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Note

A note on 2-factors with two components

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Dedicated to Professor Hikoe Enomoto on the occasion of his sixtieth birthday

Abstract

In this note, we consider a minimum degree condition for a hamiltonian graph to have a 2-factor with two components. Let G be a graph of order $n \geqslant 3$. Dirac's theorem says that if the minimum degree of G is at least $\frac{1}{2}n$, then G has a hamiltonian cycle. Furthermore, Brandt et al. [J. Graph Theory 24 (1997) 165–173] proved that if $n \geqslant 8$, then G has a 2-factor with two components. Both theorems are sharp and there are infinitely many graphs G of odd order and minimum degree $\frac{1}{2}(|G|-1)$ which have no 2-factor. However, if hamiltonicity is assumed, we can relax the minimum degree condition for the existence of a 2-factor with two components. We prove in this note that a hamiltonian graph of order $n \geqslant 6$ and minimum degree at least $\frac{5}{12}n+2$ has a 2-factor with two components.

Keywords: Hamiltonian cycle; 2-factor; Minimum degree

1. Introduction

In this note, we study how hamiltonicity affects a minimum degree condition for the existence of a 2-factor with two components. Our starting point is Dirac's [6] theorem.

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Theorem 1.1. *Every graph of order $n \geqslant 3$ and minimum degree at least $\frac{1}{2}n$ has a hamiltonian cycle.*

A hamiltonian cycle is a 2-factor with one component. This interpretation naturally leads us to consider a minimum degree condition for a graph to have a 2-factor with a specified number of components. Actually, Brandt et al. [4] studied this problem, and proved that the same bound of minimum degree as in Dirac's Theorem guarantees the existence of such a 2-factor.

Theorem 1.2 (*Brandt et al. [4]*). *Let k be a positive integer. Then every graph of order $n \geqslant 4k$ and minimum degree at least $\frac{1}{2}n$ has a 2-factor with k components.*

The notion of hamiltonicity has been extended in several other directions. One extension is pancyclicity. A graph of order $n \geqslant 3$ is said to be *pancyclic* if it has a cycle of length l for every integer l between 3 and n. Bondy [3] studied a minimum degree condition for a graph to be pancyclic, and gave the same bound as in Dirac's theorem.

Theorem 1.3 (*Bondy [3]*). *Let G be a graph of order $n \geqslant 3$ and minimum degree at least $\frac{1}{2}n$. Then G is pancyclic unless n is even and $G \simeq K_{n/2,n/2}$.*

The bound $\frac{1}{2}n$ is sharp for all of Theorems 1.1, 1.2 and 1.3. The complete bipartite graph $G = K_{k,k+1}$ has order $|G| = 2k + 1$ and minimum degree $k = \frac{1}{2}(|G| - 1)$. But G has no 2-factor. This example shows that if we lower the bound by $\frac{1}{2}$, then we have infinitely many counterexamples.

Theorems 1.1, 1.2 and 1.3 indicate that in terms of sufficient conditions based on minimum degree, we cannot observe any difference among hamiltonicity, pancyclicity and the existence of a 2-factor with a specified number of components. However, Amar et al. [1] considered a variation for studying a relationship between hamiltonicity and pancyclicity. They proved that if hamiltonicity is assumed, then the minimum degree condition for pancyclicity can be relaxed.

Theorem 1.4 (*Amar et al. [1]*). *Let G be a hamiltonian graph of order n. If the minimum degree of G is at least $(2n + 1)/5$, then G is pancyclic or bipartite.*

A possible interpretation of Theorems 1.1 and 1.4 is that the bound $\frac{1}{2}n$ is required to force G to have a cycle of length n, and that once the existence of a cycle of length n is assured (and G is not bipartite), then a smaller bound of minimum degree guarantees the existence of cycles of other lengths.

Now we turn our attention to a 2-factor with specified number of components. Considering the relationship between hamiltonicity and pancyclicity, we may suspect that if hamiltonicity is assumed, then a bound of minimum degree smaller than that in Theorem 1.2 guarantees the existence of a 2-factor with specified number of components. Here we are interested in a smaller coefficient of n. More specifically, we make the following conjecture.

