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Note

Toughness, degrees and 2-factors

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Abstract

In this paper we generalize a Theorem of Jung which shows that 1-tough graphs with $\delta(G) \geqslant \frac{|V(G)|-4}{2}$ are hamiltonian. Our generalization shows that these graphs contain a wide variety of 2-factors. In fact, these graphs contain not only 2-factors having just one cycle (the hamiltonian case) but 2-factors with k cycles, for any k such that $1 \leqslant k \leqslant \frac{n-16}{4}$.

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1. Introduction

The study of 2-factors, 2-regular spanning subgraphs, or in other words, the disjoint union of cycles that span the vertex set of a graph, has long been fundamental in graph theory. Historically, two questions have been at the forefront of this study. Under what conditions will a 2-factor exist? Is this 2-factor a single cycle (the hamiltonian problem)? However, harder questions about the actual structure of 2-factors have also been considered. For example, Aigner and Brandt [1] showed that if G has order n and minimum degree $\delta(G) \geqslant \frac{2n-1}{3}$, then G contains any graph of maximum degree 2. This verified a conjecture of Sauer and Spencer [6]. In this paper, we consider the question when a 1-tough graph contains a 2-factor with exactly k cycles. We begin with the classic result by Dirac [3] later extended in [2].

Theorem 1 (Dirac [3]). *If G is a graph of order $n \geqslant 3$ with $\delta(G) \geqslant n/2$, then G is hamiltonian.*

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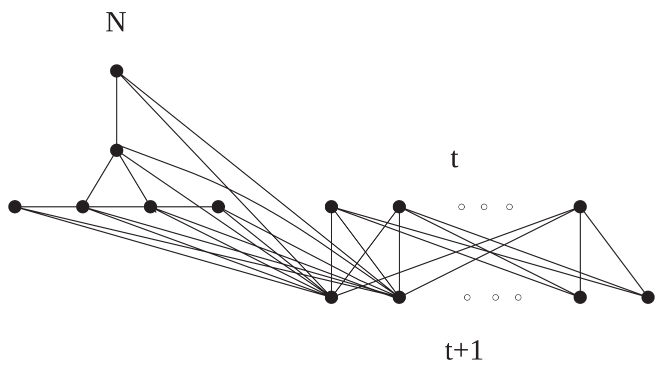

Fig. 1. Sharpness example.

Theorem 2 (Brandt et al. [2]). *If G is a graph of order $n\geqslant 3$ with $\delta(G)\geqslant n/2$, then G contains a 2-factor with exactly k cycles, for $1\leqslant k\leqslant n/4$.*

Jung [5] strengthened Theorem 1 under the condition G is 1-tough.

Theorem 3 (Jung [5]). *A 1-tough graph G order $n\geqslant 11$ with $\delta(G)\geqslant((n-4)/2)$ is hamiltonian.*

We extend Jung's result in a manner somewhat similar to Theorem 2.

Theorem 4. *If G has order n and $\delta(G)\geqslant((n-4)/2)$, then (1) if G is disconnected of order $n\geqslant 8$, then G contain a 2-factor with k cycles for $2\leqslant k\leqslant\lceil((n-4)/3)\rceil$, (2) if G is connected of order $n\geqslant 8$, but not 1-tough, then G contains a 2-factor with k cycles for $2\leqslant k\leqslant\lceil((n-4)/3)\rceil$, (3) if G is 1-tough of sufficiently large order n with $\delta(G)\geqslant((n-t)/2)$, $(0\leqslant t\leqslant 4)$, then G contains a 2-factor with k cycles where $1\leqslant k\leqslant n/4-t$.*

The sharpness of part (3) is demonstrated by the same graph that shows the sharpness of Theorem 3. The net is the graph obtained by attaching a new edge at each corner of a triangle. Now consider the graph with two sets, one consisting of $t+1$ independent vertices and the other one consisting of t independent vertices and a copy of the net. Now complete the graph by inserting all possible edges between the two sets (see Fig. 1). This graph has order $n=2t+7$ and minimum degree $t+1=((n-5)/2)$. It is also 1-tough, but has no hamiltonian cycle, in fact, it has no 2-factors at all. Thus, the minimum degree condition is sharp.

