq 1$ and 0 otherwise. By the maximaility of k_2 either v_i or v_{i+1}, say v_i, can be adjacent to at most k_3 vertices of S_3. Suppose $S_{v_{i+1}}$ is not empty and let $w \in S_{v_{i+1}}$. Since v_i and w have no common adjacency in $G - (S \cup S_2 \cup S_3)$, we have the following inequality:

$$n + 4k - m - 6 \leq d(v_i) + d(w) \leq k - 2 - \epsilon + k_1 + (k - k_1)/2 + 2k_2 + 3k_3$$
$$+(n - (k + k_2 + 2k_3) - 1).$$

This implies that $m \geq 5k/2 - 2 + \epsilon + k_1/2 \geq 5k/2 - 1$, which contradicts the fact that $m \leq (5k - 3)/2$, which would complete the proof of Claim 1.

Hence we can assume that $S_{v_{i+1}}$ is empty and that S_{v_i} is not empty with $w \in S_{v_i}$. If v_{i+1} has k_3 or fewer adjacencies in S_3, then the above count would complete the argument. Consequently, it must be the case that v_{i+1} has more than k_3 adjacencies in S_3. But this implies that for some j, v_{i+1} is adjacent to both interior vertices, say x and y of the 3-path joining v_j and v_{j+1}. If w is adjacent to both x and y then each of the 3-paths (v_i, w, x, v_{i+1}) and (v_i, w, y, v_{i+1}) would yield a new path system of the same length, and the appropriate choice leaves both v_j and v_{j+1} having neighbors in $G - S - S_2 - (S_3 - \{x, y\} \cup \{w\})$. By using this alternate set of 1-, 2- and 3-paths and applying the previous argument, a contradiction occurs. Hence, whenever v_{i+1} is adjacent to both interior vertices of a 3-path in S_3, it must be the case that w can be adjacent to at most one of those vertices. Subsequently, Since v_{i+1} and w have no common adjacency in $G-(S\cup S_2\cup S_3)$ we get the following inequality:

$$n+4k-m-6\leq d(v_{i+1})+d(w)\leq k-2-\epsilon+k_1+(k-k_1)/2+2k_2+3k_3$$
$$+(n-(k+k_2+2k_3)-1).$$

This result is a contradiction, and with all cases exhausted the proof of Claim 1 follows. □

Select a cycle C of maximum length $p\leq m$ that encounters the vertices of S in the required order. Let $H=G-C$.

Claim 2. $p>k$.

This is clearly true except when $k_1=k$. In this case, by assuming G is edge maximal, the set S induces K_k. Since $\delta(G)\geq 4k-m-4\geq k$, each vertex of S has an adjacency in $G-S$. Assume that $v_{iu}i\in E(G)$ for $i=1,2$ with $u_1,u_2\in G-S$. Either $u_1=u_2$, $u_1u_2\in E(G)$, or $u_1u_2\notin E(G)$. In the later case, since u_1 and u_2 have at least $4k-m-4$ common adjacencies, either u_1 or u_2 can be inserted into C or they have at most $k/2$ common adjacencies on C. Thus, they have a common adjacency in $G-S$. Each of these cases now yields a cycle C of length either $k+1$, $k+2$ or $k+3<2k\leq m$, which verifies Claim 2. □

Claim 3. $p=m$.

Assume that $p<m$. Note that each vertex of C has an adjacency in H. If this is not so, then select a vertex $x\in C$ with no adjacency in H, and let y be a vertex in H. The vertex y cannot be adjacent to two consecutive vertices of C, and so we have

$$n+4k-m-6\leq d(x)+d(y)\leq (p-1)+(n-p-1)+p/2.$$

This implies that $8k-2m-8\leq p\leq m-1$. Since $m\leq (5k-3)/2$, the displayed inequality implies that $(8k-7)/3\leq m\leq (5k-3)/2$. This yields a contradiction, unless $k=5, m=11$, and $p=10$. In this special case the graph H is complete, each vertex $y\in H$ is adjacent to every other vertex of C and not adjacent to x, and x is adjacent to all of the other vertices of C. Also, $|H|\geq 2$, since G is k-ordered hamiltonian. This implies the vertices of S alternate on C and are precisely the vertices of C not adjacent to the vertices of H, since a C_{11} could be constructed using two vertices of H and avoiding a vertex of C. However, a C_{11} can be formed using the edges x^-y, yx^+ and the edge xx^{++}, a contradiction in this special case. Hence, we may assume that each vertex of C has an adjacency in H.

