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Minimal Degree and (*k*, *m*)-Pancyclic Ordered Graphs

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Abstract. Given positive integers $k \le m \le n$, a graph G of order n is (k, m)-*pancyclic ordered* if for any set of k vertices of G and any integer r with $m \le r \le n$, there is a cycle of length r encountering the k vertices in a specified order. Minimum degree conditions that imply a graph of sufficiently large order n is (k, m)-pancylic ordered are proved. Examples showing that these constraints are best possible are also provided.

1. Introduction

In this paper we will deal only with finite graphs without loops or multiple edges. Notation will be standard, and we will generally follow the notation of Chartrand and Lesniak in [2]. Given a vertex x on a cycle C with an orientation, then the successor of x on C will be denoted by x^+ and the predecessor by x^-. For a graph G we will use G to represent the vertex set $V(G)$ and the edge set $E(G)$ when the meaning is clear. Given a subset (or subgraph) H of a graph G and a vertex v, then $d_H(v)$ will denote the degree of v relative to H. Given a subset H of vertices of a graph G, the subgraph induced by H will also be denoted by H when it does not lead to confusion. Thus, for example, $G - H$ will denote a set of vertices and also a subgraph, depending on the context. To shorten several of the expressions let $\epsilon_p = 2\lceil p/2\rceil - p$ for any positive integer p. Thus, $\epsilon_p = 0$ or 1 depending on whether p is even or odd, and note also that $\epsilon_p = p - 2\lfloor p/2\rfloor$.

Various degree conditions have been investigated which imply that a graph has hamiltonian type properties. The most common degree condition is the minimum degree of a graph G, which will be denoted by $\delta(G)$. Another common degree condition studied is the sum of degrees of nonadjacent vertices. For a graph G, let $\sigma_2(G) \ge s$ mean that $d(u) + d(v) \ge s$ for each pair of nonadjacent vertices in G. A graph G of order n is called *pancyclic* whenever G contains a cycle of each length r for $3 \leq r \leq n$. A stronger related property is *vertex pancyclic* which requires for any specified vertex v of G, there are cycles of length 3 through n containing v.

The following was introduced by Gary Chartrand [private communication] but first used by Ng and Schultz [7]. A graph G is *k-ordered* (*hamiltonian*) if given any ordered set S of k vertices, there is a (hamiltonian) cycle that contains S and the vertices of S are encountered on the cycle in the specified order. Additional results on $\delta(G)$ and $\sigma_2(G)$ that imply a graph G is k-ordered or k-ordered hamiltonian can be found in [6] and [5]. Here, we investigate a generalization of both k-ordered and pancyclic graphs given in the following:

Definition 1. *Let $0 \leq k \leq m$ be fixed integers and G be a graph of order $n \geq m$. The graph G is (k, m)-pancyclic ordered if for any ordered set S_k of k vertices there is a cycle C_r of length r containing S_k and encountering the vertices of S_k in the specified order for each $m \leq r \leq n$.*

Dirac [3] proved that any graph G of order n with $\delta(G) \geq n/2$ is hamiltonian, and Ore in [O60] showed that if $\sigma_2(G) \geq n$ the graph is also hamiltonian. Bondy [1] proved that if $\sigma_2(G) \geq n+1$, then G is pancyclic. Kierstead, Sárközy, and Selkow verified the following result on a minimum degree condition for a graph to be k-ordered hamiltonian.

Theorem 1 [6]. *Let $k \geq 2$ and G a graph of order $n \geq 11k - 3$. If*

$$\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1.$$

then G is k-ordered hamiltonian.

The graph F_1 in Fig. 1, which is $K_{2\lfloor k/2 \rfloor - 1} + (K_{\lceil (n-2\lfloor k/2 \rfloor + 1)/2 \rceil} \cup K_{\lfloor (n-2\lfloor k/2 \rfloor + 1)/2 \rfloor})$, verifies that Theorem 1 is sharp. The graph F_1 is not k-ordered and $\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 2$ (see [6]).

The following, which is a result on pancyclic ordered graphs involving the sum of degrees of nonadjacent vertices, was proved in [4].

Theorem 2. *Let $4 \leq k \leq m \leq n$ be positive integers, and let G be a graph of order n. The graph G is (k,m)-pancyclic ordered if $\sigma_2(G)$ satisfies any of the following conditions:*

(i) $\sigma_2(G) \geq 2n - 3$	*when*	$k \leq m < \lfloor 3k/2 \rfloor$,
(ii) $\sigma_2(G) \geq 2n - 4$	*when*	$\lfloor 3k/2 \rfloor \leq m < \lceil (5k-2)/3 \rceil$,
(iii) $\sigma_2(G) \geq 2n - 5$	*when*	$\lceil (5k-2)/3 \rceil \leq m < 2k$,
(iv) $\sigma_2(G) \geq n + 4k - m - 6$	*when*	$2k \leq m \leq (5k-3)/2$,
(v) $\sigma_2(G) \geq n + (3k-9)/2$	*when*	$m > (5k-3)/2$.

Also, all of the conditions on $\sigma_2(G)$ are sharp.

We will prove in the following minimum degree analogue of Theorem 2 for pancyclic ordered graphs. Note that Theorem 3 is not a direct consequence of

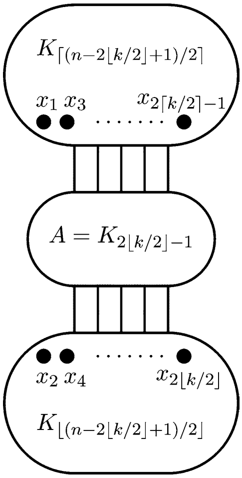

Fig. 1. F_1

Theorem 2, since the minimum degree conditions are less than one-half the σ_2 conditions in the last two cases, which are the most substantial cases. As a consequence the proof techniques are different, and in fact are more complicated and technical.

