Forbidden Subgraphs and the Hamiltonian Theme

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ABSTRACT

Let $F$ be the unique graph with degree sequence $1, 1, 3, 3, 3$. We show that every connected graph $G$ that contains no induced subgraph isomorphic to $K_{1,3}$ or $F$ is traceable. Moreover, if $G$ is 2-connected then $G$ is hamiltonian.

1. Introduction.

In this article we consider finite simple graphs without loops or multiple edges. A graph is connected if each pair of vertices is joined by a path, while a graph is $n$-connected if the removal of fewer than $n$ vertices results in a connected graph. The distance $d(x,y)$ between vertices $x$ and $y$ of a connected graph $G$ is the least number of edges in an $x$-$y$ path.

The set of vertices, the distance from the vertex $x$ to $d(x,S) = \min\{d(x,s) | s \in S\}$. The diameter, $\text{diam } G$, of a connected graph $G$ is the maximum distance between two vertices of $G$. If $S$ is a subset of the vertex set $V(G)$ of a graph $G$,

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then the subgraph induced by \( S \) is denoted by \( \langle S \rangle \). The neighborhood, \( N(x) \), of a vertex \( x \) is the set of all vertices adjacent to \( x \). A graph \( G \) is \textit{locally connected} if \( \langle N(x) \rangle \) is connected for each \( x \in V(G) \). The graph \( G \) is \textit{traceable (hamiltonian)} if it contains a path (cycle) through all its vertices. Such a path (cycle) is called a hamiltonian path (cycle).

![Figure 1](image)

\[ K_{1,3} \quad F \]

Let \( c(G) \) denote the number of components of the graph \( G \). A graph is \textit{1-tough} if \( c(G - S) \leq |S| \) for every nonempty proper subset \( S \) of \( V(G) \). The \textit{complement} of the graph \( G \), denoted \( \overline{G} \), is the graph with vertex set \( V(G) \) and \( e \) is an edge of \( \overline{G} \) if and only if \( e \) is not an edge of \( G \).

The literature abounds with results concerning traceable and hamiltonian graphs. Recent studies have related the idea of forbidden subgraphs with other properties to obtain sufficient conditions for a graph to be hamiltonian. The object of this paper is to investigate the graphs \( K_{1,3} \) and \( F \) (cf. Figure 1) and their relation to the hamiltonian theme.

\textit{Theorem A.} (Oberly and Sumner [4]). A connected, locally con-
nected graph that contains no induced subgraph isomorphic to $K_{1,3}$ is hamiltonian.

**Theorem B.** ([2]). A 2-connected graph with diameter at most 2 that contains no induced subgraph isomorphic to $K_{1,3}$ is hamiltonian.

**Theorem C.** (Jung [3], cf. Bermond [1]). If $G$ is 1-tough then either $G$ is hamiltonian, or its complement $\overline{G}$ contains the graph $F$ as a subgraph.

In terms of forbidden induced subgraphs Theorem C has the following natural Corollary.

**Corollary D.** If $G$ is 2-connected and contains no hamiltonian cycle, then $G$ has an induced subgraph isomorphic to $K_{1,3}$ or to a spanning subgraph of $F$.

In this paper we prove the following theorems.

**Theorem 1.** A connected graph that contains no induced subgraph isomorphic to $K_{1,3}$ or $F$ is traceable.

**Theorem 2.** A 2-connected graph that contains no induced subgraph isomorphic to $K_{1,3}$ or $F$ is hamiltonian.

Before beginning the proof of Theorem 1, a few observations will be helpful.

2. **Observations.**

Throughout the next two sections $G$ denotes a connected graph of diameter $d$ with no induced subgraphs isomorphic to $K_{1,3}$ or $F$. It was shown in [2] that if $d \leq 2$ then $G$ is traceable; hence, we assume that $d \geq 3$. Let $P : v_0, v_1, \ldots, v_{d-1}, v_d$ be a path of length $d = d(v_0, v_d)$. Define the following subsets of $V(G)$:

$$U_i = \{x \in V(G) | x v_i, x v_{i+1} \in E(G) \text{ and } x v_{i-1}, x v_{i+2} \notin E(G)\}$$

for $0 \leq i \leq d - 1$, and
\[ V_j = \{ x \in V(P) \mid xv_j, xv_{j+1}, xv_{j+2} \in E(G) \text{ and } xv_{j-1}, xv_{j+3} \notin E(G) \} \text{ for } 0 \leq j \leq d - 2. \]

Also define
\[
A_0 = \{ x \in V(P) \mid xv_0 \notin E(G) \text{ and } xv_1 \notin E(G) \}, \\
A_d = \{ x \in V(P) \mid xv_d \notin E(G) \text{ and } xv_{d-1} \notin E(G) \}.
\]

If \( u, w \in U_i (0 \leq i \leq d - 2) \) and \( uw \) is not an edge of \( G \) then \( \langle u, w, v_{i+1}, v_{i+2} \rangle \cong K_{1,3} \). Thus \( U_i (0 \leq i \leq d - 2) \) is complete. If \( i = d - 1 \) then \( \langle u, w, v_{d-1}, v_{d-2} \rangle \cong K_{1,3} \) unless \( uw \) is an edge of \( G \). Thus \( U_{d-1} \) is complete as well. A similar argument shows \( V_j \) is complete \( (0 \leq j \leq d - 2) \).

If \( a \in U_i \) and \( b \in V_i (1 \leq i \leq d - 2) \) and \( ab \notin E(G) \) then \( \langle a, b, v_{i-1}, v_i \rangle \cong K_{1,3} \); thus \( U_i \cup V_i \) is complete \( (1 \leq i \leq d - 2) \). Similarly \( U_{i+1} \cup V_i \) is complete \( (0 \leq i \leq d - 3) \). We have shown:

(A). The graphs \( U_i (0 \leq i \leq d - 1) \), \( V_j (0 \leq j \leq d - 2) \), \( U_i \cup V_i \) \( (1 \leq i \leq d - 2) \), and \( U_{i+1} \cup V_i \) \( (0 \leq i \leq d - 3) \) are complete.