The sharpness of parts (1) and (2) can be seen by first taking two copies of $K_{n/2}$ ($n/2\equiv 2 \bmod 3$) and deleting a matching from each. Now each component has the proper minimum degree and the two factors are trivial to construct for (1). For part (2) merely join one vertex from each copy with an edge and repeat the 2-factor construction.

In order to prove Theorem 4, we will make use of the following result from [4].

Theorem 5. *Let k be a positive integer and let G be a balanced bipartite graph of order $2n$ where $n\geqslant\max\{51,k^2/2+1\}$. If $\deg u+\deg v\geqslant n+1$ for every $u\in V_1$ and $v\in V_2$, then G contains a 2-factor with exactly k cycles.*

Let $N(x)$ and $\bar{N}(x)$ denote the neighbors and nonneighbors of the vertex x, respectively. If C_i is a cycle in G, then let $|V(C_i)|=c_i$. If $V(G)$ is partitioned into sets $S_1,\ldots,S_k$ and the graph induced by each S_i, denoted $\langle S_i\rangle$, contains a spanning cycle, we say that $V(G)$ is partitioned into cycles $C_1,\ldots,C_k$. For a given path (or segment of a cycle) with a given orientation, denote the predecessor and successor of the vertex x according to this orientation as x^- and x^+, respectively. Moreover, we denote the lth successor of x by $x^{(l)+}$. In other words, we define $x^{(l)+}$ by $x^{(1)+}=x^+$ and $x^{(l)+}=(x^{(l-1)+})^+$ for $l\geqslant 2$. Let $P=a_0a_1\ldots a_l$ be a path (or a segment of a cycle). Then the subpath $a_ia_{i+1}\ldots a_{j-1}a_j$ $(i\leqslant j)$ is denoted by $a_i\overrightarrow{P}a_j$. The same subpath, traversed in the opposite direction, is denoted by $a_j\overleftarrow{P}a_i$. Finally, the vertex x is insertible on a cycle C whenever x is adjacent to consecutive vertices of C, thus allowing C to be extended to include x.

If a set of mutually disjoint k cycles $C_1,\ldots,C_k$ and a (possibly empty) path P cover $V(G)$, then $(C_1,\ldots,C_k,P)$ is called a $(k,1)$-partition. If $\mathfrak{C}=(C_1,\ldots,C_k,\ P)$ is a $(k,1)$-partition, and there is no $(k,1)$-partition $(C_1',\ldots,C_k',P')$ with $|V(P')|<|V(P)|$, then $\mathfrak{C}$ is said to be a maximum $(k,1)$-partition. Since we allow a path to be empty, a 2-factor with k components forms a maximum $(k,1)$-partition.

Lemma 1. *Let G be a graph which has a $(k,1)$-partition, and let $(C_1,\dots,C_k,P)$ be a maximum $(k,1)$-partition. Let x and y be the starting vertex and the terminal vertex of P, respectively. Then*

(1) *if $|V(P)|\geqslant 3$, then $\deg_{C_t} x + \deg_{C_t} y \leqslant \lfloor \frac{1}{2}|V(C_t)|\rfloor$ for each t with $1\leqslant t\leqslant k$, and*
(2) *if $|V(P)| = 2$, then $\deg_{C_t} x + \deg_{C_t} y \leqslant \lfloor \frac{2}{3}|V(C_t)|\rfloor$ for each t with $1\leqslant t\leqslant k$.*

Proof (*Sketch*). A standard adjacency of x implies a nonadjacency of y argument can be used. □

Lemma 2. *Let $k\geqslant 2$ be an integer and G a graph of order $n\geqslant 19$ with $\delta(G)\geqslant \frac{1}{2}(n-4)$. Suppose G has a maximum $(k,1)$-partition $(C_1,\dots,C_t,P)$, then $|V(P)|\leqslant 1$.*

Proof (*Sketch*). Assume $|V(P)|\geqslant 2$. Let $H=\langle V(P)\rangle$ and $K=\langle \bigcup_{t=1}^{k} V(C_t)\rangle$. Let x and y be the starting vertex and the terminal vertex of P, respectively.