Select a vertex $x\in C$ such that $x^+\notin S$. For convenience let $y=x^+$, $z=x^{++}$, and $w=x^-$. If possible, choose x not in S. Let $u\in H$ such that $xu\in E(G)$. We will show that $zu\in E(G)$. If not, then observe that z and u have no common adjacency in H, since this would imply the existence of a cycle of length $p+1$ with the required properties. Hence, we have the following inequality:

$$n + 4k - m - 6 \le d(z) + d(u) \le (p-1) + (n-p-1) + (p-1)/2.$$

This implies that $8k - 2m - 7 \le p \le m - 1$. Hence, $(8k-6)/3 \le m \le (5k-3)/2$, and consequently $k \le 3$, a contradiction. Therefore $N_H(x) = N_H(z)$, and in particular $zu \in E(G)$.

Now suppose $xz \notin E(G)$. Let $N = N_H(x)$ and $N^+ = N \cup \{y\}$. Then

$$|N_H(x)| \ge (n + 4k - m - 6 - 2(p-2))/2 = (n + 4k - m - 2p - 2)/2 \ge 1.$$

If two vertices of N^+ are adjacent, then a cycle of length $p+1$ results, so we can assume that N^+ is an independent set. For vertices u_1 and u_2 in N^+ we have the following inequality:

$$n + 4k - m - 6 \le d(u_1) + d(u_2) \le p + 2(n - p - ((n + 4k - m - 2p - 2)/2)).$$

This implies $8k \le p + 2m + 8$, and so $k \le 5$. Moreover, if $k = 4$, then $m = 8$ and $p \le 7$, which gives a contradiction. When, $k = 5$, it follows that $m = 11$, $p = 10$, there are $(n-13)/2$ vertices in N, and each vertex in N^+ is adjacent to all vertices of the $(n-7)/2$ vertices of $H - N$ as well as to every other vertex of C. Also, both x and z are adjacent to all of the remaining $p - 2 = 8$ vertices of C. Note that if $v \in H - N$, then $d(v) \ge n + 3 - d(x) \ge 8 + (n-13)/2 = (n+3)/2$. Thus, $d_{H-N}(v) \ge (n+3)/2 - 5 - (n-13)/2 = 3$. Hence, there is a path (w, v_1, v_2, u, z) with $v_1, v_2 \in H - N$ that can be used to form a C_{11} from C, if $x \notin S$. Hence we can assume that x, and by symmetry z, are both in S. Also, there is a path (x, u_1, v, u_2, z) with $u_1, u_2 \in N^+$ and $v \in H - N$. Hence if $z^+ \notin S$, then there is the required C_{11}, since $zz^{++} \in E(G)$. This implies that $z^+ \in S$, and by symmetry $w^- \in S$. Since there are three consecutive vertices of S in C_{10}, it follows that our choice for x could have been made with both x and x^+ not in S, which would imply the existence of a C_{11} as required. It follows that $xa \in E(G)$.