Suppose \( x \in V(G) \) and \( x \) is adjacent to some \( v_i \in V(P) \) \( (1 \leq i \leq d - 1) \). Then \( \langle x, v_i, v_{i-1}, v_{i+1} \rangle \cong K_{1,3} \) unless one of the edges \( xv_{i-1}, xv_{i+1} \) or \( v_{i-1}v_{i+1} \) is in \( G \). If \( v_{i-1}v_{i+1} \) is in \( G \) then \( d(v_0, v_d) < d \), a contradiction. Hence, at least one of \( xv_{i-1} \) and \( xv_{i+1} \) is in \( G \), that is, \( x \) is adjacent to at least two consecutive vertices of \( P \). Since \( d(v_0, v_d) = d \), no vertex is adjacent to four vertices of \( P \) and by (A) all adjacencies to \( P \) must be consecutive. Thus the sets \( U_i (0 \leq i \leq d - 1) \) and \( V_j (0 \leq j \leq d - 2) \) are all distinct. Thus:

(B). Any vertex of \( G \) adjacent to a vertex of \( P \) lies in exactly one of \( U_i (0 \leq i \leq d - 1) \), \( V_j (0 \leq j \leq d - 2) \) or \( A_k (k = 0, d) \).

Now suppose the vertex \( x \) is not adjacent to a vertex of \( P \), but is adjacent to \( y \) where \( yv_i (2 \leq i \leq d - 2) \) is an edge of \( G \). By (B), the vertex \( y \) is adjacent to \( v_{i-1} \) or
\[ v_{i+1} \text{. Say } v_{i-1} \in E(G) \text{ (a similar argument applies if } v_{i+1} \in E(G)) \text{. If } v_{i+1} \text{ is in } G \text{ then } \langle \{x,y,v_{i+1},v_{i-1}\} \rangle \cong K_{1,3} \text{ unless at least one of } x_{i-1} \text{ or } x_{i+1} \text{ is in } G \text{. If } v_{i+1} \text{ is not in } G \text{, then } \langle \{x,y,v_{i-2},v_{i-1},v_{i+1}\} \rangle \cong F \text{ unless } x_{j} \in E(G) \text{ for some } j \in \{i-2,i-1,i,i+1\} \text{. If } y \in V_{0} \text{, then } \langle \{x,y,v_{0},v_{2}\} \rangle \cong K_{1,3} \text{ unless at least one of } x_{0} \text{ and } x_{2} \text{ is in } G \text{. In any case, } x \text{ must be adjacent to a vertex of } P \text{, a contradiction. We have shown:}

\text{Every vertex that is adjacent to a vertex in } U_{i} (1 \leq i \leq d-2) \text{ or } V_{j} (0 \leq j \leq d-2) \text{ is contained in one of } U_{k} (0 \leq k \leq d-1), V_{k} (0 \leq k \leq d-2), A_{0} \text{ or } A_{d} \text{.}

\text{Let } X = \{x \in V(G) | d(x,V(P)) > 1\} \text{. Clearly } V(G) \text{ is partitioned by } X, A_{0}, A_{d}, \bigcup_{i=0}^{d-1} U_{i}, \bigcup_{j=0}^{d-2} V_{j} \text{ and } V(P) \text{. The next two observations concern the structure of } X \text{.}

\text{Note by (C), if } x \in X \text{ then } x \text{ is not adjacent to any vertex of } U_{i} (1 \leq i \leq d-2) \text{ or } V_{j} (0 \leq j \leq d-2) \text{. Let } d(x,V(P)) = 2 \text{. Then there exists } y \in A_{0}, U_{0}, A_{d} \text{ or } U_{d-1} \text{ such that } xy \in E(G) \text{. Since } y \in A_{0} \cup U_{0} \text{ or } y \in A_{d} \cup U_{d-1} \text{, we may assume without loss of generality } y \in A_{0} \cup U_{0} \text{. If } y \in U_{0} \text{ and } A_{0} \neq \emptyset \text{ then for all } a \in A_{0}, xa \in E(G) \text{; otherwise } \langle \{v_{0},v_{1},v_{2},x,y,a\} \rangle \cong F \text{ implying } ay \in E(G) \text{ and } \langle \{y,v_{1},a,x\} \rangle \cong K_{1,3} \text{.}

\text{Suppose } X \neq \emptyset \text{.}

\text{Case 1. If } A_{0} \neq \emptyset \text{ then define the following sets:}

\[ S_{1} = \{x \in X | \text{ for some } y \in A_{0}, xy \in E(G)\}, \]
\[ S_{i} = \{x \in X | \text{ for some } y \in S_{i-1}, xy \in E(G)\} - \bigcup_{k=1}^{i-1} S_{k} \text{ (} 1 < i \). \]

\text{Case 2. If } A_{0} = \emptyset \text{ then define the following sets:}

\[ T_{1} = \{x \in X | \text{ for some } y \in U_{0}, xy \in E(G)\}, \]
\[ T_{i} = \{x \in X | \text{ for some } y \in T_{i-1}, xy \in E(G)\} - \bigcup_{k=1}^{i-1} T_{k} \text{ (} 1 < i \). \]
(D). Let \( X \neq \emptyset \). If \( A_0 \neq \emptyset \) then \( S_1 \neq \emptyset \) and if \( A_0 = \emptyset \) then \( T_1 \neq \emptyset \).

Choose any two vertices \( x, y \in A_0 \). Then \( \langle v_0, v_1, x, y \rangle \) shows that \( xy \) is in \( \mathcal{G} \). Hence, \( \langle A_0 \rangle \) and, similarly, \( \langle A_d \rangle \) are complete.