Suppose $P = xy$. Using Lemma 1, a direct count bounding $\deg_G x + \deg_G y$ from both sides shows $n\leqslant 14$, a contradiction. Therefore, $|V(P)|\geqslant 3$. Let $\varepsilon_t = \frac{1}{2}|V(C_t)| - \lfloor \frac{1}{2}|V(C_t)|\rfloor$ $(1\leqslant t\leqslant k)$. Then, by Lemma 1

$$\deg_G x + \deg_G y \leqslant \sum_{t=1}^{k}\left(\frac{1}{2}|V(C_t)| - \varepsilon_t\right) = \frac{1}{2}|V(K)| - \sum_{t=1}^{k}\varepsilon_t.$$

Assuming H is not hamiltonian, and bounding $\deg_H x + \deg_H y$, produces a contradiction. Therefore, H is hamiltonian and a direct count bounding $\deg_G x + \deg_G y$ shows that $|V(H)|\geqslant \frac{1}{3}n - \frac{2}{3}$.

Let C_0 be a hamiltonian cycle of H. For each $e_t \in E(C_t)$ $(1\leqslant t\leqslant k)$,

$$(C_0, C_1, \dots, C_{t-1}, C_{t+1}, \dots, C_k, C_t - e_t)$$

is a $(k,1)$-partition. Since $(C_1,\dots,C_k,P)$ is maximum, $|V(C_t)|\geqslant |V(P)| = |V(H)|$. If $k\geqslant 3$,

$$n\geqslant |V(H)| + |V(C_1)| + |V(C_2)| + |V(C_3)| \geqslant 4|V(H)| \geqslant 4\left(\frac{n-2}{3}\right),$$

or $n\leqslant 8$, a contradiction. Hence $k=2$. Since $|V(H)|\leqslant |V(C_j)|$ $(j=1,2)$, $|V(H)|\leqslant \frac{1}{3}n$, and we may conclude that $\deg_H x\geqslant |V(H)|-2$ and $\deg_H y\geqslant |V(H)|-2$.

For each $v\in V(H)$, $v^+\overrightarrow{C}_0 v$ is a hamiltonian path of H. By applying the same argument as above to this path instead of P, we have $\deg_H v\geqslant |V(H)|-2$. Since $n\geqslant 19$ and $|V(H)|\geqslant \frac{n-2}{3}$, this implies $\delta(H)\geqslant \frac{|V(H)|+1}{2}$ and hence H is hamiltonian-connected.

Now $N_K(u) \neq \emptyset$ for each $u\in V(H)$. Say $z\in N_{C_1}(u)$, and say $z'\in N_{C_1}(v)-\{z\}$, then since H is hamiltonian-connected, there exists a hamiltonian path Q of H starting from u and ending at v. Both $(u\overrightarrow{Q}vz'\overleftarrow{C}_1 zu, C_2, z'^+\overrightarrow{C}_1 z^-)$ and $(u\overrightarrow{Q}vz'\overrightarrow{C}_1 zu, C_2, z'^-\overleftarrow{C}_1 z^+)$ are $(2,1)$-partitions of G, hence $|z'^+\overrightarrow{C}_1 z^-|\geqslant |V(P)|\geqslant \frac{n-2}{3}$ and $|z'^-\overleftarrow{C}_1 z^+|\geqslant \frac{n-2}{3}$. Then, $|V(C_1)|\geqslant \frac{2n+2}{3}$ and $|V(C_1)|+|V(C_2)|+|V(H)|\geqslant n+3$, a contradiction. Therefore, $N_{C_1}(v)\subset\{z\}$ for each $v\in V(H)-\{u\}$. Applying the same argument to C_2 and as $V(H)\geqslant 3$, H has a vertex w with $\deg_{C_1} w\leqslant 1$ and $\deg_{C_2} w\leqslant 1$. Then $\deg_G w\leqslant |V(H)|-1+2\leqslant \frac{1}{3}n+1 < \frac{1}{2}(n-4)$ since $n\geqslant 19$, a contradiction, and the lemma follows. □