Theorem 3. *Let $4 \le k \le m \le n$ be positive integers, and let G be a graph of sufficiently large order n. The graph G is (k, m)-pancyclic ordered if $\delta(G)$ satisfies any of the following conditions:*

(i) $\delta(G) = n - 1$ *when* $k \le m < \lfloor 3k/2 \rfloor$,
(ii) $\delta(G) \ge n - 2$ *when* $\lfloor 3k/2 \rfloor \le m < 2k$,
(iii) $\delta(G) \ge n/2 + 2$, *when* $m = 10$ *or* 11, $k = 5$ *and* n *even.*
(iv) $\delta(G) \ge n/2 + 7/2$, *when* $m = 12, k = 6$ *and* n *odd.*
(v) $\delta(G) \ge \lceil n/2 \rceil + \lfloor k/2 \rfloor + t$ *when* $m = 3k - 2t - 6 - \epsilon_n$ for $-1 < t \le (k - 6 - \epsilon_n)/2$
(vi) $\delta(G) \ge \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ *when* $m \ge max\{2k, 3k - 4 - \epsilon_n\}$, *unless* $m = 11$, $k = 5$ *and* n *even.*

Also, all of the conditions on $\delta(G)$ are sharp.

2. Proofs

We begin with a proof of two lemmas that show that the degree condition $\delta(G) \ge \lceil n/2 \rceil + \lfloor k/2 \rfloor + t$ for appropriate t assures the existence of a "small cycle" containing specified vertices in a given order.

Lemma 1. *If* $4 \le k \le n$, *S is an ordered set of k vertices, and G is a graph of sufficiently large order n with* $\delta(G) \ge \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$, *then G contains a cycle C encountering the vertices of S in the designated order such that the distance in C between consecutive vertices of S is at most 5.*

Proof. Denote the ordered set by $S = \{x_1, x_2, \cdots, x_k\}$. Assume that the conclusion is not true. We can assume that G is edge maximal with this property. Thus, the addition of any edge to G will result in the existence of the required cycle. With no loss of generality we can assume that $x_1x_2 \notin E(G)$, and so $G + x_1x_2$ has a cycle C' with the property claimed. Let N_i for $i = 1, 2$ be the neighborhood of x_i in $G - C'$. By assumption, $N_1 \cap N_2 = \emptyset$, and there are no edges between N_1 and N_2.

For $i = 1, 2, |N_i| \ge \lceil n/2 \rceil + \lfloor k/2 \rfloor - 5k \ge n/2 - 9k/2$, and by a straightforward counting argument $|G - C' - (N_1 \cup N_2)| \le 4k$, $|N_i| \le n/2 - k/2$, and each vertex in N_i is adjacent to at least $n/2 - 17k/2$ vertices of N_i. This implies, since n is large, each N_i is nearly a complete graph. Associate with each vertex y of $G - (S \cup N_1 \cup N_2)$ either N_1 or N_2 depending on which set has the larger number of adjacencies of y. Add to N_i the vertices associated with N_i to obtain the superset N_i'. Thus N_1' and N_2' is a partition of $G - S$. Clearly each vertex in N_i' is adjacent to nearly $n/4$ vertices of N_i. Hence, since n is sufficiently large, any pair of vertices in the same N_i' will have a path between them of length at most 3, even after some function of k, say $8k$, vertices are deleted.

Since G is k-ordered, there is a cycle that encounters the vertices of S in the correct order. Let D be a smallest such cycle. For any pair of consecutive vertices x_j and x_{j+1} of S the path P_j in D between x_j and x_{j+1} will either start and end in the same N_i' or will start in N_1' and end in N_2', or conversely. In the first case, the minimality of D will imply that the path P_j will be of length at most 5 (using a path of length at most 3 in N_i'). In the second case the path P_j will be of length at most 9 (using paths of length at most 3 in each of N_1' and N_2' along with an edge joining N_1' and N_2'.

This implies there is a cycle D that encounters the vertices of S in the correct order and has length at most $9k$. Let $P_j = (x_j = y_1, y_2, \cdots, y_r = x_{j+1})$ be the path between x_j and x_{j+1} in D. If $r \ge 7$, then consider the three vertices y_1, y_4, y_7. Their neighborhoods in $G - D$ are disjoint, for otherwise the path P_j, and thus the cycle D, could be shortened. Thus, for $i = 1, 4$, or 7,

$$d(y_i) \le (n - 8k)/3 + 8k \le n/3 + 16k/3.$$

This gives a contradiction, since n is large and $d(y_i) \ge \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$. This completes the proof of the Lemma 1.

Lemma 2. *If* $4 \le k \le n$, *S is an ordered set of k vertices*, $-1 \le t \le (k - 6 - \epsilon_n)/2$, *and G is a graph of sufficiently large order n with* $\delta(G) \ge \lceil n/2 \rceil + \lfloor k/2 \rfloor + t$, *then G contains a cycle of length at most* $\max\{2k, 3k - 2t - 6 - \epsilon_n\}$ *encountering the vertices S in the designated order.*

Proof. Associated with the cycle C of Lemma 1 are k paths between the consecutive vertices of S. Select C to be of minimal length relative to the conditions of Lemma 1. Let m_i be the number of these paths of C containing i vertices not in S. We can assume that the cycle C is chosen such that m_0 is a large as possible, relative to the restriction on m_0 that m_1 is a large as possible, and relative to the restriction on m_1 that m_2 is as large as possible. Since Lemma 1 implies that all such paths are of length at most 5, $m_i = 0$ for $i \geq 5$. Also, clearly $m_0 + m_1 + m_2 + m_3 + m_4 = k$.