Note that $wy \notin E(G)$, since this would imply the existence of a cycle of length $p+1$ with the required properties. The vertex $u \in N$ is not adjacent to w, since this would give a cycle of length $p+1$ with the required properties. If w and u have a common adjacency in H, then there is also a cycle of length $p+1$ using the edge xz. However, no common adjacency in H of w and u implies the following inequality:

$$n + 4k - m - 6 \le d(w) + d(u) \le n - p - 1 + (p-2) + p/2.$$

This implies that $8k - 2m - 6 \le p \le m - 1$. This gives the inequality $(8k-5)/3 \le m \le (5k-3)/2$, which implies $k \le 1$, a contradiction. Thus we can conclude there is a cycle of length m that encounters S in the correct order, verifying Claim 3. □

Assume there exist cycles of every length from m to p containing S in the correct order, but there is no cycle of length $p+1$ with this property. Let $C = C_p$ be a required cycle of length p, and let $H = G - C$. The k vertices of S divide the vertices of C into k disjoint intervals except for endvertices, each starting and ending with a vertex of S.

Claim 4. Some vertex in H has at least two adjacencies in some interval, or every vertex of C has an adjacency in H.

Suppose there is a vertex x in C having no adjacencies in H, and every vertex $y \in H$ has at most one adjacency in every interval of C. It follows that y has at most k adjacencies in C. This gives the following inequality:

$$n + 4k - m - 6 \leq d(x) + d(y) \leq (n - p - 1) + k + (p - 1).$$

This implies $m \geq 3k - 4$, and hence $3k - 4 \leq (5k - 3)/2$, and so $k \leq 5$. Thus, if $k > 5$ the claim follows. Thus, we have a contradiction unless $k = 4$ or 5. Observe also that this implies that any vertex in C adjacent to no vertex of H must be adjacent to all of C.

In the case when $k = 4$ (5), the vertex y has precisely four (five) adjacencies in C and none of these can be in S, since vertices of S are each in two intervals. Thus, the vertices of S have no adjacencies in H, so they are adjacent to all other vertices of C. Also, the vertices of H form a complete graph. Further, each of the intervals in C has at least three vertices. Let u_1 and u_2 be vertices of C in consecutive intervals with adjacencies w_1 and w_2 in H (possibly $w_1 = w_2$) such that no vertex between them has any adjacencies in H. Note that all vertices between u_1 and u_2 are adjacent to all other vertices of C. Let P be the path from u_2 to u_1^- on C and Q the reverse order path from u_2^- to u_1. Then, there is a cycle of length $p + 1$ or $p + 2$ using (u_1, w_1, w_2, u_2), P, $u_1^- u_2^-$ and Q that encounters S in the correct order. There is a chord of length two from each vertex of S in C, so the cycle of length $p + 2$ can be shortened if necessary. This gives a contradiction, which completes the proof of Claim 4. □

Claim 5. If each vertex of C has an adjacency in H, then there is a vertex of H with $k + 1$ adjacencies in C and hence two adjacencies in some interval of C.

First consider the case when there is some interval in C with at least 4 vertices. Let x_1, x_2, x_3, x_4 be four consecutive vertices in that interval. Let y_i be an adjacency of x_i in H for $1 \leq i \leq 4$. If $y_i = y_j$, $i \neq j$, then we are done, so the y_i's are distinct. If y_1 and y_4 have a common adjacency in H, then there is a cycle of length $p + 1$ with the required property, a contradiction. If $y_1 y_4 \notin E(G)$, then the following inequality follows:

$$n + 4k - m - 6 \leq d(y_1) + d(y_4) \leq (n - p - 2) + p/2 + p/2 < n,$$

a contraction. Hence $y_1 y_4 \in E(G)$. If x_2 or x_3 is insertible in the cycle obtained from C by replacing x_2 and x_3 with y_1 and y_4, then there is a cycle of length $p + 1$, contradicting our choice of C. A cycle of length $p + 1$ also results if y_1 and x_3 have a common adjacency in H. Since, neither of these occur, we have the following inequality:

$$n + 4k - m - 6 \leq d(y_1) + d(x_3) \leq (n - p - 1) + p/2 + p/2 < n,$$

which gives a contradiction unless $y_1 x_3 \in E(G)$. However, $y_1 x_3 \in E(G)$ implies there is a vertex in H with two adjacencies in an interval of C.