Proof of Theorem 4. For (1) note that G can have only two components and each must be very dense, hence construction of the 2-factors is easy. For (2) note that if G is connected but not 1-tough, G contains a cut vertex, and again the two components must be very dense.

For (3) we proceed by induction on t. For $t=0$ we apply Theorem 2. Also, for $k=1$ the result follows from Theorem 3. Hence, we may assume that $t\geqslant 1$ and $k\geqslant 2$. Thus inductively, we assume that for any 1-tough graph with $\delta(G)\geqslant \frac{n-(t-1)}{2}$ $(1\leqslant t<4)$ the result follows for all k in the appropriate range. Now let G be 1-tough with $\delta(G)\geqslant \frac{n-t}{2}$ and consider the graph $G+w$, for some new vertex w. This graph is clearly 1-tough and $\delta(G+w)\geqslant \frac{n-t}{2}+1 = \frac{n+1-(t-1)}{2}$. This implies by the induction hypothesis that $G+w$ contains a 2-factor with $k+1$ cycles where $1\leqslant k+1\leqslant \frac{n+1}{4}-(t-1)$.

Thus, G contains k cycles, say $C_1, C_2, \dots, C_k$ $(t\leqslant k\leqslant n/4-t)$ and a path P (where $|V(P)|=p$) that partition $V(G)$. Over all such collections of k cycles and a path, choose one with $c_1+\cdots+c_k$ a maximum. Without loss of generality we may assume that $|V(C_1)|\geqslant |V(C_2)|\geqslant\cdots\geqslant |V(C_k)|$ and hence, $n\geqslant 3k+p$.

By Lemma 2, $p=1$. Thus, we have disjoint cycles $C_1, C_2, \dots, C_k$ $(k\geqslant 2)$ and x not on any cycle, such that, $V(G)=V(C_1)\cup\cdots\cup V(C_k)\cup\{x\}$. If x is insertible on any cycle, then the desired 2-factor exists. Thus, we may assume this fails to occur.

Since deg $x \geqslant \frac{n-4}{2}$, the adjacencies of x must nearly alternate on each cycle, with a few possible minor exceptions. These are that x might once be nonadjacent to four consecutive vertices of one cycle, we call this a 4-gap with respect to x, or x may have one 3-gap and one 2-gap, or x may have three 2-gaps. Let $N_0 = \{v \in \bigcup V(C_i)(1 \leqslant i \leqslant h) | v^-, v^+ \in N(x) \text{ and } v \notin N(x)\}$. Vertices of N_0 are the 1-gap vertices with respect to x.

Any $w \in N_0 \cap V(C_i)$ may be replaced on C_i by x, with w then replacing x in the system. Thus, we may assume that w has adjacency conditions analogous to x. Denote this property as $w \approx x$. □

Claim 1. *There are no chords between vertices of N_0 on the same cycle.*

Suppose $a_1 b_1 \in E(G)$ for $a_1, b_1 \in N_0 \cap V(C_i)$, some i i, $1 \leqslant i \leqslant k$. Then C_i may be extended to include x as $x, a_1^-, a_1^{--}, \ldots, b_1^+, b_1, a_1, a_1^+, \ldots, b_1^-, x$. This cycle, along with the remaining $k-1$ cycles form the desired 2-factor, a contradiction.