Let $P_1 = (y_0, y_1, \ldots, y_r)$ and $P_2 = (z_0, z_1, \ldots, z_s)$ be two of these paths. Thus, $r, s \leq 5$ and $y_0, y_r, z_0, z_s \in S$. If $r > 2$, then the neighborhoods of y_0 and y_r in $G - C$, say N'_0 and N'_r respectively, are disjoint and their union spans all but at most $9k$ vertices of $G - C$. Of course, the same is true for z_0 and z_s when $s \geq 2$. If $r = 5$, then y_1 and y_2 have no adjacencies in N'_r and y_3 and y_4 have no adjacencies in N'_0, since this would contradict the minimality of the length of C. This implies y_1 and y_2 are adjacent to nearly all of the vertices of N'_0 and the same is true for y_3 and y_4 relative to N'_r. If $r = 4$, then y_1 is adjacent to nearly all of the vertices of N'_0 and no vertices of N'_r, y_3 is adjacent to nearly all of the vertices of N'_r and no vertices of N'_0, and y_2 has many adjacencies in either N'_0 or N'_r, and possibly both. Clearly, the same is true for P_2 and s and the corresponding neighborhoods N''_0 and N''_s relative to P_2. When $r, s > 2$, because we can reverse the order of one of the paths, there is no loss of generality in assuming that there are large sets N_0 and N_1 of order approximately $n/4$ such that $N_0 \subset N'_0 \cap N'_0$ and $N_1 \subset N''_r \cap N''_s$. If $r \geq 4$ then y_0, y_1 (and y_2 if $r = 5$) are adjacent to nearly all of the vertices of N_0, and also $s \geq 4$ then z_0, z_1 (and z_2 if $s = 5$) are adjacent to nearly all of the vertices of N_0. The symmetric condition is true for N_1. Also, with no loss of generality we can assume the remaining y_i and z_j will have a large number of adjacencies in either N_0 or N_1.

The minimal length of the cycle C places restrictions on the number of edges between the interior vertices of one of these paths of C and the endvertices of another of these paths. Let $q_r(s)$ be the maximum number of edges between the interior vertices of a path of C of length s and the endvertices of a path of C of length $r \geq s$.

Claim 1. $q_5(s) \leq 2$ for $s \leq 5$.

Proof. For the path P_1 assume that $r = 5$ and for the path P_2 assume that $1 \leq s \leq 5$. If $s \leq 2$, then the result is obvious, since there is at most one interior vertex in a path of length at most 2. If $s = 3$, then y_0 and y_5 will have at most 2 adjacencies in $\{z_1, z_2\}$ unless they have a common adjacency z_ℓ for $\ell = 1$ or 2. However, if this occurs then P_1 can be replaced by the path (y_o, z_ℓ, y_5) of length 2 and P_2 can be replaced by a path $(z_0, w_1, y_2, y_3, w_2, z_4)$ of length 5 with $w_1 \in N_0$ and $w_2 \in N_1$. This contradicts the minimality of the length of C.

Consider the case when $s = 4$. If y_0 is adjacent to z_3, then there is a path of length 3, namely (y_0, z_3, w'_1, y_5) with $w'_1 \in N_1$, that can replace P_1. There is a similar path if y_5 is adjacent to z_1. If y_0 and y_5 are adjacent to consecutive vertices in the interior of P_2, then there is also a path of length 3 that can replace P_1. If y_0 and y_5

have as many as 3 adjacencies in the interior of P_2, then one of the three previous situations must occur. Since the path P_2 can be replaced by a path $(z_0, w_1, y_2, y_3, w_2, z_4)$ of length 5 with $w_1 \in N_0$ and $w_2 \in N_1$, this gives a contradiction to the length of C. Hence, y_0 and y_5 have at most 2 adjacencies in the interior of P_2.

The case when $s = 5$ is completely analogous to the $s = 4$ case. There is a path of length at most 4 that can replace P_1 if y_0 is adjacent to either z_3 or z_4, y_5 is adjacent to either z_1 or z_2, or y_0 and y_5 have adjacencies in the interior of P_2 within a distance 2. If y_0 and y_5 have at least 3 adjacencies in the interior of P_2, then one of these situations will occur. Since the path P_2 can be replaced by a path $(z_0, w_1, y_2, y_3, w_2, z_4)$ of length 5 with $w_1 \in N_1$ and $w_2 \in N_2$, this gives a contradiction to the length of C. This completes the proof of Claim 1. □

Claim 2. $q_4(s) \leq 2$ for $s \leq 4$.

Proof. The argument for Claim 2 mimics the proof for Claim 1. The result if obvious when $s = 2$, since there is at most one interior vertex. When $s = 3$, assume that y_0 and y_4 have a total of at least 3 adjacencies in the interior of P_2. Then, y_0 and y_4 have a common adjacency in the interior of P_2, and so there is a path of length 2 that can replace P_1. As before, since y_2 has an adjacency in either N_0 or N_1, there is a path of length 5 that can replace P_2 and is disjoint from the path of length 2. This new path system has the same length as the original system, but has one more path of length 2, a contradiction. Consider the case when $s = 4$. There is a path of length at most 3 that can replace P_1 if either y_0 is adjacent to z_3, y_4 is adjacent to z_1, or y_0 and y_4 have a common adjacency or are adjacent to consecutive vertices in the interior of P_2. If y_0 and y_4 have at least 3 adjacencies in the interior of P_2, then one of these situations will occur. There is a path of length 5 than can replace P_2 and is disjoint from any of the paths just described. The new path system is no longer than the original system, but has one more path of length at most 3. This gives a contradiction, which completes the proof of Claim 2. □

Claim 3. If $m_3 = m_4 = 0$ and $m_2 > 0$, then there is a path of length 3 associated with C whose endvertices have at most 2 adjacencies to the interior vertices of any of the other paths associated with C.