We may now assume there is no interval of C with four vertices. It follows that $p = m = 2k$, the vertices of S alternate, and each interval has precisely three vertices. Let x_1, x_2, x_3 be the three consecutive vertices in some interval. Let y_1 be an adjacency of x_1 in H. If $y_1x_3 \in E(G)$ or $y_1x_2 \in E(G)$, then we have proved the claim, so neither is an edge in G. Also, if y_1 and x_3 have a common adjacency in H, then there is a cycle of length $p+1$, a contradiction. Therefore, we have the following inequality:

$$n + 2k - 6 \leq d(y_1) + d(x_3) \leq (n - 2k - 1) + (2k - 1) + (k - 1).$$

This implies $k \leq 3$, a contradiction. Therefore, we can conclude there is a vertex in H with at least $k+1$ adjacencies in C and hence at least two adjacencies in some interval of C, completing the proof of Claim 5. □

Select two vertices x and y in one of the intervals of C which have a common adjacency, say $z \in H$, that are at a minimum distance along C. Let A be the vertices of C strictly between x and y in this interval. Thus, none of the vertices A are in S.

Claim 6. Some vertex in A has an adjacency in H.

Suppose not and consider the cycle obtained from C by replacing the path with vertices in A by the path (x, z, y). If all of the vertices of A can be inserted into this cycle, then the required cycle of length $p+1$ exists, which gives a contradiction. If not, then insert as many vertices as possible, and assume we are left with a set $\emptyset \neq B \subseteq A$ of vertices that cannot be inserted. Select a vertex $w \in B$. If $b = |B|$ and w has no adjacency in H, then we have the following inequality:

$$\begin{aligned} n + 4k - m - 6 &\leq d(w) + d(z) \leq ((b-1) + (p-b+1)/2) + (((n-p-1) \\ &\quad + (p-b+1)/2) < n, \end{aligned}$$

a contradiction, completing the proof of Claim 6.

Claim 7. $|A| = 1$

Suppose $|A| \geq 2$. If all of the vertices in A are insertible in the path $C - A$, then the required cycle of length $p+1$ is obtained. Assume not, and let $(x_1, x_2, \cdots, x_s)$ be the path of C using the vertices in A. Let x_t be the first vertex of A starting from x_1 that is not insertible. Observe that x_t and z must have a common adjacency in H, since if this is not true then we get the following inequality:

$$n + 4k - m - 6 \leq d(x_t) + d(z) \leq (n - p - 1) + (a - 1) + (p - a + 1) < n,$$

a contradiction. Let z_t be such a common adjacency. If $t > 1$, then the required cycle of length $p+1$ is obtained by using the path (y, z, z_t, x_t) to replace A and inserting all of the remaining vertices of A except for x_1. Hence, we must have that x_1 is not insertible, and so $t = 1$. Likewise, x_s is not insertible, and there is a vertex $z_s \in H$ that is a common adjacency of x_s and z. If $s = 2$, then the required cycle of length $p+1$ can be obtained by using the path (x, z, z_2, x_2, y) and avoiding the vertex x_1. The required cycle can also be obtained if all of the vertices of A strictly between x_1 and x_s can be inserted. Thus, we can assume that $s > 2$, and let x_r be the first vertex past x_1 that is not insertible. Associated with x_r is the vertex $z_r \in H$ that is commonly adjacent to z and x_r. Again, the required cycle is obtained by using the path $(x, z, z_r, x_r, \cdots)$, inserting the vertices strictly between x_1 and x_r and avoiding x_1. Therefore, we can conclude that $|A| = 1$, completing the proof of Claim 7. □

Claim 8. No vertex of H can have three adjacencies in one interval.

