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LINEAR FORESTS AND ORDERED CYCLES

GUANTAO CHEN¹, RALPH J. FAUDREE², RONALD J. GOULD³,
MICHAEL S. JACOBSON⁴, LINDA LESNIAK⁵ AND FLORIAN PFENDER^{3,6}

¹Georgia State University, Atlanta, GA 30303

²University of Memphis, Memphis, TN 38152

³Emory University, Atlanta, GA 30322

⁴University of Louisville, Louisville, KY 40292 and

⁵Drew University, Madison, NJ 07940

⁶Technische Universität Berlin, Berlin, Germany

Abstract

A collection $L = P^1 \cup P^2 \cup \dots \cup P^t$ ($1 \leq t \leq k$) of t disjoint paths, s of them being singletons with $|V(L)| = k$ is called a (k, t, s) -linear forest. A graph G is (k, t, s) -ordered if for every (k, t, s) -linear forest L in G there exists a cycle C in G that contains the paths of L in the designated order as subpaths. If the cycle is also a hamiltonian cycle, then G is said to be (k, t, s) -ordered hamiltonian. We give sharp sum of degree conditions for nonadjacent vertices that imply a graph is (k, t, s) -ordered hamiltonian.

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1. INTRODUCTION

Over the years hamiltonian graphs have been widely studied. A variety of related properties have also been considered. Some of the properties are weaker, for example traceability in graphs, while others are stronger, for example hamiltonian connectedness. Recently a new strong hamiltonian property was introduced in [7] and further studied in [5], [2], and [3].

We say a graph G on n vertices, $n \geq 3$ is k -ordered for an integer k , $1 \leq k \leq n$, if for every sequence $S = (x_1, x_2, \dots, x_k)$ of k distinct vertices in G , there exists a cycle that contains all the vertices of S in the designated

order. A graph is *k-ordered hamiltonian* if for every sequence S of k vertices there exists a hamiltonian cycle which encounters S in its designated order. Hu, Tian and Wei [4] considered a different question; when is it possible to find a long cycle passing through a collection of paths?

In this paper we combine these two ideas. In order to treat this in generality, we say L is a (k, t, s) -linear forest if L is a collection $L = P^1 \cup P^2 \cup \dots \cup P^t$ ($1 \leq t \leq k$) of t disjoint paths, s of them being singletons such that $|V(L)| = k$. A graph G is (k, t, s) -ordered if for every (k, t, s) -linear forest L in G there exists a cycle C in G that contains the paths of L in the designated order as subpaths. Further, if the paths of L are each oriented and C can be chosen to encounter the paths of L in the designated order and according to the designated orientation on each path, then we say G is *strongly (k, t, s) -ordered*. If C is a hamiltonian cycle then we say G is *(k, t, s) -ordered hamiltonian* and *strongly (k, t, s) -ordered hamiltonian*, respectively. Note that saying G is (s, s, s) -ordered is the same as saying G is s -ordered.

We will think of all cycles being directed. For a cycle C and vertices $x, y \in V(C)$, we denote the $x - y$ path on C following the direction of C by xCy .

As usual, we will denote the minimum degree of a graph G by $\delta(G)$, and the minimum degree sum of two non adjacent vertices in a graph G by $\sigma_2(G)$.

We will say that a graph G on at least $2k$ vertices is *k-linked*, if for every vertex set $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$ of $2k$ vertices, there are k disjoint $x_i - y_i$ paths. The property remains the same if we allow repetition in T , and ask for k internally disjoint $x_i - y_i$ paths. Thus, as an easy consequence, every k -linked graph is k -ordered and $(2k - s, k, s)$ -ordered.

An important theorem about k -linked graphs is the following theorem of Bollobás and Thomason [1]:

Theorem 1. *Every $22k$ -connected graph is k -linked.*

The following lemmas will be used later.

Lemma 1. *If a $2k$ -connected graph G has a k -linked subgraph H , then G is k -linked.*

Proof. Let $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$ be a set of $2k$ vertices in $V(G)$. Since G is $2k$ -connected, there are $2k$ disjoint paths from T to $V(H)$.