Proof. Assume this is not true. Then, the endvertices of each path of length 3 have at least 3 common adjacencies in the interior of some other path of length 3 associated with C, and so the endvertices have a common adjacency in the second path. Identify with each path P of length 3 a second path Q for which the endvertices of P have a common adjacency in the interior of Q. This results in a cycle of paths $Q_1, Q_2, \ldots Q_b$ with $b \geq 2$ such that the relation between Q_i and Q_{i+1} taken modulo b is the same as the relationship between P and Q. Replacing the b paths $Q_1, Q_2, \ldots, Q_b$ with the corresponding b paths of length 2 results in a cycle of length less than that of C. This contradiction completes the proof of Claim 3. □

Select two vertices x and y that are endvertices of one of the paths of C of maximum length, say r. If $r \le 2$, then $|C| \le 2k$, giving the required cycle. If $r \ge 3$, then by Claims 1, 2 and 3 the pair x and y of endvertices of a path of C have at most two adjacencies in the interior of any of the paths associated with C. Therefore, by counting the number of adjacencies of x and y is each of the paths of C,

$$\begin{aligned}&2(\lceil n/2\rceil + \lfloor k/2\rfloor + t) \le d(x) + d(y)\\ &\le (n - k - m_1 - 2m_2 - 3m_3 - 4m_4) + 2(m_1 + m_2 + m_3 + m_4) + 2(k-3) + \alpha,\end{aligned}$$

where α is the number of vertices in S adjacent in C to either x or y. This implies that

$$\epsilon_n - \epsilon_k + 2t + 6 \le m_1 - m_3 - 2m_4 + \alpha.$$

Therefore,

$$\begin{aligned}|C| &= k + m_1 + 2m_2 + 3m_3 + 4m_4\\ &= k + 2(m_0 + m_1 + m_2 + m_3 + m_4) - 2m_0 - m_1 + m_3 + 2m_4.\end{aligned}$$

Hence,

$$|C| = 3k - 2m_0 - m_1 + m_3 + 2m_4 \le 3k - 2t - 6 - \epsilon_n - 2m_0 + \alpha + \epsilon_k.$$

Since $m_0 \ge \alpha$, it follows that $|C| \le 3k - 2t - 6 - \epsilon_n$ if either k is even or $m_0 > 0$. Thus the proof is complete except for the case when k is odd and $m_0 = 0$.

Consider the special case when k is odd and $m_0 = 0$. Since k is odd and n is large, some pair of consecutive vertices of S will have a large number (greater than $3k$) of common adjacencies in $G - S$. With no loss of generality we can assume that x_1 and x_2 are the vertices. Consider the new graph $G^* = G + x_1x_2$, which satisfies the same initial conditions as G. A repeat of the previous arguments results in a cycle C^* in G^* satisfying

$$|C^*| = 3k - 2m_0^* - m_1^* + m_3^* + 2m_4^* \le 3k - 2t - 6 - \epsilon_n - 2m_0^* + \alpha^* - \alpha' + \epsilon_k,$$

where $\alpha^* = 1$ if x or y is incident to the edge x_1x_2 and 0 otherwise, and the same is true for α', which represents the change in the sum of the degrees of x and y from G to G^*. In this case $|C^*| \le 3k - 2t - 6 - \epsilon_n - 1$. However, if the edge x_1x_2 is replaced by a path of length 2 from x_1 to x_2 in G that follows from their large common neighborhood, there is a cycle C in G of length at most $3k - 2t - 6 - \epsilon_n$. This completes the proof of Lemma 2 □

An immediate corollary of Lemma 2 is the special case when $t = -1$, (e.g. $\delta(G) \ge \lceil n/2\rceil + \lfloor k/2\rfloor - 1$).

Corollary 1. *If* $4 \le k \le n$, *S is an ordered set of k vertices, and G is a graph of sufficiently large order n with* $\delta(G) \ge \lceil n/2\rceil + \lfloor k/2\rfloor - 1$, *then G contains a cycle of length at most* $\max\{2k, 3k - 4 - \epsilon_n\}$ *encountering the vertices S in the designated order.*

The corollary is particularly noteworthy since it shows that the minimum degree condition implying k-ordered hamiltonian is a candidate to be the minimum degree condition that implies (k, m)-pancyclic ordered for m approximately $3k$.

Before giving a proof of Theorem 3, some additional notation will be introduced. Given a path P, a vertex $x \notin P$ is *insertible* in P, if it is adjacent to two consecutive vertices of P. Thus, if $d_P(x) > \lceil |P|/2 \rceil$, then it is insertible. Also, given two vertices of a cycle C, an edge between these two vertices is an *ℓ-chord* if the distance between the vertices on C is ℓ.

Proof. **(Theorem 3)** We start with the sharpness of each of the conditions. Let S be an ordered set of k vertices in the graph $K_n - \lfloor k/2 \rfloor K_2$ in which the $\lfloor k/2 \rfloor$ missing edges are between pairs of consecutive vertices of S. There will be no cycle of length $m < \lfloor 3k/2 \rfloor$ that encounters the vertices of S in the correct order. Since $\delta(K_n - \lfloor k/2 \rfloor K_2) = n - 2$, this verifies the sharpness of (i). Consider the graph $H = K_n - E(C_k)$, and let S be the ordered set of k vertices associated with the cycle C_k. Note that $\delta(H) = n - 3$ and any cycle of H that encounters the vertices of S in the correct order will have at least $2k$ vertices. This verifies the sharpness of (ii). For the sharpness in (iii), consider the graph $G = \overline{K}_{n/2-1} + ((K_5 - E(C_5) \cup K_{n/2-4})$, and the ordered set S of 5 vertices from the missing cycle C_5 in G. There is a cycle of length 10 but no cycle of length 11 in G that encounters the vertices of S in the correct order. Futhermore, $\delta(G) = n/2 + 1$. For the sharpness of (iv) consider F_2 (see Figure 2) in the case when n is odd, $k = 6$ and $t = -1$. In this graph there is no cycle of length 12 that encounters the vertices of S, derived from the vertices of the missing C_6 in the correct order. Futhermore, $\delta(G) = (n + 5)/2$. Next, consider the graph F_2 with k even. For $-1 \leq t \leq (k - 6 - \epsilon_n)/2$, the smallest cycle in F_2 that