Let \( x, y \in S_1 \). By definition of \( S_1 \) there exist \( x', y' \in A_0 \) such that \( xx' \) and \( yy' \) are in \( \mathcal{G} \). If \( x' = y' \) then \( \langle v_0, x', x, y \rangle \) implies that \( x \) is adjacent to \( y \). If \( x' \neq y' \) then \( xy \in E(\mathcal{G}) \) for otherwise \( \langle v_0, v_1, x', y', x, y \rangle \neq F \).

We conclude that \( \langle S_1 \rangle \) is complete. Also, observe that if \( A_0 = \emptyset \) then a similar argument shows that \( \langle T_1 \rangle \) is complete.

We shall show that \( \langle S_i \rangle \) is complete for each \( i \). Assume that \( \langle S_j \rangle \) is complete for all \( 1 \leq j \leq k \), and let \( x, y \in S_{k+1} \). By definition there exist \( x', y' \in S_k \) such that \( xx', yy' \in E(\mathcal{G}) \). Also, there are \( x'', y'' \in S_{k-1} \) such that \( x'x'', y'y'' \in E(\mathcal{G}) \) (if \( k = 1 \) then \( x'', y'' \in A_0 \)). If \( x' = y' \) then \( \langle x'', x', x, y \rangle \rangle = K_{1,3} \) unless \( x \) is adjacent to \( y \). If \( x' \neq y' \) then, since \( \langle S_i \rangle \) is complete, \( x' \) is adjacent to \( y' \). Consider \( \langle x'', x', y', x \rangle \rangle \); it follows that \( x''y' \) or \( xy' \) is in \( \mathcal{G} \). In the former case, by our choice of \( x'' \) there is a vertex \( p \) such that \( px'' \in E(\mathcal{G}) \) and \( p \) is not adjacent to any of \( x', y', x, y \). Now \( \langle p, x'', x', y', x, y \rangle \rangle \) shows that \( x \) is adjacent to \( y \) or, as in the former case, \( xy' \) is in \( \mathcal{G} \). The graph \( \langle x, y, y', y'' \rangle \rangle \) implies that \( x \) is adjacent to \( y \). Thus we have:

(E). Let \( X \neq \emptyset \). If \( A_0 \neq \emptyset \) then \( \langle S_i \rangle \) is complete for all \( i \). If \( A_0 = \emptyset \) then \( \langle T_i \rangle \) is complete for all \( i \).

Let \( x_1 \in S_1 \). Since \( d(x_1, v_d) \leq d \) and \( d(v_0, v_d) = d \), it follows from (C) that there exists a vertex \( x_i \) in some \( S_i \) such that \( x_i \) is adjacent to \( z \) for some \( z \in U_{d-1} \cup A_d \). Choose \( i \) maximum with this property. If \( i \geq 3 \), let \( x_{i-1} \in S_{i-1} \) be adjacent to \( x_1 \), and \( x_{i-2} \in S_{i-2} \) be adjacent
to $x_{i-1}$. In the case that $i = 2$, let $x_{i-2} \in A_0$ and in the case that $i = 1$, let $x_{i-1} \in A_0$ and $x_{i-2} = v_0$.

Suppose that $S_{i+1} \neq \emptyset$, say $x_{i+1} \in S_{i+1}$. Since $x_{i+1}$ is not adjacent to $z$, $\langle \{x_{i-1}, x_{i}, x_{i+1}, z\} \rangle$ implies $z$ is adjacent to $x_{i-1}$. Consideration of $\langle \{x_{i-2}, x_{i-1}, x_{i}, x_{i+1}, z, v_d\} \rangle$ yields an adjacency between $x_{i-2}$ and $z$, since $x_{i-2}, x_{i-1}, x_i$, and $x_{i+1}$ are not adjacent to $v_d$. Now a contradiction arises from $\langle \{x_{i-2}, x_i, z, v_d\} \rangle$. Therefore, $S_j = \emptyset$ for all $j \geq i + 1$.

(F). Let $i$ be the maximum such that $S_i \neq \emptyset$. Then there are $x \in S_i$ and $z \in A_d$, or $z \in U_{d-1}$ if $A_d = \emptyset$, such that $x$ is adjacent to $z$. If $A_0 = \emptyset$ then the preceding statement holds with $T_i$ replacing $S_i$.

Let $Y = \{y \in X | y \notin S_k\}$. Suppose $Y \neq \emptyset$. Since $G$ is connected there exist $y \in Y$ and $y' \in U_{d-1} \cup A_d$ such that $yy' \in E(G)$. Choose $x_i$ and $z$ as guaranteed by (F). By an argument similar to that establishing (D), $z$ and $y'$ are both in $A_d$, or both in $U_{d-1}$ when $A_d = \emptyset$. If $z = y'$ then $\langle \{x_i, y, z, v_d\} \rangle$ implies that $y$ is adjacent to $x_i$, which contradicts $y \in Y$. If $z \neq y'$ then $\langle \{x_i, y, y', z, v_d, v_{d-1}\} \rangle$ leads to a contradiction when $A_d \neq \emptyset$, while $\langle \{x_i, y, y', z, v_{d-2}, v_{d-1}\} \rangle$ gives a contradiction if $A_d = \emptyset$.

(G). If $A_0 \neq \emptyset$ then $X = \cup S_k$. If $A_0 = \emptyset$ then $X = \cup T_k$.

3. Proof of Theorem 1.

For $C, D \subseteq V(G)$ write $C - D$ whenever there exist vertices $c \in C$ and $d \in D$ such that $cd \in E(G)$. Let $C_1, C_2, \ldots, C_n$ be a partition of $V(G)$ satisfying: $\langle C_i \rangle$ is complete and $C_i \sim C_{i+1}$. Then the sequence $C_1, C_2, \ldots, C_n$ is used to denote a hamiltonian path of $G$ in which the vertices of $C_i$ are traced consecutively and precede the vertices
of $C_{i+1}$ in the hamiltonian path. Also, if $C_i = \{v\}$ we write $v$ in place of $C_i$.

Let $i$ be chosen as in (F).