Choose the paths from T to $V(H)$ such that each path contains exactly one element of $V(H)$ (if $x_i \in T \cap V(H)$ then the corresponding path consists only of this one vertex). Now we can connect these paths in the desired way inside H , since H is k -linked. ■

Lemma 2. *If G is a graph, $v \in V(G)$ with $d(v) \geq 2k - 1$, and if $G - v$ is k -linked, then G is k -linked.*

Proof. Let $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$ be a set of $2k$ vertices in $V(G)$. If $v \notin T$, we can find disjoint $x_i - y_i$ paths inside $G - v$. Thus we may assume that $v = x_1$. If $y_1 \in N(v)$, we can find disjoint $x_i - y_i$ paths for all $i \geq 2$ in $G - v - y_1$, since $G - v - y_1$ is $(k - 1)$ -linked. Adding the path vy_1 completes the desired set of paths in G . If $y_1 \notin N(v)$, then there exists a vertex $x'_1 \in N(v) - T$, since $d(v) \geq 2k - 1$. We can find disjoint $x_i - y_i$ paths for $i \geq 2$ and a $x'_1 - y_1$ path in $G - v$, which we can then extend to an $x_1 - y_1$ path in G . ■

Further, we will use a Theorem of Mader [6] about dense graphs:

Theorem 2. *Every graph G with $|V(G)| = n \geq 2k - 1$, and $|E(G)| \geq (2k - 3)(n - k + 1) + 1$ has a k -connected subgraph.*

Corollary 3. *Every graph G with $|V(G)| = n \geq 2k - 1$, and $|E(G)| \geq 2kn$ has a k -connected subgraph.*

2. DEGREE CONDITIONS

In this section we examine minimum degree conditions sufficient to insure a graph is either (k, t, s) -ordered hamiltonian or strongly (k, t, s) -ordered hamiltonian. Sharp results for $s = t = k$ were shown in [5], [2] and [3]:

Theorem 4 [5]. *Let $k \geq 2$ be a positive integer and let G be a graph of order n , where $n \geq 11k - 3$. Then G is k -ordered hamiltonian if $\delta(G) \geq \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor - 1$.*

Theorem 5 [3]. *Let $k \geq 3$ be a positive integer and let G be a graph of order $n \geq 2k$. If $\sigma_2(G) \geq n + \frac{3k-9}{2}$, then G is k -ordered hamiltonian.*

As a first step, we prove the following theorem:

Theorem 6. Let s, t, k be integers with $0 \leq s < t < k$ or $s = t = k \geq 3$. If G is a (strongly) (k, t, s) -ordered graph on $n \geq k$ vertices with

$$\sigma_2(G) \geq \begin{cases} n + k - t & \text{if } s = 0, \\ n + k - t + s - 1 & \text{if } s > 0, \end{cases}$$

then G is (strongly) (k, t, s) -ordered hamiltonian.

As a corollary, we obtain the following theorem.

Theorem 7. For $k \geq 1$ and $1 \leq t \leq k$, if G is a (strongly) (k, t, s) -ordered graph on $n \geq k$ vertices with $\delta(G) \geq \frac{n+k-t+s}{2}$, then G is (strongly) (k, t, s) -ordered hamiltonian.

In the same spirit, we will prove another theorem, which is not needed for our main result, Theorem 10.

Theorem 8. Let s, t, k be integers with $1 < t/2 < s \leq t \leq k$. If G is a (strongly) (k, t, s) -ordered graph on $n \geq 11k$ vertices with

$$\sigma_2(G) \geq n + k - \frac{t+3}{2},$$

then G is (strongly) (k, t, s) -ordered hamiltonian.

Proof of Theorem 6 and Theorem 8. Since G is (strongly) (k, t, s) -ordered, we may choose a longest cycle C containing the paths of a given (k, t, s) -linear forest L in the designated order and with the designated orientations (if there are any) on each path. We need to show that C is hamiltonian.

Let $L = P^1 \cup P^2 \cup \dots \cup P^t$, and $x_1, \dots, x_t, y_1, \dots, y_t \in V(C)$, such that $P^i = x_i C y_i$ for all $1 \leq i \leq t$. Note that $x_i = y_i$ if P_i is a singleton. Let $R^i = y_i C x_{i+1}$ for $1 \leq i \leq t-1$, and $R^t = y_t C x_1$. Let $R = \bigcup_i R^i$.

Suppose C is not hamiltonian and let H be a component of $G - C$.

Claim 1. No R^i contains more than one vertex adjacent to H .