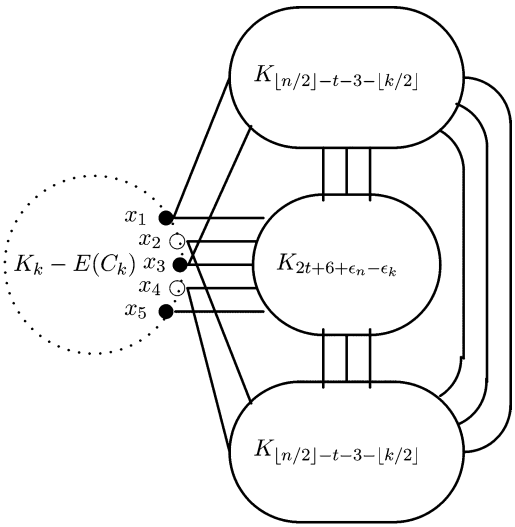

Fig. 2. F_2

encounters the vertices of S derived from the missing cycle C_k in the correct order has length $3k - 2t - 6 - \epsilon_n$. Futhermore, $\delta(F_2) = \lceil n/2 \rceil + \lfloor k/2 \rfloor + t$. If k is odd note that x_k has the same neighborhood as x_1 in F_2. For $t \geq 0$ this verifies the sharpness for (v). The sharpness of the bound on m for (vi) follows from example F_2 and the bound on δ follows from fact that $\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$ is required for G to be k-ordered hamiltonian, given in Theorem 1.

If $\delta(G) \geq n - 1$, then G is complete and is clearly (k, k)-pancyclic ordered. This verifies (i). If $\delta(G) \geq n - 2$, then $G = K_n - pK_2$ for $0 \leq p \leq \lfloor n/2 \rfloor$. Therefore, for $n \geq \lfloor 5k/2 \rfloor$ it is easy to find a cycle of length m for $m \geq \lfloor 3k/2 \rfloor$ that encounters, in the appropriate order, any ordered set of k vertices of G. This verifies (ii).

We will now deal with cases $(iii), (iv), (v)$ and (vi) with the smallest possible value of m in each case. Assume that $\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor + t$, for $-1 \leq t \leq (k - 6 - \epsilon_n)/2$. Let $S = \{x_1, x_2, \ldots, x_k\}$ be an ordered set of k vertices of G that implies that G is not (k, m)-pancyclic ordered. We will show that this leads to a contradiction. Assume that G is an edge maximal graph with respect to not being (k, m)-pancyclic ordered relative to the set S. By Lemma 2 we know there is a cycle of length at most m that encounters the vertices of S in the required order. Select a cycle D of maximal length $p \leq m$ that encounters the vertices of S in the required order. Let $H = G - D$. Once the existence of the necessary small cycles has been verified, which will be accomplished in Claims 1 and 2, the existence of larger cycles will follow in the successive claims. From that point on, it will be sufficient to assume that $\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$, except when $m = 2k + 1$, n is even, and k is odd, or when $m = 2k$, n is odd, and k is even. In these cases $\delta(G) \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor$ is sufficient, and this will complete the special cases of (iii) and (iv).

Claim 1. $p \geq 2k$.

Proof. Assume that $p < 2k$. Since n is large, for each vertex $x \in D$, nearly half of the vertices of H are adjacent to x. Thus, there is a vertex $y \in H$ such that $xy \in G$. By assumption y is not adjacent to two consecutive vertices of D, and so $x^+y \notin G$. This implies that y can be chosen such that $x^{++}y \in G$. First consider the case when $p \leq 2k - 3$. Then, y and x^+ have no common adjacencies in H. Therefore,

$$2(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1) \leq d(y) + d(x^+) \leq (p - 1) + \lfloor p/2 \rfloor + (n - p - 1).$$

This implies $\epsilon_n + 2\lfloor k/2 \rfloor - \lfloor p/2 \rfloor \leq 0$. Since $p < 2k - 2$, this gives $\epsilon_n + 2\lfloor k/2 \rfloor - (k - 2) \leq 0$, a contradiction. Thus, $|D| \geq 2k - 2$.

If $p = 2k - 2$, then x can be chosen so that $x^+ \notin S$. Thus x^+ and y can be interchanged. This implies that x^+ has at most $\lfloor p/2 \rfloor$ adjacencies in D. Since y and x^+ have no common adjacencies in H, this gives the inequality

$$2(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1) \leq d(y) + d(x^+) \leq 2\lfloor p/2 \rfloor + (n - p - 1) < n,$$

a contradiction. Hence we may assume that $p = 2k - 1$.

If x can be chosen so that both x^+ and x^{++} are not in S, then observe that x^{++} can be interchanged with some vertex in H, and thus has at most $\lfloor p/2 \rfloor$ adjacencies in D. Note also that x^{++} and y have no common adjacencies in H, which gives the inequality

$$2(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1) \leq d(y) + d(x^{++}) \leq 2\lfloor p/2 \rfloor + (n - p) < n,$$

a contradiction. Hence we can assume the vertices of S alternate with vertices not in S on D except there are precisely two that are adjacent.