If $A_0 = \emptyset$ and $A_d \neq \emptyset$ then $U_0, v_0, U_1, v_1, \ldots, v_{d-2}, v_{d-1}, U_d, A_d, T_1, T_{i-1}, \ldots, T_1$ represents a hamiltonian path of $G$. If $A_0 = \emptyset$ and $A_d = \emptyset$ then $U_0, v_0, U_1, v_1, \ldots, v_{d-2}, v_{d-1}, v_d, U_{d-1}, T_1, T_{i-1}, \ldots, T_1$ is a hamiltonian path. Let $A_0 \neq \emptyset$ and $A_d \neq \emptyset$. If one of $v_{d-2} = \emptyset$, $U_{d-1} = \emptyset$, $v_{d-2} \sim U_{d-1}$ holds then $A_0, v_0, U_0, v_1, v_0, U_1, \ldots, v_{d-1}, v_{d-2}, U_{d-1}, v_d, U_d, A_d, S_i, S_{i-1}, \ldots, S_1$ represents a hamiltonian path. If none of the three conditions hold then consider

(a). $A_0, v_0, U_0, v_1, V_0, U_1, \ldots, v_{d-1}, v_d, U_{d-1}, A_d, S_i, S_{i-1}, \ldots, S_1$;

(b). $A_0, v_0, U_0, v_1, V_0, U_1, \ldots, v_{d-1}, U_{d-1}, v_{d-2}, A_d, S_i, S_{i-1}, \ldots, S_1$.

If $v_{d-2} \sim A_d$ then (b) yields a hamiltonian path. If $U_{d-1} \sim A_d$ then (a) gives a hamiltonian path. An induced $K_{1, 3}$ occurs if neither $v_{d-2} \sim A_d$ nor $U_{d-1} \sim A_d$. Finally, if $A_0 \neq \emptyset$ and $A_d = \emptyset$ then $A_0, v_0, U_0, v_1, V_0, U_1, \ldots, v_{d-1}, v_{d-2}, v_d, U_{d-1}, S_i, S_{i-1}, \ldots, S_1$ is a hamiltonian path.

Note that we are able to trace $G$, under the appropriate conditions as listed above, even when subsets of $V(G)$ are empty. Also observe that whenever $X \neq \emptyset$, that is, $S_1 \neq \emptyset$ or $T_1 \neq \emptyset$, then $G$ is in fact hamiltonian.

The graph $G$ is traceable and the proof of Theorem 1 is complete.

Clearly, connectedness is a necessary hypothesis in Theorem 1. Also it is easy to construct nontraceable graphs containing an induced $K_{1, 3}$ or $F$; of course, these can be contained in traceable graphs.
4. Proof of Theorem 2.

Let \( G \) be a 2-connected graph that contains no induced subgraph isomorphic to \( K_{1,3} \) or \( F \). Fix a pair \( v_0, v_d \in V(G) \) such that \( d(v_0, v_d) = d = \text{diam } G \). The proof of Theorem 1 allows us to assume that every \( v_0 - v_d \) path \( P \) of length \( d \) possesses property (\( \star \)):

\[
X = \{ x \not\in P \mid d(P, x) > 1 \} = \emptyset. \tag{\( \star \)}
\]

Also, by Theorem B, we may assume that \( d \geq 3 \).

Let us suppose that \( d > 3 \). We claim that there are \( v_0 - v_d \) paths \( P \) and \( Q \) such that \( P \) has length \( d \), \( V(P) \cap V(Q) = \{v_0, v_d\} \) and \( G(Q) = ((V(G) - V(Q)) \cup \{v_0, v_d\}) \) is connected with neither \( v_0 \) nor \( v_d \) a cutvertex of \( G(Q) \).

For each \( v_0 - v_d \) path \( P \) of length \( d \) let \( U_i \) \( (0 \leq i \leq d - 1) \), \( V_i \) \( (0 \leq i \leq d - 2) \), \( A_0 \) \( \) and \( A_d \) be defined as in Section 2. Relabel the sequence of sets \( A_0, U_0, V_0, \ldots, \)

\( U_1, V_1, \ldots, U_{d-2}, V_{d-2}, U_{d-1}, A_d \) by \( W_0, W_1, W_2, \ldots, W_{2i+1}, \)

\( W_{2i+2}, \ldots, W_{2d-3}, W_{2d-2}, W_{2d-1}, W_{2d} \). Let \( D(P) \) be the collection of paths \( R^*: v_0, x_1, \ldots, x_k \) such that \( x_i \in W_{j_i} \) and \( j_k = \max\{j_1, j_2, \ldots, j_k\} \). Choose \( R \in \bigcup D(P) \) such that \( j_k \) is maximum. Say \( R \in D(P) \) where \( P: v_0, v_1, \ldots, v_d \) and \( R: v_0, x_1, \ldots, x_k \). We show that \( 2d - 3 \leq j_k \leq 2d \). Otherwise, one of the following cases holds.

Case 1. \( j_k = 0 \). Then \( x_k \in A_0 \). This is impossible as \( G \) is 2-connected.

Case 2. \( j_k = 2t + 1 \) \( (0 \leq t \leq d - 3) \). Then \( x_k \in U_t \). If \( y \in V_t \) then, by (A), \( x_k \) is adjacent to \( y \), contradicting our choice of \( R \). Thus, \( V_t = \emptyset \). As \( v_{t+1} \) is not a cutvertex, there exist \( x, y \in V(G) \) such that \( xy \in E(G) \) and one of the following holds:
(i). \( x \in U_t, y \in U_{t+1}, \)
(ii). \( x \in U_t, y \in V_{t+1}, \)
(iii). \( x \in U_t, y \in U_{t+2}, \)
(iv). \( x \in V_{t-1}, y \in U_{t+1}, \)
(v). \( x \in U_{t-1}, y \in U_{t+1}, \)
(vi). \( x \in A_0, \)
\[ y \in U_1 \cup U_2 \quad (t = 0), \]
\[ y \in U_1 \cup U_2 \quad (t = 2t+1). \]

(Observe that there must be an edge \( xy \) for some \( x \in \bigcup_{i=0}^{2t+1} W_i, \)
\[ y \in \bigcup_{i=2t+2}^{2d} W_i. \] All possibilities, other than (i) – (vi), either contradict \( d(v_0, v_1) = d \), or give rise to an induced \( K_{1,3} \) or \( F \) when \( t = 0 \) or \( t = d - 3 \).