Suppose there exists an interval R^i with at least two vertices adjacent to H . Without loss of generality we may assume that R^1 is such an interval. Pick two of these vertices v_1, v_2 such that there are no other adjacencies of H in

$v_1 C v_2 \subset R^1$. Note that $r = |v_1 C v_2| - 2 \geq 1$, otherwise C can be extended by at least one vertex.

Let $u_1 \in N(v_1) \cap H$, let $u_2 \in N(v_2) \cap H$. Note that we allow $u_1 = u_2$. Consider now $X = (N(u_1) \cup N(u_2)) \cap C$. There cannot be two vertices consecutive on R in X , otherwise C can be extended by at least one vertex. Further, X does not contain any vertices of $v_1^+ C v_2^-$ by our choice of v_1, v_2 . Note that $R \setminus v_1^+ C v_2^-$ consists of $t - s + 1$ paths, and $|C \setminus R| = k - 2t + s$, thus

$$\begin{aligned} d(u_1) + d(u_2) &\leq 2|X| + d_H(u_1) + d_H(u_2) \\ &\leq 2 \left(|H| - 1 + \frac{|R| - r + t - s + 1}{2} + k - 2t + s \right). \end{aligned}$$

Now concentrate on v_1^+ and v_2^- . There cannot be two consecutive vertices in $R \setminus v_1^+ C v_2^-$, such that one is adjacent to v_1^+ and the other adjacent to v_2^- , otherwise the whole segment $v_1^+ C v_2^-$ could be inserted between those two vertices, and a longer cycle through u_1 could be found. Thus,

$$d(v_1^+) + d(v_2^-) \leq 2 \left(r - 1 + \frac{|R| - r + 1 + t - s}{2} + k - 2t + s + n - |C| - |H| \right).$$

But now,

$$\begin{aligned} 2(n + k - t) &\leq d(v_1^+) + d(u_1) + d(v_2^-) + d(u_2) \\ &\leq 2(n + k - t - 1 + |R| + k - 2t + s - |C|) = 2(n + k - t - 1), \end{aligned}$$

a contradiction. Therefore, there can be at most one vertex adjacent to H in each R^i .

To prove Theorem 6, observe that the degree condition forces G to be complete or $(k - t + s + 1)$ -connected. If G is complete we are done. So we may assume that G is $(k - t + s + 1)$ -connected. Since $|C - R| = k - 2t + s$, there are at least $t + 1$ vertices adjacent to H in R . Thus, there exists an R^i with two such vertices, a contradiction proving Theorem 6.

To prove Theorem 8, we first prove the following claim.

Claim 2. H is the only component of $G - C$.

Otherwise, let H' be a different component, let $v_1 \in H, v_2 \in H'$. For $i = 1, 2$, let

$$\begin{aligned} a_i &= |\{v \in N(v_i) \cap (C \setminus L)\}|, \\ b_i &= |\{v \in N(v_i) : v = x_j \text{ or } v = y_j \text{ for some } j \text{ with } x_j \neq y_j\}|, \\ c_i &= |\{v \in N(v_i) : v = x_j = y_j \text{ for some } j\}|. \end{aligned}$$

We know that $a_i + b_i + 2c_i \leq t$, since by Claim 1, v_i can have at most one neighbor in each R_j . Further, $b_i \leq 2(t - s)$. Thus,

$$\begin{aligned} 2d(v_1) &\leq 2(|H| - 1 + k - 2t + s + a_1 + b_1 + c_1) \\ &= 2|H| + k + a_1 + k - t - 2 + (b_1 - 2(t - s)) + (a_1 + b_1 + 2c_1 - t) \\ &\leq 2|H| + k + a_1 + k - t - 2. \end{aligned}$$

Similarly,

$$2d(v_2) \leq 2|H'| + k + a_2 + k - t - 2.$$

Therefore,

$$n + k - \frac{t + 3}{2} \leq d(v_1) + d(v_2) \leq |H| + |H'| + k + \frac{a_1 + a_2}{2} + k - t - 2 \leq n + k - t - 2,$$

a contradiction, proving the claim.

The degree condition forces G to be complete or $(k - \frac{t-1}{2})$ -connected. If G is complete we are done. So we may assume that G is $(k - \frac{t-1}{2})$ -connected. Since $|C - R| = k - 2t + s$, there are at least $\frac{3t+1}{2} - s$ neighbors of H in R .

Claim 3. For some i , $1 \leq i \leq t$, the following is true: $x_i = y_i$ and H has two neighbors in $y_{i-1}Cx_{i+1}^- \setminus x_i$.