Given an $x \in S$ such that $x^+ \notin S$, then it has been shown that there is a path (x, y, y', x^+) whose interior vertices are in H. Thus, for any $z \in S - x$ with z^- and z^+ not in S, $zz^{++}, zz^{--} \notin G$. Hence, $d_D(z) \leq p - 3$. If $z^{++}w \in G$ with $w \in H$, then w and z have no common adjacency in H, since this would give the existence of a required cycle of length $p + 1$. This implies

$$2(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1) \leq d(w) + d(z^{++}) \leq n - p + \lfloor p/2 \rfloor + p - 3.$$

This implies $\epsilon_n + 2\lfloor k/2 \rfloor - \lfloor p/2 \rfloor + 1 \leq 0$. Since $p \leq 2k - 1$, this gives $\epsilon_n + 2\lfloor k/2 \rfloor - (k - 1) + 1 \leq 0$, a contradiction. Thus, $|D| \geq 2k$, and this completes the proof of Claim 1. □

Note that Lemma 2 and Claim 1 imply that the degree condition in Case (*iv*) is sufficient to get a cycle of length 12. The remainder of the cycles for Case (*iv*) will follow from the case $m = 13$, which is part of Case (*vi*).

Claim 2. $p = m$.

Proof. First consider the case when $m = 2k + 1$, and assume that Claim 2 is not true. By Claim 1 we know that $p = 2k$. If there is a vertex $x \in S$ such that both $x^+, x^{++} \notin S$, then the same proof used in the case $p = 2k - 1$ of Claim 1 can be used here. Hence, on D we can assume that the vertices of S alternate with vertices not in S. As in the proof of the Claim 1, for any $x \in S$, there is a $y \in H$ that is adjacent to both x and x^{++}. Hence y and x^+ can be interchanged, implying that x^+ has at most k adjacencies on D. It follows immediately that y and x^+ have a common adjacency, say $w \in H$, which results in a cycle with $2k + 2$ vertices. This implies that any $z \in S$ cannot be adjacent to either z^{--} or z^{++}, since this gives a cycle of length $2k + 1$. Thus, each vertex $x \in S$ has at most $2k - 3$ adjacencies on D. There is no common adjacency of y and x^{++} in H, since this gives a required cycle of length $2k + 1$. Thus, the following inequality holds, where $\alpha = 1$ when n is even and k is odd, and $\alpha = 0$ otherwise;

$$2(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1 + \alpha) \leq d(y) + d(x^{++}) \leq n - 2k + k + 2k - 3.$$

This implies $\epsilon_n - \epsilon_k + 1 + 2\alpha \leq 0$. When $\alpha = 0$, this gives a contradiction when n is odd or when k is even, and it clearly gives a contradiction when $\alpha = 1$. Thus, we can assume that $p \geq 2k + 1$ and $m > 2k + 1$.

Select consecutive vertices $y_1, y_2, y_3, y_4 \in D$ such that y_2 and y_3 are not in S. Since $p \leq 3k$, there is a vertex $z_1 \in H$ adjacent to y_1 and another vertex $z_2 \in H$ adjacent to y_4. If z_1 and z_2 have a common adjacency, say $z \in H$, then the path (y_1, z_1, z, z_2, y_4) can replace the path (y_1, y_2, y_3, y_4) of D to get a cycle of length $p+1$ with the required property, a contradiction. If this does not occur, then

$$2(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1) \leq d(z_1) + d(z_2) \leq 2\lfloor p/2 \rfloor + (n-p) \leq n,$$

a contradiction. This proves Claim 2. □

Note that Lemma 2, Claim 1, and Claim 2 imply that the degree condition in Case (*iii*) is sufficient to get cycles of length 10 and 11. The remainder of the cycles for Case (*iii*) will follow from the case $m = 12$, which is part of Case (*vi*).

Assume there exist cycles of every length from m to $p < n$ that encounter S in the correct order, but there is no cycle of length $p+1$ with this property. Let $C = C_p$ be such a cycle, and let $H = G - C$. The k vertices of S divide the vertices of C into k disjoint intervals except for endvertices, each starting and ending with a vertex of S.

Claim 3. *Some vertex of C has no adjacencies in H.*

Proof. Assume that this is not true. If $p = 2k$, then the argument of Claim 1 implies the existence of of a cycle of length $2k+1$. Thus, we can assume that $p \geq 2k+1$. We can select consecutive vertices $y_1, y_2, y_3, y_4 \in C$ such that y_2 and y_3 are not in S. First consider the case when there is a vertex $z_1 \in H$ adjacent to y_1 and another vertex $z_2 \in H$ adjacent to y_4. In this case the proof used in Claim 2 can be used here.

We now consider the only other possibility when $z_1 = z_2$. Observe that y_2 and y_3 have no common adjacency in H. Also, if y_2 is adjacent to a vertex of C that precedes (or succeeds) an adjacency of y_3 in C, then the edge y_2y_3 can be inserted into C at a location other than between y_1 and y_4. This cannot occur, since this would result in a cycle of length $p+1$ containing z_1 with the required property. Thus,

$$2(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1) \leq d(y_2) + d(y_3) \leq (n-p) + p \leq n,$$

a contradiction. This completes the proof of Claim 3. □

Select two vertices y and y', if they exist, that are at a minimum distance along C in one of the intervals of C and have a common adjacency, say $z \in H$. Let A be the vertices of C strictly between y and y' in this interval and let $a = |A|$. Thus, none of the vertices A are in S.

Claim 4. *Some vertex in A has an adjacency in H.*

Proof. Suppose not and consider the cycle obtained from C by replacing A by the path (y, z, y'). If all of the vertices of A can be inserted into the path from y' to y in the cycle C, then the required cycle of length $p+1$ exists, which gives a contradiction. If not, then insert as many vertices as possible, and assume you are left with a set $\emptyset \neq B \subseteq A$ of vertices that cannot be inserted into the path with vertices $C - B$. Select a vertex $w \in B$. Since w has no adjacency in H and if we let $b = |B|$, then we have the following inequality:

$$2(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1) \leq d(w) + d(z)$$
$$\leq ((b-1) + (p-b+1)/2) + ((n-p-1) + (p-b+1)/2) < n,$$

a contradiction, completing the proof of Claim 4. □

Claim 5. $|A| = 1$.