Conjecture 1. For each integer k with $k \geqslant 2$, there exist real numbers a_k and c_k and an integer n_k such that $a_k < \frac{1}{2}$ and every hamiltonian graph of order $n \geqslant n_k$ and minimum degree at least $a_k n + c_k$ has a 2-factor with k components.

In this note, we prove this conjecture for $k = 2$.

Theorem 1.5. *Let G be a hamiltonian graph of order $n \geqslant 6$ and minimum degree at least $\frac{5}{12}n + 2$. Then G has a 2-factor with two components.*

We give a proof of the above theorem in Section 2. We do not think Theorem 1.5 is best-possible. We discuss the detail and give some remarks in Section 3.

Before proceeding we establish some notation. For graph-theoretic terminology not explained in this note, we refer the reader to [5]. Let G be a graph. We denote the minimum degree of G by $\delta(G)$. For a vertex x of G, we denote by $N(x)$ and $\deg x$ the neighborhood of x and the degree of x in G, respectively. Given a vertex x on a cycle C with an orientation, $\overrightarrow{C}$, then the successor of x on C will be denoted by x^+ and the predecessor by x^-. Further, let $N^+(x)$ denote the set of successors of the neighbors of x and $N^-(x)$ denote the set of predecessors of neighbors of x. Given a pair of vertices u, v in C, we denote by $u\overrightarrow{C}v$ the subpath in C that starts from u, traverses in the direction of $\overrightarrow{C}$ and ends at v. The subpath of C that starts from u and ends at v, but traverses in the opposite direction, is denoted by $u\overleftarrow{C}v$.

2. Proof of the main theorem

In this section, we prove Theorem 1.5.

Proof of Theorem 1.5. If $n = 6$, then $\delta(G) \geqslant 5$, which implies $G = K_6$ and clearly G has a 2-factor with two components. If $n = 7$, then $\delta(G) \geqslant 5$ and G is obtained from K_7 by removing at most three independent edges, and it is easy to see that G has a required 2-factor.

Suppose $8 \leqslant n \leqslant 24$, then $\delta(G) \geqslant \frac{5}{12}n + 2 \geqslant \frac{1}{2}n$ and hence G has a 2-factor with two components by Theorem 1.2. Therefore, we may assume $n \geqslant 25$.

Assume G has no 2-factor with two components. Let $C = x_1x_2x_3 \cdots x_nx_1$ be a fixed hamiltonian cycle of G. For $x \in V(C)$ and $e = uu^+ \in E(C)$, (x, e) is said to be an *insertion* if $\{u, u^+\} \subset N(x)$. By the definition if (x, uu^+) is an insertion, then $x \neq u, u^+$. Let I be the set of all insertions. For $x \in V(G)$ and $e \in E(C)$, we define $i(x)$ and $j(e)$ by $i(x) = |\{e \in E(C): (x, e) \in I\}|$ and $j(e) = |\{x \in V(C): (x, e) \in I\}|$. Note

$$\sum_{x \in V(G)} i(x) = \sum_{e \in E(C)} j(e) = |I|. \tag{1}$$

For $x \in V(C)$ and $uu^+ \in E(C)$, $(x, uu^+) \in I$ if and only if $x \in N(u) \cap N(u^+)$, and it follows that $j(uu^+) = |N(u) \cap N(u^+)|$.

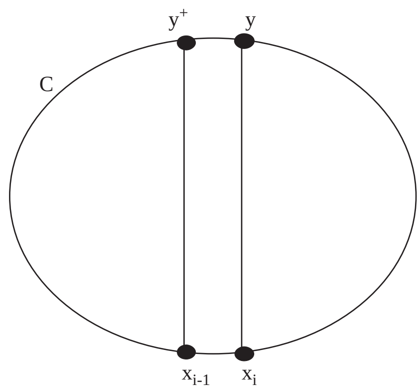

Fig. 1. A 2-factor with two cycles.

Claim 1.

$$|I| > n^2/8.$$

Proof. Observe, for each $y \in N(x_i) - \{x_{i-3}, x_{i-1}, x_{i+1}\}$ it must be the case that $y^+ \notin N(x_{i-1})$, for otherwise a 2-factor with 2 cycles would result (See Fig. 1).