Let h_i count the number of neighbors of H in $y_{i-1}Cx_i \cup y_iCx_{i+1}^-$. We know that $h_i \in \{0, 1, 2\}$ for all $1 \leq i \leq t$. Further, $\sum_i h_i \geq 3t + 1 - 2s - (t - s)$, since the sum counts every neighbor of H in $\{x_i : x_i \neq y_i\}$ once and all other neighbors of H in R twice. Thus, at least $(t - s) + 1$ of the h_i are equal to 2. Therefore, $h_i = 2$ for some i with $x_i = y_i$. The vertex x_i cannot be one of the two neighbors of H by Claim 1, establishing the claim.

Let i be as in Claim 3, let $y \in y_{i-1}C x_i^-$ and $z \in y_i^+ C x_{i+1}^-$ be the two neighbors of H . If $y^+ z^+ \in E$, then $y H z C^- y^+ z^+ C y$ is a longer cycle. Thus, $y^+ z^+ \notin E$ and, since y^+ and z^+ are not in $N(H)$,

$$|C| \geq 2 + \frac{d(y^+) + d(z^+)}{2} > \frac{n+k}{2} - \frac{t}{4} + 1.$$

This implies that

$$|R| = |C| - k + 2t - s > \frac{n-k}{2} > 5k.$$

Now let $u \in H, v \in C - N(H)$. Then

$$\begin{aligned} d(v) \geq n+k - \frac{t+3}{2} - d(u) &\geq n+k - \frac{t+3}{2} - (k-2t+s) - t - |H| \\ &\geq |C| - 1 - s + \frac{t-1}{2}. \end{aligned}$$

Therefore, v is adjacent to all but at most $\frac{s}{2}$ vertices on C .

For the final contradiction we differentiate two cases.

Case 1. Suppose $y^+ \neq x_i$ or $z^+ \neq x_{i+1}$.

Let $w \in \{y^+, z^+\} - \{x_i, x_{i+1}\}$. Let $N = N(x_i) \cap N(x_{i+1}) \cap N(w)$. Since none of the vertices x_i, x_{i+1}, w is adjacent to H , each is adjacent to all but at most $\frac{s}{2}$ vertices of the cycle. Thus, $|N| \geq |C| - \frac{3s}{2}$.

Claim 4. For some $j, |N \cap y_j C x_{j+1}| \geq 4$.

Otherwise,

$$5k < |R| \leq 3t + |R| - |N| \leq 3t + \frac{3s}{2},$$

a contradiction.

Let j be as in the last claim, and let $v_1, v_2, v_3, v_4 \in N \cap y_j C x_{j+1}$ be the first four of these vertices in that order.

If $v_4 \in y^+ C x_i$, define a new cycle as follows: $C' = z C^- v_4 x_{i+1} C y H z$.

If $v_4 \in z^+ C x_{i+1}$, let $C' = z C^- x_i v_4 C y H z$.

Otherwise observe that there is at most one neighbor x of H in $v_1 C v_4$.

For $j \neq i$, define the new cycle C' as follows:

If $x \in v_1Cv_2$, let $C' = zC^-x_i v_3 x_{i+1} Cv_2 w v_4 CyHz$.

If $x \in v_3Cv_4$, let $C' = zC^-x_i v_2 x_{i+1} Cv_1 w v_3 CyHz$.

Otherwise, let $C' = zC^-x_i v_2 Cv_3 x_{i+1} Cv_1 w v_4 CyHz$.

For $i = j$, a very similar construction works:

Let $C' = zC^-v_4 w v_1 C^-x_i v_2 Cv_3 x_{i+1} CyHz$.

In any case, no vertex in $C - C'$ is adjacent to H , so all of them have high degree to C and thus high degree to $R \cap C'$. Therefore, we can insert them one by one into C' creating a longer cycle, a contradiction, completing Case 1.

Case 2. Suppose $y^+ = x_i, z^+ = x_{i+1}$.

Let $N' = N(x_i) \cap N(x_{i+1})$. Then $|N'| \geq |C| - s$.

Claim 5. For some l , $|N' \cap y_l C x_{i+1}| \geq 5$.

Otherwise,

$$5k < |R| \leq 4t + |R| - |N'| \leq 4t + s,$$

a contradiction.