If one of (i) – (iv) holds then \( x = x_k \) or, by (A), \( xx_k \) is in \( G \). Then one of \( v_0, x_1, \ldots, x_k, x, y \) (if \( x \not\in V(R) \)) or \( v_0, x_1, \ldots, x_i, y \) (if \( x = x_i \)) contradicts the choice of \( R \).

Suppose that (v) holds. Examining \( \langle \{v_{t+1}, v_{t+2}, v_{t+3}^x, x_k^y \} \rangle \) shows that \( xx_k \) or \( x_k y \) is in \( G \). It is obvious that either edge gives a contradiction.

If (vi) holds then \( v_0, x, y \) contradicts the choice of \( R \).

**Case 3.** \( j_k = 2t + 2 \) (\( 0 \leq t \leq d - 3 \)). Then \( x_k \not\in V_t \). As before, (A) implies that \( U_{t+1} = \emptyset \).

Assume first that \( V_{t+1} = \emptyset \). Since \( v_t \) is not a cut-
vertex of \( G \) and \( d(v_0, v_1) = d \) there must exist an edge \( xy \) such that

(i). \( x \in U_t \cup V_t, y \in U_{t+2}, \)
(ii). \( x \in U_{d-3}, y \in A_d (t = d-3), \)
(iii). \( x \in V_{d-3}, y \in A_d (t = d-3), \)
(iv). \( x \in A_0, y \in U_2 (t = 0). \)

If (i) holds then one of the paths \( v_0, x_1, \ldots, x_k, x, y \) (if \( x \not\in V(R) \)) and \( v_0, x_1, \ldots, x_i, y \) (if \( x = x_i \)) contradicts the choice of \( R \). If (ii) holds then \( \langle \{v_{d-4}, v_{d-3}, v_{d-2}, v_{d-1}, x, y \} \rangle = F \). We argue in a similar way if (iv) holds.

In the event (iii) holds, then \( \langle \{x, v_{d-3}, v_{d-1}, y \} \rangle = K_{1,3} \).
Let \( z \in V_{t+1} \). Consider the \( v_0 \rightarrow v_d \) path
\[
P' : v_0, v_1, \ldots, v_{t+1}, z, v_{t+3}, \ldots, v_d
\]
of length \( d \). Define the sets \( U'_1, V'_1 \) for the path \( P' \).
Note that \( U'_1 = U_1 \) (\( i \neq t+1, t+2 \)), \( V'_1 = V_1 \) (\( i \neq t, t+1, t+2 \)),
\( x_k \in U'_t \) and \( v_{t+2} \in V'_{t+1} \). Now \( x_k \) is adjacent to \( v_{t+2} \), so
\[\langle\{v_0, x_1, \ldots, x_k, v_{t+2}\}\rangle\] contains a path contradicting our choice of \( R \).

As a consequence of these cases, \( x_k \in U_{d-2}, V_{d-2}, U_{d-1} \),
or \( A_d \). Let us show that \( x_k \in U_{d-2} \) leads to a contradiction.
If \( x_k \in U_{d-2} \) then \( V_{d-2} = \emptyset \). If both \( A_d \) and \( U_{d-1} \) were
empty then \( v_{d-1} \) would be a cutvertex of \( G \), an impossibility.
If \( A_d = \emptyset \) and \( U_{d-1} \neq \emptyset \) then, because \( v_{d-1} \) is not a cut-
vertex, we obtain, as in case \( 2 \), a path terminating in \( U_{d-1} \).
This contradicts our choice of \( R \). So assume \( A_d \neq \emptyset \). As \( G \)
is 2-connected there exist \( y \in A_d \) and \( x \) in one of \( A_0, U_1 \),
or \( V_1 \) such that \( x \) is adjacent to \( y \). If
\( x \in U_1 \) (\( 1 \leq i \leq d - 3 \)) then \( \langle\{x, y, v_{i+1}, v_{i+2}\}\rangle \cong F \); if
\( x \in A_0 \cup U_0 \) then \( d(v_0, v_d) < 4 \); if \( y \in V_1 \) (\( 0 \leq i \leq d - 3 \)) then
\( \langle\{x, y, v_{i+1}, v_{i+2}\}\rangle \cong K_{1,3} \). Hence, \( A_d \sim (U_{d-2} \cup U_{d-1}) \). If
\( A_d \neq U_{d-2} \) then, again arguing as in case \( 2 \), we obtain a
path terminating in \( U_{d-1} \), contrary to \( x_k \in U_{d-2} \). Thus,
\( A_d \sim U_{d-2} \) which again invalidates our choice of \( R \).

We conclude that \( x_k \in U_{d-1}, V_{d-2}, \) or \( A_d \). Let \( Q \) be
the path obtained by adjoining \( v_d \) to \( R \). Then
\( V(Q) \cap V(P) = \{v_0, v_d\} \) and, because of property (\( *) \), \( G(Q) \) is
connected. It remains to show that \( Q \) may be chosen so that
neither \( v_0 \) nor \( v_d \) are cutvertices of \( G(Q) \).

Suppose that \( v_0 \) is a cutvertex of \( G(Q) \). As a result
of property (\( *) \), \( \langle V(G(Q)) \rangle - \{v_0\} \) has two components, one
being \( \langle V(G(Q)) \rangle - A_0 \). Since \( G \) is 2-connected, some \( x \in A_0 \)
and \( y \in V(Q) \) are adjacent in \( G \). Choose \( i \) maximum such
that \( x_i \) is adjacent to a vertex of \( A_0 \) and let
\[ V(Q') = \{ v_0, x_i, x_{i+1}, \ldots, x_k, v_d \} \cup A_0. \] Then \( v_0 \) is not a cut-vertex of \( G(Q') \). A similar argument allows us to adjust \( Q' \) if \( v_d \) is a cutvertex of \( G(Q') \).