Also, note that there are at least $5n/12 + 2 - i(x_i)$ vertices $y \in N(x_i)$ so that $y^+ \notin N(x_i)$, which follows from the definition of $i(x_i)$. Hence we have

$$|(V(G) - N(x_i)) \cap (V(G) - N(x_{i-1}))| \geqslant \tfrac{5}{12}n - 1 - i(x),$$

which implies $|N(x_i) \cup N(x_{i-1})| \leqslant \frac{7}{12}n + 1 + i(x_i)$. But this yields

$$\begin{aligned} j(x_i x_{i-1}) &= |N(x_i) \cap N(x_{i-1})| = |N(x_i)| + |N(x_{i-1})| - |N(x_i) \cup N(x_{i-1})| \\ &\geqslant \frac{5}{12}n + 2 + \frac{5}{12}n + 2 - \left(\frac{7}{12}n + 1 + i(x_i)\right) > \frac{n}{4} - i(x_i). \end{aligned}$$

Hence, we see that

$$\sum_{i=1}^{n} \left(\frac{n}{4} - i(x_i)\right) < |I|.$$

This gives by Eq. (1):

$$\frac{n^2}{4} - |I| < |I|.$$

Thus, we see that $|I| > n^2/8$, completing the proof of Claim 1.

Consequently, by averaging over the vertices, it follows that there is a vertex x_i such that $i(x_i) > n/8$. Let $X = N(x_i) - \{x_{i-1}, x_{i+1}\}$, $Y = N^+(x_{i+1}) - \{x_{i+3}, x_i\}$ and $Z = N^-(x_{i-1}) - \{x_{i-3}, x_i\}$. Clearly, $|A|, |B|, |C| \geqslant 5n/12$. □

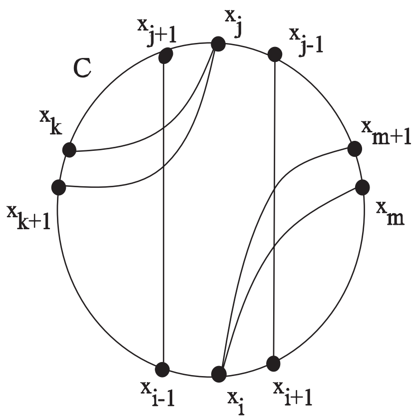

Fig. 2. An example of insertion.

Claim 2.

$$X \cap Y = X \cap Z = \emptyset.$$

Proof. Assume $X \cap Y \neq \emptyset$ and let $v \in X \cap Y$. It follows that $C_1 = x_{i+1} \overrightarrow{C} v^{-} x_{i+1}$ and $C_2 = x_i v \overrightarrow{C} x_1$ forms a 2-factor of G with two components, a contradiction. An analogous argument shows that $X \cap Z = \emptyset$. □

Claim 3.

$$|Y \cap Z| \geqslant \frac{n}{4}.$$

Proof. Since $|X|, |Y|, |Z| \geqslant 5n/12$ and $|X \cup Y \cup Z| \leqslant n$, the previous claim implies that $|Y \cap Z| \geqslant 3\left(\frac{5n}{12}\right) - n$. This implies that $|Y \cap Z| \geqslant n/4$, concluding the proof of Claim 3.

Recall that x_i was chosen so that $i(x_i) > n/8$ and the sets X, Y, Z were defined with respect to x_i. □

Claim 4. *For each $x_j \in Y \cap Z$ we have $i(x_j) = 0$.*

Proof. Let $x_j \in Y \cap Z$ and assume $i(x_j) > 0$, with $(x_j, x_k x_{k+1}) \in I$. Note by Claim 2 that $k \neq i - 1$ and $k \neq i$. Also, since $i(x_i) > n/8 \geqslant 2$ we may assume that $(x_i, x_m x_{m+1}) \in I$ with $k \neq m$. Consider the two cycles $x_{i+1} \overrightarrow{C} x_{j-1} x_{i+1}$ and $x_{j+1} \overrightarrow{C} x_{i-1} x_{j+1}$ and by inserting x_i between x_m and x_{m+1} and inserting x_j between x_k and x_{k+1}, a 2-factor with two cycles results, a contradiction (See Fig. 2). Thus, it follows that $i(x_j) = 0$ for each $x_j \in Y \cap Z$. □

Claim 5. *There is a vertex x_i with $i(x_i) > n/6$.*

Proof. From Claim 1, we know that $|I| > n^2/8$. Furthermore, from Claims 3 and 4 we have that there are at most $3n/4$ vertices x having $i(x) > 0$. Consequently, we have a vertex x_i with

$$i(x_i) > \frac{n^2/8}{3n/4} = \frac{n}{6}$$

and the claim follows.