Assume there is a vertex $z \in H$ with adjacencies x_1, x_2, x_3. By Claim 7 we know that there is precisely one vertex on C between x_1 and x_2 and between x_2 and x_3. Denote these vertices by y_1 and y_2. Neither y_1 nor y_2 is insertible, since this would give the desired cycle of length $p+1$. Also, $y_1y_2 \notin E(G)$ for the same reason. Therefore, y_1 and y_2 have a common adjacency in H, which we will denote by z', since if this did not occur the following inequality results:

$$n + 4k - m - 6 \leq d(y_1) + d(y_2) \leq (n - p - 1) + p/2 + p/2 \leq n - 1,$$

a contradiction. This implies that x_2 is not insertible for the same reason as y_1 and y_2. Observe that x_2 and z cannot have a common adjacency in H, since this gives a cycle of length $p+1$ avoiding y_1 and using z and the common adjacency. The same argument implies that y_2 and z' do not have a common adjacency in H. This implies the following inequality involving x_2, y_2, z, z':

$$2(n+4k-m-6) \leq d(x_2) + d(z') + d(y_2) + d(z) \leq 2(n-p-1) + 4(p/2) \leq 2(n-1),$$

a contradiction. Therefore, no vertex of H can have three adjacencies in an interval of C, completing the proof of Claim 8. □

The previous observations have placed many restrictions on the graph G. No vertex of H can have three adjacencies in any interval of C, and when there are two adjacencies in some interval they are at a distance two on C. In fact, if a vertex of H is adjacent to t vertices of S, then the vertex can have at most $2k - t$ adjacencies in C, since the vertices of S are in two intervals. We also know that at least one vertex of C has no adjacencies in H. If two vertices are in the same interval and are at a distance three apart on the cycle C, then both cannot have adjacencies in H, for otherwise a count similar to that at the end of Claim 8 would produce a contradiction. Consequently, C has at most $2k < m \leq p$ vertices, a contradication. Thus, by Claim 4, some vertex of H has two adjacencies in some interval. Also, each pair $u, v \in H$ of nonadjacent vertices have

$$d_H(u) + d_H(v) \geq n + 4k - m - 6 - p = |H| + 4k - m - 6 \geq |H| + 3/2,$$

which implies they have at least two common adjacencies in H.

Claim 9. If $y_1, y_2 \in H$ each have two adjacencies in the same interval of C, then they have the same two adjacencies.

Assume that $(x_1, x_2, \cdots, x_t)$ are the vertices of an interval, and that $y_1x_i, y_1x_{i+2}, y_2x_j, y_2x_{j+2} \in E(G)$ with $i < j$. Previous observations imply that x_{i+1}, x_{j+1} have adjacencies in H. Hence, to avoid having two vertices in the interval with adjacencies in H that are at a distance on C, we must have $j \geq i+6$. Either $y_1 y_2 \in E(G)$ or there is a $y \in H$ such that (y_1, y, y_2) is a path in H. Let $A = \{x_{i+3}, x_{i+4}, \cdots, x_{j-1}\}$, which is a set with at least three vertices, and let P be the path containing the remaining vertices of C. Starting with x_{i+3} and using the order on A, insert one at a time the vertices of A into P or the path obtained from P from inserting vertices of A. If all of the vertices of A can be inserted, then a C_{p+1} cycle can be constructed using the path from x_{i+2} to x_j containing y_1 and y_2, and replacing P with a path with the appropriate number of vertices of A inserted. If all of the vertices of A cannot be inserted, then let x_q be the first vertex that cannot be inserted. Let $B = \{x_q, x_{q+1}, \cdots, x_{j-1}\}$ with $b = |B|$. There must be some common adjacency, say $z \in H$, of x_q and y_1, for otherwise the following inequality results:

$$n + 4k - m - 6 \leq d(y_1) + d(x_q) \leq n - p - 1 + (b-1) + 2((p-b+1)/2) = n - 1,$$

a contradiction. A C_{p+1} can be constructed using the path (x_{i+2}, y_1, z, x_q) and by inserting all but one of the vertices of $A - B$. This gives a contradiction that completes the proof of Claim 9. □