Claim 2. *Suppose $a_1 \in V(C_i) \cap N_0$ and $b_1 \in V(C_j) \cap N_0$, $i \neq j$, and suppose $a_1 b_1 \in E(G)$. Further, suppose that C_i has only 1-gaps with respect to x. Then $|V(C_i)| \leqslant 6$.*

Suppose $|V(C_i)| \geqslant 7$. Then, $|V(C_i)|$ is even, say $|V(C_i)| = 2m \geqslant 8$. Let C_i be $a_1, z_1, a_2, z_2, \ldots, a_m, z_m, a_1$. Then, as $x \approx a_r$ $m-1 \leqslant \deg_{C_i} a_r \leqslant m$ for each $a_r \in N_0 \cap V(C_i)$ and any chord of C_i from a_r is of the form $a_r z_s$, where $a_r \in N_0$ and $z_s \in N(x)$. Thus, at least one of z_1 or z_m has a chord to some a_r. If say z_1 has such a chord, then $x, z_r, \ldots, z_m, a_1, b_1, b_1^+, \ldots, b_1^-, x$ and $z_1, a_2, a_2^+, \ldots, a_r, z_1$ and the remaining $k-2$ cycles form the desired 2-factor, a contradiction.

Thus if any vertex of N_0 with positive degree in $\langle N_0 \rangle$ is on a 1-gap cycle C_i, then C_i must be a cycle of order 4 or 6.

Claim 3. *Any vertex of N_0 does not have adjacencies to both N_0 vertices on three 4-cycles.*

Suppose not, say some $v \in V(C_1) \cap N_0$ is adjacent to $b_1, b_2 \in V(C_2) \cap N_0$, $c_1, c_2 \in V(C_3) \cap N_0$ and $d_1, d_2 \in V(C_4) \cap N_0$ where C_2, C_3 and C_4 are 4-cycles. Let C_2 be w_1, b_1, w_2, b_2, w_1 and let C_3 be y_1, c_1, y_2, c_2, y_1 and C_4 be r_1, d_1, r_2, d_2, r_1. Also let C_1 be $v, z_1, a_1, \ldots, a_m, z_m, v$.

If z_1 is adjacent to some $a_j \in N_0 \cap V(C_1)$ and $a_j^+ = z_j \in N(x) \cap C_1$, then the cycles $z_1, z_1^+, \ldots, a_j, z_1$ and $z_j, z_j^+, \ldots, v, b_1, w_2, b_2, w_1, x, z_j$ extend the 2-factor, a contradiction. Also z_1 must have no adjacencies to C_2 or C_3, or we again complete the 2-factor, a contradiction.

Now, as the degree of z_1 in $G - V(C_2 \cup C_3 \cup C_4)$ is at least $\frac{n-4}{2}$ and there are only $n-12$ remaining vertices, we see that z_1 is insertible in at least four places. By a counting argument similar to that of Claim 2, any cycle containing a gap of more than one may contain at most two vertices of N_0. By examining the possible gaps in such a cycle, we find that $|V(C_1)| \leqslant 13$. Since z_1 is not adjacent to the predecessor of any $z_i \in N(x) \cap V(C_1)$, an examination of the possible gaps cases shows that z_1 is insertible in at most three places on C_1. Hence it must be insertible off of C_1. Continuing in this manner we see that each of $a_1, \ldots, a_m$ are insertible on other cycles or we construct a 2-factor with k cycles. If each of the a_i are insertible at distinct locations, we do so. If not, we consider inserting them in segments whose ends are insertible at the same locations. In either case, we insert all the a_i vertices elsewhere and then use the cycle x, z_m, v, z_1, x to complete the 2-factor, a contradiction completing the proof.

Claim 4. *No vertex $v \in N_0$ has adjacencies to all three N_0 vertices of a 6-cycle.*

Suppose not, say $v \in N_0 \cap V(C_1)$ was adjacent to all three vertices of $V(C_2) \cap N_0$ in the 6-cycle $C_2 : w_1, b_1, w_2, b_2, w_3, b_3, w_1$ and let C_1 be $z_1, a_1, \ldots, a_m, z_m, v, z_1$.