Now to complete the proof of Theorem 1.5, let x_i be a vertex with $i(x_i) > n/6$, and of X, Y and Z be defined as above. Without loss of generality, we may assume $i = 1$. Also let $x_j \in Y \cap Z$. Since $i(x_j)=0$, clearly if $x_k \in N(x_j)$ with $k < j-1$, then $x_{k+1} \notin N(x_j)$. Furthermore, if $x_{k+1} \in N(x_1)$ then the cycle $x_1 \overrightarrow{C} x_k x_j \overleftarrow{C} x_{k+1} x_1$ and the cycle $x_{j+1} \overrightarrow{C} x_n x_{j+1}$ would form a 2-factor with two cycles. If $x_k \in N(x_j)$ with $k > j+1$ then as above it follows that $x_{k-1} \notin N(x_j)$ and $x_{k-1} \notin N(x_1)$. Also note that $x_j \notin N(x_j)$ and $x_j \notin N(x_1)$. Hence, it follows that there is a set W of size at least $5n/12 + 2$ with $W \cap N(x_1) = W \cap N(x_j) = \emptyset$. Furthermore, since $i(x_1) > n/6$, it is easy to see that there are more than $\frac{n}{6}$ vertices in $N(x_1)$ that cannot be in $N(x_j)$. Consequently, this implies that

$$\deg x_j < n - \left(\frac{5n}{12} + 2\right) - \frac{n}{6} = \frac{5n}{12} - 2,$$

which contradicts the original hypothesis, and concludes the proof of the Theorem. □

3. Concluding remarks

We do not think Theorem 1.5 is best possible. Actually, we do not know whether a linear bound of minimum degree in Conjecture 1 is appropriate. We cannot even construct a hamiltonian graph of minimum degree at least five which has no 2-factor with two components. But there exist infinitely many hamiltonian graphs of minimum degree four which have no 2-factor with two components. Let $K_5 - e$ denote the graph obtained from K_5 by deleting one edge, and let G be a graph obtained from a cycle by replacing each vertex with a copy of $K_5 - e$ so that resulting graph is 4-regular (See Fig. 3). Then G is hamiltonian, but G has no 2-factor with two components. Actually, G has only one non-hamiltonian 2-factor that consists of $\frac{1}{5}|V(G)|$ cycles.

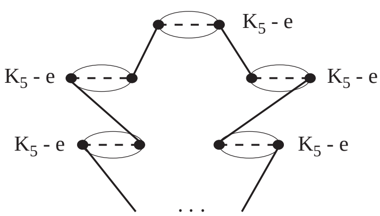

Fig. 3. A hamiltonian graph with no 2-factor with two components.

A graph G is said to be 1-*tough* if $\omega(G - S) \leqslant |S|$ for every nonempty subset S of $V(G)$. Trivially, a hamiltonian graph is 1-tough. Since a number of sufficient conditions for hamiltonicity have been relaxed for 1-tough graphs and 1-toughness is easier to use than hamiltonicity, one may think that a possible approach to Conjecture 1 is to replace the assumption that the given graph is hamiltonian with a weaker assumption that it is 1-tough. However, this approach does not work. The following example is due to Bigalke and Jung [2]. Let N, referred to as the *net*, be the unique graph of order six having the degree sequence $(3, 3, 3, 1, 1, 1)$. For an integer k with $k \geqslant 2$, let $G_k = kK_1 + ((k-1)K_1 \cup N)$. Then G_k is a 1-tough graph with minimum degree $k = \frac{1}{2}|G| - \frac{5}{2}$, but G_k does not have a 2-factor. Therefore, we cannot relax the coefficient of $|G|$ for 1-tough graphs without allowing infinitely many exceptions.

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