Let u be a vertex of C that does not have an adjacency in H, and assume that $d_C(u) = p - r$. Thus, u is not adjacent to r vertices of C including itself. Then, each vertex of H has at least $n + 4k - m - 6 - (p - r) - (n - p - 1) = 4k - m - 5 + r$ adjacencies in C. Therefore, if $r \geq 4$, then $4k - m - 5 + r \geq 4k - (5k-3)/2 - 1 = (3k+1)/2 > 3k/2$. This implies that each vertex of H will have a pair of adjacencies in more than half of the k intervals of C, and so each pair of vertices $h_1, h_2 \in H$ will have two adjacencies in some common interval. By Claim 9 the vertices h_1 and h_2 will have the same two adjacencies in this interval and they will be at a distance 2 on C. Hence, $h_1 h_1 \notin E(G)$, since otherwise there would be the required C_{p+1}. Thus, we can assume that H has no edges. This contradicts the fact that each vertex in H has at least 2 adjacencies in H. Hence we can assume that $r \leq 3$.

Claim 10. $r \leq 2$.

Suppose not, and let u be a vertex of C such that $d(u) = d_C(u) = p - 3$. If each vertex of H is nonadjacent to some other vertex of H, then each vertex of H will have $4k - m - 1 \geq (3k+1)/2$ adjacencies in C. This implies that each vertex of H will have a pair of adjacencies in more that half of the k intervals of C, and so H will have no edges. This contradicts the fact that each vertex in H has at least two adjacencies in H. Hence we can assume that some vertex of H, say x, is adjacent to all of the other vertices of H. Note that each vertex of H has at least $(3k-1)/2$ adjacencies in C, and thus has pairs of adjacencies in at least $(k-1)/2$ of the k intervals of C. This implies that there is no triangle in H, since at least some pair of vertices in the triangle will have a common pair of adjacencies in some interval of C. Thus, there are no edges in the neighborhood of x in H, so H is just a star centered at x. This gives a contradiction if $|H| \geq 3$, since some pair of nonadjacent vertices of H will not have two common adjacencies in H. This leaves only the case when H is just an edge, say xy. Let v be a vertex of C between two adjacencies of x in some interval of C. Thus, v is not adjacent to x and also it cannot be inserted into C. Therefore, we have the following inequality:

$$n+2 \leq n+4k-m-6 \leq d(x)+d(v) \leq p/2+p/2+2 \leq n,$$

a contradiction that completes the proof of Claim 10.

Claim 11. $r \leq 1$.

Suppose not, say $d(u)=d_C(u)=p-2$. Thus, each vertex of H has degree at least $4k-m-3 \geq \lceil (3k-3)/2 \rceil$ relative to C and is adjacent to two vertices in each of at least $3k-m-3 \geq \lceil (k-3)/2 \rceil$ intervals of C. Also, note that each vertex $v \in H$ has at least $n+4k-m-6-(p+(n-p-1)) = 4k-m-5 \geq \lceil (3k-7)/2 \rceil$ adjacencies in H in common with any vertex of C that is between two adjacencies of v in some interval of C. Thus, $|H| \geq 4k-m-4 \geq \lceil (3k-5)/2 \rceil$. Thus, on the average each interval has at least $(4k-m-4)(3k-m-3)/k \geq (\lceil (3k-3)/2 \rceil \lceil (k-3)/2 \rceil)/k$ vertices of H with a pair of adjacencies in that interval. Observe that if a vertex v of H is nonadjacent to t other vertices of H, then v has degree at least $4k-m-3+t \geq \lceil (3k-3)/2 \rceil + t$ relative to C and is adjacent to two vertices in each of at least $3k-m-3+t \geq \lceil (k-3)/2 \rceil + t$ intervals of C.