Let l be as in the last claim, and let $z_1, z_2, z_3, z_4, z_5 \in N' \cap y_l C x_{i+1}$ be the first five of these vertices in that order. At most one of them is adjacent to H , say z_2 . Now a very similar argument as in the last case gives the desired contradiction, just replace x_i by z_1 , x_{i+1} by z_5 , and w by z_4 . One possible cycle would then be (for $l < j < i$): $C' = zC^-x_i z_2 C z_3 x_{i+1} C z_1 v_2 Cv_3 z_5 Cv_1 z_4 v_4 CyHz$. ■

Theorem 9. If $s = t = k \geq 3$ or $0 \leq s < t < k$, and G is a graph of order $n \geq \max\{178t + k, 8t^2 + k\}$ with

$$\sigma_2(G) \geq \begin{cases} n + k - 3 & \text{if } s = 0, \\ n + k + s - 4 & \text{if } 0 < 2s \leq t, \\ n + k + \frac{t-9}{2} & \text{if } 2s > t, \end{cases}$$

then G is strongly (k, t, s) -ordered.

Proof of Theorem 9. To simplify the proof, we will first use an induction argument on k . The statement is obviously true for the base cases

($s = 0, t = 1, k = 2$) and ($s = t = k = 3$), since G then is 2-connected. Suppose the statement is true for all $k \leq k_0$. We need to show the statement for $k = k_0 + 1$. So, let G be a graph of order $n \geq \max\{178t + k, 8t^2 + k\}$ satisfying the degree condition for some triple (k, t, s) . We need to show that for any (k, t, s) -linear forest L in G , we can find a cycle passing through it in the designated order and direction. Let L be such a forest. Delete all inner vertices of the paths from $V(G)$, and replace the paths by edges to create a new graph G' and a new linear forest L' . If there are any paths of three or more vertices in G , this will reduce the order of G and the order of L . Finding a cycle in G' through L' yields a cycle in G through L . Since $k' = 2t - s, n' = n - (k - k') \geq \max\{178t + k', 8t^2 + k'\}$, and

$$\sigma_2(G') \geq \sigma_2(G) - 2(k - k') \geq \begin{cases} n' + k' - 3 & \text{if } s = 0, \\ n' + k' + s - 4 & \text{if } 0 < 2s \leq t, \\ n' + k' + \frac{t-9}{2} & \text{if } 2s > t, \end{cases}$$

there is such a cycle in G' if $k' < k$, by the induction hypothesis. Thus, we may assume that $k' = k$, and so $L = L'$, meaning that L consists only of paths with one or two vertices.

Claim 1. G has a t -linked subgraph H .

All vertices of G with $d(v) < \frac{n}{2}$ have to be adjacent. If there are at least $2t$ of them, this clique is H . Otherwise $|E(G)| \geq (n - 2t)\frac{n}{4} \geq 44tn$, which implies by Corollary 3 that G contains a $22t$ -connected subgraph H . By Theorem 1, H is t -linked:

Claim 2. G is t -linked (and thus $(2t - s, t, s)$ -ordered) or $V(G) = V(A) \cup V(B)$, where $|A| \leq |B| + 2t - 1$, B is t -linked, and A is either t -linked or complete.

If G is $2t$ -connected, then G is t -linked by Lemma 1. So assume there is a cut set K with $|K| < 2t$. Let A' and B' be two components of $G - K$ with $|A'| \leq |B'|$. Let $v \in A', w \in B'$. Then

$$n + 2t - s - 3 \leq d(v) + d(w) \leq |A'| + |B'| + 2|K| - 2 \leq n + 2t - 3,$$

so u and v can miss a total of at most s possible adjacencies. Since $|B'| > \frac{n}{2} - t$, this ensures B' to be $22t$ -connected and thus t -linked. If A' is complete,

we are done. Otherwise, the degree sum condition insures $|A'| \geq \frac{n-2t-s+1}{2}$, so A' is $2t$ -connected and thus t -linked. To find A and B , we now partition the vertices of K as follows one-by-one: Add any vertex $u \in K$ with degree $d_{B'}(u) \geq 2t-1$ to B' , and add the remaining vertices to A' . The result will be as desired, as can be seen step by step: If u has high ($\geq 2t-1$) degree to B' , adding it to B' will leave B' t -linked by Lemma 2. If u has low degree to B' , it must be either adjacent to all of A' or have high degree to A' by the degree sum condition. In both cases, A' stays complete (if $|A'| < 2t$), or A' stays t -linked (note that a complete graph on $2t$ vertices is t -linked), again by Lemma 2. This proves the claim.