Proof. Suppose instead $|A| \geq 2$. If all of the vertices in A are insertible in the path $C - A$, then the required cycle of length $p+1$ is obtained. Assume not, and let $(y_1, y_2, \ldots, y_s)$ be the path of C using vertices in A. Let y_q be the first vertex of A starting from y_1 that is not insertible. Observe that y_q and z must have a common adjacency in H, since if this is not true then we get the following inequality:

$$2(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1) \leq d(y_q) + d(z) \leq (n-p-1) + (a-1) + (p-a+1) < n,$$

a contradiction. Let z_q be such a common adjacency. If $q > 1$, then the required cycle of length $p+1$ is obtained by using the path $(z, z_q, y_q, \ldots)$ to replace the vertices in the path $(y_1, y_2, \ldots, y_{q-1})$ and inserting the vertices $\{y_1, y_2, \ldots, y_{q-2}\}$ of A into $C - A$. Hence, we must have that y_1 is not insertible, and so $q = 1$. Likewise, y_s is not insertible, and there is a vertex $z_s \in H$ that is a common adjacency of y_s and z. If $s = 2$, then the required cycle of length $p+1$ can be obtained by using the path (y, z, z_2, y_2, y') and avoiding the vertex y_1. The required cycle can also be obtained if all of vertices of A strictly between y_1 and y_s can be inserted. Thus, we can assume that $s > 2$, and let y_r be the first vertex past y_1 that is not insertible. Associated with y_r is the vertex $z_r \in H$ that is commonly adjacent to z and y_r. Again, the required cycle is obtained by using the path $(y, z, z_r, y_r, \cdots)$, inserting the vertices strictly between y_1 and y_r and avoiding y_1. Therefore, we can conclude that $|A| = 1$, completing the proof of Claim 5. □

Claim 6. *No vertex of H can have 3 adjacencies in one interval.*

Proof. Assume there is a vertex $z \in H$ with adjacencies y_1, y_2, y_3. By Claim 5 we can assume that there is precisely one vertex on C between y_1 and y_2 and one between y_2 and y_3. Denote these vertices by w_1 and w_2. Neither w_1 nor w_2 is insertible, since this would give the desired cycle of length $p+1$. Also, $w_1 w_2 \notin G$ for the same reason. Therefore, w_1 and w_2 have a common adjacency in H, which we will denote by z', since if this did not occur the following inequality results:

$$2(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1) \leq d(w_1) + d(w_2) \leq (n-p) + p/2 + p/2 \leq n,$$

a contradiction. This implies that y_2 is not insertible for the same reason as w_1 and w_2. Observe that y_2 and z cannot have a common adjacency in H, since this gives a cycle of length $p+1$ avoiding w_1 and using z and the common adjacency. The same argument implies that w_2 and z' do not have a common adjacency in H. This implies the following inequality involving y_2, w_2, z, z':

$$4(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1) \leq d(y_2) + d(z') + d(w_2) + d(z) \leq 2(n-p) + 4(p/2) \leq 2n,$$

a contradiction. Therefore, no vertex of H can have three adjacencies in an interval of C, completing the proof of Claim 6. □

Claim 7. *Two vertices at a distance 3 in the same interval of C cannot both have adjacencies in H.*

Proof. Assume the claim is not true and let $(y_1, y_2, \cdots, y_s)$ be the vertices in some interval such that y_i has an adjacency $z_1 \in H$ and y_{i+3} has adjacency $z_2 \in H$. Observe that $z_1 \neq z_2$ by Claim 5. Also, z_1 and z_2 have a common adjacency in H, say z, by the same count appearing in the first displayed inequality of Claim 6. Replacing the path $(y_i, y_{i+1}, y_{i+2}, y_{i+3})$ by the path $(y_i, z_1, z, z_2, y_{i+3})$ gives the required path with $p+1$ vertices. This contradiction completes the proof of Claim 7. □

Claim 8. *If $z_1, z_2 \in H$ each have two adjacencies in some interval of C, then they have the same two adjacencies in that interval.*

Proof. Assume the claim is not true, let $(y_1, y_2, \ldots, y_s)$ be the vertices in some interval, and suppose that z_1y_i, z_1y_{i+2}, $z_2y_j, z_2y_{j+2} \in G$ with $i < j$. Observe that $z_1 \neq z_2$ by Claim 6. Also, z_1 and z_2 have a common adjacency in H, say z. Both y_{i+1} and y_{j+1} have adjacencies in H by Claim 4. Therefore, by Claim 7, $j \geq i+6$. Let $A = \{y_{i+3}, y_{i+4}, \ldots, y_{j-1}\}$, which has at least 3 vertices, and let P be the path containing the remaining vertices of C. Starting with y_{i+3} and using the natural order of A insert one at a time the vertices of A into P or into the present path obtained from P from inserting the previous vertices of A. If all of the vertices of A can be inserted, then a C_{p+1} cycle can be constructed using the path $(y_{i+2}, z_1, z, z_2, y_j)$ and inserting all of the vertices of A into P. If all of the vertices of A cannot be inserted, then let y_q be the first vertex that cannot be inserted. Let $B = \{y_q, y_{q+1}, \ldots, y_{j-1}\}$ with $b = |B|$. There must be some common adjacency, say z', of y_q and z_1, for otherwise the following inequality results:

$$\begin{aligned} 2(\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1) &\leq d(y_q) + d(z_1) \leq (n-p-1) + (b-1) + 2((p-b+1)/2) \\ &< n, \end{aligned}$$

a contradiction. By Claim 7, $q \geq i+6$. A C_{p+1} can be constructed by using the path (y_{i+2}, z_1, z', y_q) and inserting all of the vertices of $A - B$ except for y_{q-1}. This gives a contradiction that completes the proof of Claim 8. □