If z_1 is adjacent to b_i then we could insert b_i between v and z_1 and replace b_i on C_2 with x, completing the 2-factor, a contradiction. If z_1 is adjacent to say w_1 then $v, b_1, w_2, b_2, w_3, b_3, v$ and $z_m, x, w_1, z_1, \ldots, z_m$ completes the 2-factor, a contradiction. Thus, as before, z_1 must be insertible on another cycle. Also as before, $a_1, \ldots, a_m$ must all be insertible. Thus, we can again construct a 2-factor with k cycles, a contradiction.

Claim 5. *No vertex of N_0 can have two adjacencies to N_0 vertices on two distinct 6-cycles of the system.*

If not, then under these conditions any such v has a 3-gap on both other cycles, a contradiction.

Claim 6. *If $F = \langle N_0 \rangle$, then $\Delta(F) \leqslant 14$.*

This follows from the gap structure for $v \in N_0$ and the earlier claims.

Now consider the following partition of $V(G) = S \cup L \cup R$ where

$$S = \{v \in V(G) | v \text{ is in an } m\text{-gap}, m \geqslant 2\}, \quad L = N_0 \cup \{x\}, \quad \text{and} \quad R = N(x).$$

Using this structure we now construct a 2-factor with k-cycles. Based on the m-gap ($m \geqslant 2$) structure there are several cases and each is handled similarly. For this reason we present only one representative case. The other cases are similar. For convenience let $|S| = s$.

Thus, suppose that there are three 2-gaps with respect to x and deg $x = d = n/2 - 2$. Say $S = \{a_1, b_1, a_2, b_2, a_3, b_3\}$ and note that now $|L| = n/2 - 4$ and $|R| = n/2 - 2$.

We claim that the number of vertices in $T \subset R$ with degree less than $n/100$ to L is small, in fact, at most 15. To see this, note by Claim 8 that the minimum number of edges from L to R is $(n - s - d)(n/2 - 14)$. If R contains r vertices of degree less than $n/100$ then the maximum number of edges from R to L is $nr/100 + (d - r)(n/2 - 4)$. Thus,

$$(n - s - d)(n/2 - 14) \leqslant nr/100 + (d - r)(n/2 - 4).$$

Substituting for s and d and estimating the right hand side from above we obtain that

$$(n/2 - 4)(n/2 - 14) \leqslant nr/100 + (n/2 - r)(n/2 - 4)$$

which implies that $r \leqslant 15$.

To see how to do this, consider the vertices of our three 2-gaps. By definition of the 2-gaps they have all their adjacencies in $R = N(x)$. This creates at most three paths with both end vertices in $N(x)$. Each of the vertices of T has many neighbors in $N(x)$ and some neighbors in N_0. We select one neighbor for each vertex of T from each set. This creates at most 15 paths of order three with one end in L and one end in R. Finally, we note that to balance the sets that remain after these paths are removed we need to select more vertices from N_0. Note that we have presently selected at most 15 vertices from N_0 and at most 21 vertices of $N(x)$ as the ends of paths. In order to balance the sets that will remain we can either select another two pairs of adjacent vertices of N_0 if such pairs exists, or select four more paths of three vertices each where one vertex is from $N(x)$ and the other two are from N_0 (or a combination of both). Since n is sufficiently large and the unused vertices all have relatively high degree to the other set, all these paths can be joined to form a cycle. This is done by linking end vertices either directly, if an edge is present, or using a path containing a balanced number of vertices from L and R. Also note by carefully selecting vertices in R, we may create fewer initial paths, that is, some of the end vertices of the paths selected may coincide. However, even in the worst case, we can complete the construction of the single cycle, leaving a dense balanced spanning bipartite subgraph in what remains.

Now apply Theorem 5 to this subgraph to complete the 2-factor with exactly k cycles. As the other gap cases are handled similarly, this completes the proof.

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