Case 1. Suppose $t < 2s$.

First, we may assume that $t \geq 3$. Otherwise, $t = s \leq 2$, and there is nothing to prove. We will use A' and B' as defined in the proof of Claim 2 above. There is a vertex $v \in B'$ with $d_A(v) = 0$: For every vertex $w \in A'$ we have $d_{B'}(w) = 0$, and for every $w \in A \cap K$ we have $d_{B'}(w) \leq 2t-2$. Since there are at most $2t-1$ vertices in $A \cap K$, at most $(2t-2)(2t-1) < |B'|$ vertices can have $d_A(v) > 0$.

Therefore, by the degree sum condition, we have $d_B(w) \geq 2t-s + \frac{t-5}{2}$ for every $w \in A$. Let $L = \{x_1y_1, x_2y_2, \dots, x_t y_t\}$, where $x_i = y_i$ if the path is a singleton, and all paths are directed from x_i to y_i (remember: all paths are either edges or singletons by the induction hypothesis). We need to find paths from y_i to x_{i+1} . Let

$$\begin{aligned} L_A &= L \cap A, \\ L_B &= L \cap B, \\ L'_A &= \{x_i \in L_A | y_{i-1} \in L_B\} \cup \{y_i \in L_A | x_{i+1} \in L_B\}, \\ L'_B &= \{x_i \in L_B | y_{i-1} \in L_A\} \cup \{y_i \in L_B | x_{i+1} \in L_A\}, \\ S_A &= \{x_i \in L_A | y_{i-1} \in L_B\} \cap \{y_i \in L_A | x_{i+1} \in L_B\}, \\ S_B &= \{x_i \in L_B | y_{i-1} \in L_A\} \cap \{y_i \in L_B | x_{i+1} \in L_A\}. \end{aligned}$$

By these definitions we get

$$|L'_A| + |S_A| = |L'_B| + |S_B|.$$

For $x_i \in L'_A$, let $N'(x_i) = (N(x_i) \cap B) - (L - \{y_{i-1}\})$.

For $y_i \in L'_A$, let $N'(y_i) = (N(y_i) \cap B) - (L - \{x_{i+1}\})$.

For $X \subset L'_A$, let

$$N'(X) = \bigcup_{x_i \in X} N'(x_i) \cup \bigcup_{y_i \in X} N'(y_i).$$

For $t = s = 3$, there is nothing to prove. For $t = 3, s = 2$, we get for every nonempty $X \subset L'_A$,

$$|N'(X)| \geq 3 - |L_B| + |X| + |X \cap S_A| \geq |X| + |X \cap S_A|.$$

For $t \geq 4$ we get for every nonempty $X \subset L'_A$,

$$\begin{aligned} |N'(X)| &\geq 2t - s + \frac{t-5}{2} - |L_B| + |X| + |X \cap S_A| - |S_B| \\ &= |X| + |X \cap S_A| + |L_A| - |S_B| + \frac{t-5}{2} \\ &\geq |X| + |X \cap S_A| + |L'_A| - |S_B| + \frac{t-5}{2} \\ &= |X| + |X \cap S_A| + \frac{|L'_A| - |S_B| + |L'_B| - |S_A|}{2} + \frac{t-5}{2} \\ &\geq |X| + |X \cap S_A| + \frac{t-5}{2}. \end{aligned}$$

Thus, $|N'(X)| \geq |X| + |X \cap S_A|$, and thus by Hall's Theorem, we can find disjoint neighbors for all $x_i, y_i \in L'_A$ in $N'(x_i)$ or $N'(y_i)$, respectively. Using that B is t -linked and that A is t -linked or complete, we can now find the desired cycle.

Case 2. Suppose $s = 0$.

The degree condition forces G to be $(2t-1)$ -connected. If G is $2t$ -connected, then it is t -linked and we are done. If G has a cut set K of size $2t-1$, the degree condition forces $G-K$ to consist of two complete components A' and B' , both of which are adjacent to all vertices in K . It is easy to see that such a graph is t -linked.

Case 3. Suppose $0 < s \leq t/2$.