Since, by Claim 3, C has a vertex with no adjacencies in H, we see that $|C| \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor$ and $|H| \leq n - \lceil n/2 \rceil - \lfloor k/2 \rfloor$. Claim 6 implies that no vertex of H has more than $2k$ adjacencies in C; hence $|H| \geq \lceil n/2 \rceil + \lfloor k/2 \rfloor - 2k$. This results in the following inequalities:

$$\lceil n/2 \rceil + \lfloor k/2 \rfloor \le |C| \le n - \lceil n/2 \rceil - \lfloor k/2 \rfloor + 2k,$$

and

$$\lceil n/2 \rceil + \lfloor k/2 \rfloor - 2k \le |H| \le n - \lceil n/2 \rceil - \lfloor k/2 \rfloor.$$

Claim 9. $\lceil n/2 \rceil + \lfloor k/2 \rfloor \le |C| \le \lceil n/2 \rceil + \lfloor k/2 \rfloor + 1$.

Proof. When $|C| \ge \lceil n/2 \rceil + \lfloor k/2 \rfloor + 2$, then $|H| \le n - \lceil n/2 \rceil - \lfloor k/2 \rfloor - 2$. This implies that each vertex of H is adjacent to at least

$$\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1 - |H| + 1 \ge \epsilon_n + k - \epsilon_k + 2 \ge k + 1$$

vertices of C. Therefore, in this case each vertex of H will have two adjacencies in some interval of C. Since n is large, $|H|$ is large and so by Claims 5 and 8 there will be a set R of at least $k+3$ vertices of H that are adjacent to the same pair of vertices at a distance 2 in some interval of C. No pair of vertices of R can be adjacent, since this would imply a cycle C_{p+1}, a contradiction. This implies that $|H| \ge \lceil n/2 \rceil + \lfloor k/2 \rfloor - 1 + (k+3) - 2k$. This contradicts the fact that $|H| \le n - \lceil n/2 \rceil - \lfloor k/2 \rfloor$, which completes the proof of Claim 9. □

Claim 10. $|C| \ne \lceil n/2 \rceil + \lfloor k/2 \rfloor + \alpha$ *for* $\alpha = 0$ *or* 1.

Proof. Assume that $|C| = \lceil n/2 \rceil + \lfloor k/2 \rfloor + \alpha$ for $\alpha = 0$ or 1. Then $\delta(G)$ and $|C|$ imply that each vertex of C with no adjacencies in H will be adjacent to all of the other vertices of C except for possibly α. By Claim 7, nearly one half of the vertices of C have no adjacencies in H. Also, each vertex in H will have at least $k - 1 + \alpha$ adjacencies in C and because of Claim 6, will have no more than $2k$ adjacencies in C.

We will first show that at most one vertex in an interval of C can have adjacencies in H. Assume not. Then select two vertices y_i and y_{i+s} with $s > 0$ in some interval of C with adjacencies in H, say z_1 and z_2 respectively (possibily $z_1 = z_2$). Select s as small as possible, so that none of the vertices $\{y_{i+1}, \ldots, y_{i+s-1}\}$ between y_i and y_{i+s} have adjacencies in H. All of the vertices in $\{y_{i+1}, \ldots, y_{i+s-1}\}$ are insertible in the path of C between y_{i+s} and y_i. Thus, if $z_1 = z_2$ there is a required cycle of length $p+1$ using the path (y_i, z_1, y_{i+s}) and inserting the vertices of $\{y_{i+1}, \ldots, y_{i+s-1}\}$ into the path of C between y_{i+s} and y_i. If $z_1 \ne z_2$, then there is common adjacency, say $z' \in H$, of z_1 and z_2 since n is large. Using the path $(y_i, z_1, z', z_2, y_{i+s})$ and again inserting the vertices of $\{y_{i+1}, \ldots, y_{i+s-1}\}$ will give a cycle of length $p+1, p+2$, or $p+3$ with the required properties. However, the cycles of length $p+2$ or $p+3$ can be reduced to a cycle of length $p+1$, since there are many vertices of C adjacent to all of the other vertices of C except for possibly one giving many chords of length 2. This gives a contradiction, which implies there is a most one vertex in each of the k intervals of C with an adjacency in H. □

The previous conclusion implies that each vertex of H has between $k - 1 + \alpha$ and k adjacencies in C and is adjacent to all of the other vertices of H except for possibily $1 - \alpha$. Also, all of the vertices of C except for at most k have no adjacencies in H and are adjacent to all but at most α vertices of C. If $\alpha = 0$, then there is a vertex $z \in H$ with adjacencies y_1 and y_2 in the interior of consecutive intervals of C. The edge $y_1^- y_2^- \in G$, which implies that the cycle $C' = (y_1^-, y_2^-, y_2^{--}, \ldots, y_1, z, y_2, y_2^+, \ldots, y_1^-)$ is a cycle of length $p+1$ with the required property. If $\alpha = 1$, then there is a, in fact any, vertex $z \in H$ with adjacencies y_1, y_2 and y_3 in the interior of consecutive intervals of C. Either the edge $y_1^- y_2^- \in G$ or the edge $y_2^- y_3^- \in G$. In either case a cycle of length $p+1$ with the required property can be formed just as in the case when $\alpha = 0$. This gives a contradiction, which completes the proof of Claim 10.

Hence with all cases exhausted, it must be the case that G is (k, m)-pancyclic ordered, completing the proof of Theorem 3. □

3. Questions

In the statement of Theorem 3 the order n of the graph G is sufficiently large. This is a consequence of the proof and not of examples for small order graphs. It would be of interest to show that the statements of Theorem 3 are valid for all $n \geq 2k$. In particular it would be of interest to know if statement (vi) of Theorem 3 is valid for all $n \geq 2k$.

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