The initial claim has been shown. That \( G \) is hamiltonian follows upon showing that \( G(Q) \) contains a hamiltonian \( v_0 - v_d \) path. For convenience, let \( U_i \) (respectively, \( V_i, A_0, A_d \)) denote \( U_i - V(Q) \) (respectively, \( V_i - V(Q), A_0 - V(Q), A_d - V(Q) \)).

In order to be brief we list several observations.

\( 1. \) If \( A_0 \neq \emptyset \) then \( A_0 - (U_0 \cup V_0) \).

This holds because \( v_0 \) is not a cutvertex of \( G(Q) \) and \( A_0 - U_i \) \((3 \leq i \leq d - 1)\) contradicts \( d(v_0, v_d) = d > 4, A_0 - U_2 \) gives rise to \( F \) as an induced subgraph of \( G \), and \( A_0 - V_i \) \((1 \leq i \leq d - 2)\) yields \( K_{1,3} \).

\( 2. \) If \( U_0 \neq \emptyset \) then \( A_0 - U_0 \), or \( A_0 = \emptyset \).

Let \( z \in A_0 \) and \( x \in U_1 \cup V_0 \) be adjacent, and let \( x_0 \in U_0 \).

Then \( \{z, x, V_1, v_2, v_3, x_0\} \) implies \( zx_0 \) or \( xx_0 \) is in \( G(Q) \). If \( xx_0 \) is in \( G(Q) \) then \( \{x_0, x, z, v_2\} = K_{1,3} \) unless \( zx_0 \) is an edge. Thus \( A_0 - U_0 \).

\( 3. \) If \( A_0 \neq \emptyset, U_0 = \emptyset \) then \( a) A_0 - V_0 \) or \( b) A_0 - U_1 \).

Let \((1)'\), \((2)'\), \((3)'\) denote the corresponding facts concerning \( A_d, U_{d-1}, U_{d-2}, V_{d-2} \).

\( 4. \) If \( U_0, V_0, V_1 \neq \emptyset \) then either \( c) U_0 - V_0 \), \( d) U_0 - V_1 \), or \( e) V_0 - V_1 \).

Examining \( \{v_1, x_0, y_0, y_1\} \) for \( x_0 \in U_0, y_0 \in V_0, y_1 \in V_1 \) shows that \( 4 \) holds.

Suppose \( U_0 \neq \emptyset, U_{d-1} \neq \emptyset \). By \( 2\), \( 2' \) and in accordance with which of \( 4\), \( c\), \( d\), or \( e\) holds trace \( G(Q) \) as follows:

\( c) v_0, A_0, U_0, V_0, v_1, U_1, V_1, \ldots, U_{d-1}, A_d, V_d \)

\( d) v_0, A_0, U_0, V_1, v_1, V_0, U_1, V_2, U_2, V_2, \ldots, V_{d-1}, U_{d-1}, A_d, V_d \);
(e). $v_0, A_0, U_0, v_1, V_0, U_1, V_1, v_2, U_2, V_2, \ldots, v_{d-1}, U_{d-1}, A_d, v_d$.

(These represent hamiltonian paths whether or not $A_0$ is empty. Also, if one or more of $V_0$ or $V_1$ is empty then the appropriate one of (c) or (d) still yields a hamiltonian path.)

Suppose $U_0 = \emptyset, U_{d-1} \neq \emptyset$. Apply (2)' and whichever of (3a) or (3b) holds to trace $G(Q)$:

(3a). $v_0, A_0, V_0, v_1, U_1, V_1, \ldots, v_{d-1}, U_{d-1}, A_d, v_d$;

(3b). $v_0, A_0, U_1, V_0, v_1, V_1, v_2, U_2, V_2, \ldots, v_{d-1}, U_{d-1}, A_d, v_d$.

Suppose $U_0 = \emptyset, U_{d-1} = \emptyset$. Then one of (3a), (3b) holds and one of (3a)', (3b)' holds. We trace $G(Q)$ as follows:

(3a), (3a)'. $v_0, A_0, V_0, v_1, U_1, V_1, \ldots, v_{d-2}, U_{d-2}, v_{d-1}, U_{d-2}, A_d, v_d$;

(3a), (3b)'. $v_0, A_0, V_0, v_1, U_1, V_1, \ldots, v_{d-2}, V_{d-2}, v_{d-1}, U_{d-2}, A_d, v_d$;

(3b), (3b)'. $v_0, A_0, U_1, V_0, v_1, V_1, v_2, U_2, V_2, \ldots, v_{d-2}, V_{d-2}, A_d, v_d$.

It now remains to show that if diam $G = d = 3$ then $G$ is hamiltonian. For diameter 3 the previous approach leads to a prohibitive number of cases. We employ an alternative technique.

By Theorem A there is a $v \in V(G)$ such that $\langle N(v) \rangle$ is disconnected. Since $G$ contains no induced $K_{1,3}$ then $N(v) = A(v) \cup B(v)$ where $A(v) \neq \emptyset \neq B(v), A(v) \cap B(v) = \emptyset$ and both $\langle A(v) \rangle$ and $\langle B(v) \rangle$ are complete. Let

$C(v) = \{x \in V(G) \mid d(x, A(v)) = 1, d(x, v) = 2\}$ and $D(v) = \{x \in V(G) \mid d(x, B(v)) = 1, d(x, v) = 2\}$.

Since $G$ is 2-connected, $C(v) \cup D(v) \neq \emptyset$.

Case 1. Assume that $C(v) \not\subseteq D(v)$ and $D(v) \not\subseteq C(v)$. We wish to show $\langle C(v) - D(v) \rangle$ is complete. Let $c, c' \in C(v) - D(v)$
and choose \(a, a' \in A(v)\) such that \(ac\) and \(a'c'\) are edges of \(G\). If \(a = a'\) then \(\langle a, c, c', v \rangle\) implies \(cc'\) is in \(G\).