We will first consider the cases when $k \geq 6$. Some interval of C will have at least $(\lceil (3k-3)/2 \rceil \lceil (k-3)/2 \rceil)/k \geq \lceil (14)/6 \rceil = 3$ vertices of H with the same pair of adjacencies in the interval. These three vertices, which we will denote by T, are all nonadjacent. Thus, there is a vertex v of H with $d_H(v) \leq |H|-3$. If there is a vertex of $v \in H$ with $d_H(v) \leq |H|-5$, then v has a pair of adjacencies in at least $\lceil (k+5)/2 \rceil$ intervals of C. Since each of the vertices of H has a pair of adjacencies in at least $\lceil (k-3)/2 \rceil$ of the intervals of C, each will share a pair of vertices in some interval with v. This implies that v has no adjacencies in H, a contradiction. If $d_H(v)=|H|-4$, then select a set R of three vertices of H that are adjacent to v, and this can be done since $|H| \geq \lceil (3k-5)/2 \rceil \geq 7$. If all of the vertices of R are adjacent, then each of the intervals of C that contains their pairs of adjacencies are disjoint and are disjoint from those associated with v. This implies that $\lceil (k+3)/2 \rceil + 3\lceil (k-3)/2 \rceil \leq k$, a contradiction. Hence, some pair of vertices of R are not adjacent, and so some vertex of $u \in R$ has a pair of adjacencies in at least $\lceil (k-1)/2 \rceil$ of the intervals of C. This implies that u is not adjacent to v, a contradiction that implies that there is no vertex v with $d_H(v) \leq |H|-4$. It has already been shown that there is a triple T of nonadjacent vertices that will each have degree $|H|-3$ in H, and so all of these vertices are adjacent to the all of the vertices in $H-T$. This implies the intervals of C that contain the pairs of adjacencies of the vertices of $H-T$ are disjoint from the at least $\lceil (k+1)/2 \rceil$ intervals associated with the vertices of T. However this is not possible, since $2\lceil (k-3)/2 \rceil + \lceil (k+1)/2 \rceil > k$. Thus, no pair of vertices of the at least four vertices in $H-T$ are adjacent. This implies there is a vertex $w \in H$ with $d_H(w) \leq |H|-5$, a contradiction. Thus, we can assume that $k=4$ or 5.

When $k=5$, then $m=10$ or 11. If $m=10$, then $|H| \geq 6$, each vertex of H has at least five adjacencies in H, and at least seven adjacencies in C with pairs of adjacencies in at least two intervals of C. There are no triangles in H, since it is not possible to have three disjoint sets of intervals, each with at least two intervals. Thus, for some vertex $v \in H$, we must have $d_H(v) \leq |H|/2$, and so v will be nonadjacent to at least two additional vertices of H. Therefore, v will have pairs of adjacencies in at least four of the five intervals of C, and hence cannot be adjacent to any of the other vertices of H. This is a contradiction, so we can assume that $m = 11$.

When $k = 5$ and $m = 11$, then $|H| \geq 5$, each vertex of H has at least four adjacencies in H and at least six adjacencies in C with a pair of adjacencies in at least one interval of C. If no vertex of H has pairs of adjacencies in two different intervals, then $|H| = 5$, each vertex of H has a pairs of adjacencies in a unique interval, no vertex of S is adjacent to a vertex of H, and each interval has at least five vertices. Hence, using any interval of the cycle C, there is an extension to a C_{p+2} by simultaneously inserting two vertices of H, one of which has a pair of adjacencies in the interval. Thus, for any $z \in S$, neither zz^{++} nor zz^{--} can be in $E(G)$, since this would imply the existence of a C_{p+1}. Therefore, $d_C(z) \leq p - 3$, a contradiction.