The degree condition forces G to be $(2t-2)$ -connected. If G is $2t$ -connected, then it is t -linked and we are done. If G has a cut set K of size $2t-2$, the degree condition forces $G-K$ to consist of two complete components A'

and B' , both of which are adjacent to all vertices in K . It is easy to see that such a graph is $(2t - s, t, s)$ -ordered. If K has size $2t - 1$, G has a very similar structure. Again, it is straightforward to verify the claim. ■

Theorem 10. *If $0 \leq s \leq t \leq k$, and G is a graph of order $n \geq \max\{178t + k, 8t^2 + k\}$ with*

$$\sigma_2(G) \geq \begin{cases} n + k - 3 & \text{if } s = 0, t \geq 3, \\ n + k + s - 4 & \text{if } 0 < 2s \leq t, t \geq 3, \\ n + k + \frac{t-9}{2} & \text{if } 2s > t \geq 3, \\ n + k - 2 & \text{if } s \leq 1, t = 2, \\ n + k - 1 & \text{if } s = 0, t = 1, \\ n & \text{if } s = t \leq 2, \end{cases}$$

then G is strongly (k, t, s) -ordered hamiltonian.

Proof. Apply Theorem 6 and Theorem 9. ■

3. SHARPNESS

Theorem 6 is sharp for $s = 0$, illustrated by the following graph: Let $A = K_{\frac{n+k-t-1}{2}}$, and B be a set of $\frac{n-k+t+1}{2}$ isolated vertices. Add all edges between A and B . For n sufficiently large, G is strongly (k, t, s) -ordered, and $\sigma_2(G) = n + k - t - 1$. But G is not strongly (k, t, s) -ordered hamiltonian, since no hamiltonian cycle can contain a (k, t, s) -linear forest L which completely lies inside A : Every hamiltonian cycle has exactly $k - t - 1$ edges in A , one edge less than L .

The following graph shows sharpness of Theorem 9, $s = 0$. Let G consist of three complete graphs: $A = K_{\frac{n-k+2}{2}}$, $K = K_{k-2}$, $B = K_{\frac{n-k+2}{2}}$. Add all edges between A and K and all edges between K and B . The degree sum condition is just missed, but G is not $(k, t, 0)$ -ordered: Let $x_1 \in A$, $y_t \in B$, $\langle L - \{x_1, y_t\} \rangle = K$.

The following graph shows sharpness of Theorem 9, $t \geq 2s \geq 2$. Let G consist of four complete graphs: $S = K_s$, $T = K_{k-s}$, $A = K_{2s-1}$, $B = K_{n-k-2s+1}$. Add all edges from A , all edges between T and B . For every vertex $s_i \in S$, pick two vertices $u_i, v_i \in T$. Add all edges between S and T but the edges $s_i u_i, s_i v_i$. We have $\sigma_2(G) = n + k + s - 5$, but if we pick

$V(L) = V(S) \cup V(T)$, such that $x_{2i} = y_{2i} = s_i, x_{2i+1} = u_i, y_{2i-1} = v_i$ for all $i \leq s$, there is no cycle passing through L in the designated order and direction.

The following graph shows sharpness of Theorem 9, $2s > t$. Let G consist of four complete graphs: $S = K_{\lceil \frac{t}{2} \rceil}, T = K_{k - \lceil \frac{t}{2} \rceil}, A = K_{t-1}, B = K_{n-k-2s+1}$. Add all edges from A , all edges between T and B . For every vertex $s_i \in S$, pick two vertices $u_i, v_i \in T$, with the exception that $v_{i+1} = u_i$ for $1 \leq i \leq s - \lceil \frac{t}{2} \rceil$. Add all edges between S and T but the edges $s_i u_i, s_i v_i$. We have $\sigma_2(G) = n + k + \lceil \frac{t}{2} \rceil - 5$, but if we pick $V(L) = V(S) \cup V(T)$, such that $x_{2i} = y_{2i} = s_i, x_{2i+1} = u_i, y_{2i-1} = v_i$ for all $i \leq \lceil \frac{t}{2} \rceil$, there is no cycle passing through L in the designated order and direction.

4. NOTE ADDED IN PROOFS

Very recently, Thomas and Wollan [8] have improved the bound in Theorem 1 to the following.

Theorem 11. *If a graph G is $2k$ -connected and has at least $5k|V(G)|$ edges, then G is k -linked.*

Corollary 12. *Every $10k$ -connected graph is k -linked.*

Using these results in place of Theorem 1 will improve some of the bounds on n .

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