If \(a \neq a'\) then with any \(b \in B(v)\) we use \(\langle a, a', c, c', v, b \rangle\) to conclude that \(cc'\) is in \(G\). Hence, \(\langle C(v) - D(v) \rangle\) and similarly \(\langle D(v) - C(v) \rangle\) are nonempty complete graphs. A similar argument also shows that \(C \cap D\) can be partitioned into sets \(C'\) and \(D'\) such that \(\langle C_0 \rangle\) and \(\langle D_0 \rangle\) are complete, where \(C_0 = (C - D) \cup C'\) and \(D_0 = (D - C) \cup D'\).

Observe that if \(V(G) = \{v\} \cup A(v) \cup B(v) \cup C(v) \cup D(v)\) then \(C_0 \sim D_0\) as \(diam\ G = 3\). In this case, \(v, A, C_0, D_0, B, v\) represents a hamiltonian cycle. Thus we may assume that there are vertices at a distance 3 from \(v\). Let

\[
E(v) = \{x \in V(G) | d(x, C(v)) = 1, d(x, v) = 3\}
\]

and

\[
F(v) = \{x \in V(G) | d(x, D(v)) = 1, d(x, v) = 3\}.
\]

Observe that since \(G\) contains no induced \(K_{1, 3}\), (**) there is no element of \(E(v) \cup F(v)\) adjacent to an element of \(C(v) \cap D(v)\).

Suppose that \(E(v) \neq \emptyset\) and \(F(v) = \emptyset\) (note a similar argument will hold if \(E(v) = \emptyset\) and \(F(v) \neq \emptyset\)). Let \(d \in D(v) - C(v)\) and \(e \in E(v)\). If \(d(d, e) = 2\) then there is \(c \in C(v)\) such that \(ec\) and \(dc\) are edges in \(G\). As already observed, \(c \in C(v) - D(v)\) and \(\langle a, c, d, e \rangle \not\approx K_{1, 3}\) for any \(a \in A\) adjacent to \(c\). Therefore, \(d(d, e) = 3\) and there are vertices \(c \in C(v) - D(v)\) and \(x \in C(v) \cup D(v)\) such that \(e, c, x, d\) is a path. Again choosing \(a \in A\), with \(ac \in E(G)\), we have \(ax\) is in \(G\). Now \(\langle a, c, d, e, v, x \rangle \not\approx F\) and hence we conclude that \(E(v) \neq \emptyset \neq F(v)\).

It is straightforward to show that \(\langle E(v) \rangle\) and \(\langle F(v) \rangle\) are complete. If \(E(v) \not\approx F(v)\), then by (**) any path from \(E(v)\) to \(F(v)\) which contains elements of \(C(v) \cup D(v)\) has length at least 4. Thus, \(E(v) \sim F(v)\) and in fact, we may assume that \(E(v) \not\approx F(v)\) and \((E(v) - F(v)) \sim F(v)\). Therefore,
represents a hamiltonian cycle.

Case 2. Assume that \( C(v) = D(v) \). In this case
\[
E(v) \cup F(v) = \emptyset \quad \text{and} \quad V(G) = \{v\} \cup A(v) \cup B(v) \cup C(v).
\]
If \( \text{diam } G = 2 \) then \( G \) is hamiltonian by Theorem B. Thus, we may assume that there exist \( c \) and \( c' \in C(v) \) such that \( d(c, c') = 3 \). We shall show that Case 1, with \( c \) replacing \( v \), now applies.

Since \( c \) is not adjacent to \( c' \) in \( G \), no \( a \in A(v) \) is adjacent to both \( c \) and \( c' \). If there is an \( a \in A(v) \) adjacent to neither, then by choosing \( b, b' \in B(v) \) with \( bc \) and \( b'c' \) in \( G \) we have that \( \langle \{v, b, b', a, c, c'\} \rangle \neq F \). Thus, \( A(v) \) is partitioned as \( A \) and \( A' \) (\( A \neq \emptyset \) and \( A' \neq \emptyset \)) with \( a \in A \) if and only if \( ac \) is in \( G \) and \( a \in A' \) if and only if \( ac' \) is in \( G \). Similarly, \( B(v) \) can be partitioned into nonempty sets \( B \) and \( B' \). So \( \langle N(c) \rangle \) is disconnected and we define \( A(c), B(c), C(c), D(c) \) with \( A \subseteq A(c) \) and \( B \subseteq B(c) \).

Let \( y \in A' \). Then \( d(c, y) = 2 \) and \( d(y, A(c)) = 1 \) so \( y \in C(c) \). If \( y \in D(c) \) then by the definition of \( D(c) \) there is some \( z \in B(c) \) such that \( yz \) is in \( G \). Now \( z \notin A(v) \) as then \( z \in A \subseteq A(c) \); moreover, \( z \notin B(v) \) for otherwise \( yz \) is not in \( G \). Therefore, \( z \in C(v) \). Since \( \langle \{v, y, z, c'\} \rangle \neq K_{1,3} \) then \( zc' \) is in \( G \). But \( z \in B(c) \) implies \( zc \) is in \( G \) and hence \( d(c, c') = 2 \). This is a contradiction and we now conclude that \( C(c) \notin D(c) \) and similarly \( D(c) \notin C(c) \). Thus Case 1 applies with \( c \) replacing \( v \).

Case 3. Assume that \( D(v) \subseteq C(v) \) and that there exists \( x \in V(G) \) such that \( d(v, x) = 3 \).

Then \( E(v) \), as defined in Case 1, is nonempty and by
\[(**): E(v) = \{x \in V(G) | d(x, C(v) - D(v)) = 1, d(x, v) = 3\}.
\]
Also, \( v \) is not a cutvertex of \( G \) so \( D(v) \neq \emptyset \). As in Case 1
\( (C(v) - D(v)) \) is complete. In fact, \( (D(v)) \) is also complete.