Let $v \in H$ be a vertex with a maximum number, say t, of pairs of adjacencies in intervals of C. We know that $t \geq 2$. Let R be the set of vertices of H that are adjacent to v, and so we have that $|R| \geq 4$, and no vertex of R has a common pair of adjacencies with v in some interval of C. If $t = 5$, then each vertex of R would have a common pair of adjacencies with v in some interval of C, a contradiction. If $t = 4$, then all of the vertices of R would have their pair of adjacencies in the same interval, and so no pair of vertices of R would be adjacent. This implies each vertex of R has degree at most $|H| - 5$, which implies $t \geq 5$, a contradiction. If $t = 3$, then the pairs of adjacencies associated with the vertices of R must be in the two intervals disjoint from the intervals associated with v. Hence, two vertices, say u_1 and u_2 of R will have a common pair of adjacencies in one of these two intervals, $u_1u_2 \notin E(G)$, and so each of u_1 and u_2 will have pairs of adjacencies in both of the intervals disjoint from the intervals associated with v. This implies that neither u_1 not u_2 is adjacent to any vertex of R, a contradiction as before. If $t = 2$, then the argument used in the $t = 3$ case will imply there is a vertex of R that is nonadjacent to at least two other vertices of R, and this implies $t \geq 3$. With this contradiction we can assume that $k = 4$.

The argument for $k = 4$ and $m = 8$ is identical to the argument for $k = 5$ and $m = 11$, except that the parameters are different. In this case $|H| \geq 4$, each vertex of H has at least three adjacencies in H and at least five adjacencies in C with a pair of adjacencies in at least one interval of C. This observaton completes the proof of Claim 11. $\square$

Finally we consider the case when $r = 1$, that is $d(u) = d_C(u) = p - 1$, and so each vertex with no adjacency in H is adjacent to all of the other vertices of C. Each vertex of H has two adjacencies in some interval of C. Let $y_1 \in H$, and let (x_1, x_2, x_3) be a path in some interval of C with $y_1x_1, y_1x_3 \in E(G)$. Previous observations imply there is a $y_2 \in H$ that is adjacent to both x_2 and y_1. If x_1^- is in the same interval of C as x_1, then x_1^- has no adjacencies in H and cannot be adjacent to x_2. This contradicts the fact that $d(u) = d_C(u) = p - 1$. Hence, x_1 and likewise x_3 are in S, and if y_1 has two adjacencies in any interval, both are in S. This implies that y_1 has just k adjacencies in C. This gives a contradiction except when $k = 4$, in which case y_1 is adjacent to precisely the vertices of S. Thus, $k = 4$, all vertices of H are adjacent to precisely the vertices of S, and there are no edges in H. This gives a contradiction that completes the proof of Theorem 9. $\square$

For $m = (5k - 3)/2$, the $\sigma_2(G)$ condition of Theorem 9 is $\sigma_2(G) \geq n = (3k - 9)/2$, and this condition does not involve the variable m. Hence, a direct corollary of Theorem 9 and the example implying the sharpness of Theorem 2 is the following:

Corollary 3. *Let $k \geq 4$ and G be a graph of order n with $\sigma_2(G) \geq n + (3k - 9)/2$. Then, G is (k, m)-pancyclic ordered if $(5k - 3)/2 < m \leq n$. Also, the bound on σ_2 is sharp.*

The results of this section are summarized in the following result.

Theorem 10. *Let $4 \leq k \leq m \leq n$ be positive integers, and let G be a graph of order n. Then, the graph G is (k, m)-pancyclic ordered if $\sigma_2(G)$ satisfies any of the following conditions:*

(i) $\sigma_2(G) \geq 2n - 3$ *when* $k \leq m < \lfloor 3k/2 \rfloor$,
(ii) $\sigma_2(G) \geq 2n - 4$ *when* $\lfloor 3k/2 \rfloor \leq m < \lceil (5k - 2)/3 \rceil$,
(iii) $\sigma_2(G) \geq 2n - 5$ *when* $\lceil (5k - 2)/3 \rceil \leq m < 2k$,
(iv) $\sigma_2(G) \geq n + 4k - m - 6$ *when* $2k \leq m \leq (5k - 3)/2$,
(v) $\sigma_2(G) \geq n + (3k - 9)/2$ *when* $m > (5k - 3)/2$.

Also, all of the conditions on $\sigma_2(G)$ are sharp.

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