(Observe that if \( d \) is not adjacent to \( d' \) in \( (D(v)) \) then there exists \( a' \in A(v) \) not adjacent to \( d \) and \( b \in B(v) \) adjacent to \( d \). Now there exists \( e \in E(v), c \in C(v) - D(v) \) and \( a \in A(v) \) such that \( e, c, a, d \) is a path and \( c \) is adjacent to \( d \). (This is true because if \( c \) is adjacent to \( d \), any vertex \( a \in A(v) \) adjacent to \( c \) must be adjacent to \( d \), otherwise \( \{c, a, d, e\} \equiv K_{1,3} \). If \( c \) is not adjacent to \( d \) there exists a path \( e, c, c', d \) with \( ec' \) not in \( G \), but \( \{c, a, e, c'\} \) implies \( ac' \) is in \( G \) and \( \{c, a, c', v, e, d\} \) gives a contradiction.) Finally, \( \{a, c, d, e, a', b\} \) gives a contradiction.

Since \( d(D(v), E(v)) \leq 3 \), there is a \( c \in C(v) - D(v) \) adjacent to a vertex of \( D(v) \) (otherwise there is a \( K_{1,3} \) centered in \( A(v) \)). Since \( G \) is 2-connected there exists \( c', c'' \in C(v) - D(v) \) (note ** holds) such that both \( c' \) and \( c'' \) are adjacent to vertices of \( E(v) \) and \( c \neq c'' \). Then \( v, B(v), D(v), c, E(v), c'', C(v) - (D(v) \cup \{c, c''\}) \), \( A(v), v \) (if \( c = c' \)) or \( v, B(v), D(v), c, c', E(v), c'', C(v) - (D(v) \cup \{c, c', c''\}) \), \( A(v), v \) (if \( c \neq c' \)) represent hamiltonian cycles in \( G \).

_Case 4._ Suppose \( D(v) \subset C(v) \) and \( d(v, x) \leq 2 \) for all \( x \in V(G) \). For simplicity let \( A = A(v) \), \( B = B(v) \), etc. Then \( V(G) = \{v\} \cup A \cup B \cup C \). Observe \( (C - D) \) is complete (as in Case 1).

_Subcase A._ Suppose \( (D) \) is complete. If \( C - D \sim D \) then \( v, A, C - D, D, B, v \) represents a hamiltonian cycle. If \( C - D \not\sim D \), then since \( G \) is 2-connected, there exists at least two vertices in \( A \), say \( a_1 \) and \( a_2 \), with adjacencies in \( C - D \) (and these adjacencies are distinct unless \( |C - D| = 1 \)). Further, \( a_1 \) and \( a_2 \) have no adjacencies in \( D \) or an induced
$K_{1,3}$ would result. Since $D \subset C$, there exists $a \in A$ such that $a \sim D$, thus, $a \neq a_1$ or $a_2$. Then

$$v, a_1, C - D, a_2, A - \{a, a_1, a_2\}, a, D, B, v$$

represents a hamiltonian cycle.

**Subcase B.** Suppose $\langle D \rangle$ is not complete. Choose nonadjacent $d_1, d_2 \in D$. We note $d_1$ and $d_2$ have no common adjacencies in $A$ or $B$ (for an induced $K_{1,3}$ would result). Further, each $a \in A$ is adjacent to exactly one of $d_1$ and $d_2$ (since otherwise, for $b_1, b_2 \in B$ such that $b_1d_1, b_2d_2 \in E(G)$, $
\langle\{a, v, b_1, b_2, d_1, d_2\}\rangle \neq F$).

Fix $a \in A$ and define $D_1 = \{x \in D | ax \in E(G)\}$ and $D_2 = \{x \in D | ax \notin E(G)\}$. Clearly $\langle D_1 \rangle$ is complete or a $K_{1,3}$ centered at $a$ would exist. Further, $\langle D_2 \rangle$ is complete for otherwise there exists nonadjacent $d, d' \in D_2$. Then if $d$ and $d'$ have a common adjacency in $B$, a $K_{1,3}$ exists; while if $b, b' \in B$ such that $bd$ and $d'b'$ are edges of $G$, then $\langle\{a, v, b, b', d, d'\}\rangle \neq F$. Thus, either case produces a contradiction, and $\langle D_2 \rangle$ is complete.

Recall that each vertex in $C - D$ has an adjacency in $A$. Let $c \in C - D$ such that $ca' \in E(G)$ (some $a' \in A$). If $a'd_1 \in E(G)$ then $cd_1$ is an edge of $G$ or a $K_{1,3}$ would exist. If $a' \neq d_1$, then there exists $a'' \in A$ such that $a''d_1 \in E(G)$. Further, choose $b \in B$ such that $bd_2 \in E(G)$. Then by considering $\langle\{c, a', a'', d_1, v, b\}\rangle$ we see $bd_1 \in E(G)$ or $cd_1 \in E(G)$. But $bd_1$ contradicts the fact that $d_1$ and $d_2$ have no common adjacencies in $B$. As $c$ was arbitrary in $C - D$, $d_1$ is adjacent to each vertex in $C - D$. A similar argument shows $d_2$ is adjacent to each vertex in $C - D$. Now

$$v, B, D_1 - \{d_1\}, d_1, C - D, d_2, D_2 - \{d_2\}, A, v$$

represents a hamiltonian cycle in $G$. This completes Case 4 and the proof of the Theorem.
In Figure 2 we display various examples that demonstrate the independence of Theorem 1 and Theorem 2, as well as the need to forbid both induced subgraphs in Theorem 2.

A 2-connected nonhamiltonian graphs containing no induced $K_{1,3}$ (but containing $F$).

A 2-connected nonhamiltonian graph containing no induced $F$ (but containing $K_{1,3}$).

Figure 2a.
A hamiltonian graph that is locally connected, containing no induced $K_{1,3}$ and containing $F$.

A hamiltonian graph, containing no induced $K_{1,3}$ or $F$, that is not locally connected.

Figure 2b.
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