A Note on 2-Factors in Line Graphs

Ronald J. Gould
Emory University
Atlanta GA 30322

Emily A. Hynds
Samford University
Birmingham AL 35229

Abstract

A 2-factor of a graph G consists of a spanning collection of vertex disjoint cycles. In particular, a hamiltonian cycle is an example of a 2-factor consisting of precisely one cycle. Harary and Nash-Williams characterized graphs with hamiltonian line graphs. Gould and Hynds generalized this result, characterizing those graphs whose line graphs contain a 2-factor with exactly k ($k \geq 1$) cycles. With this tool we show that certain properties of a graph G, that were formerly shown to imply the hamiltonicity of the line graph, $L(G)$, are actually strong enough to imply that $L(G)$ has a 2-factor with k cycles for $1 \leq k \leq f(n)$, where n is the order of the graph G.

1 Introduction

All graphs considered in this paper are simple graphs. For terms or notation not defined here, see [4]. For a graph G, let $N(v)$ denote the neighborhood of vertex v. A set $S \subseteq V(G)$ is said to be *independent* if $uv \notin E(G)$ for every $u, v \in S$. The *independence number* of a graph G, denoted $\alpha(G)$, is the size of a largest independent set of vertices of G. For a set $S \subseteq V(G)$ we use $\langle S \rangle$ to denote the subgraph induced by S.

A *circuit* of G is an alternating sequence $C : v_1, e_1, v_2, e_2, ..., v_m, e_m, v_1$ of vertices and edges of G, such that $e_i = v_i v_{i+1}$, $i = 1, 2, ..., m-1$, $e_m = v_m v_1$, and $e_i \neq e_j$ if $i \neq j$. A circuit whose m vertices v_i are distinct is called a *cycle*.

We define a *dominating circuit* of a graph G to be a circuit of G with the property that every edge of G either belongs to the circuit or is adjacent to an edge of the circuit.

A *star* is the complete bipartite graph $K_{1,n}$. The vertex of degree n is termed the *center* of the star and the vertices of degree 1 are the *leaves*. If a star has center w we often denote it as S_w. Further, if we wish to specify a star centered at w with some specific leaves, say a, b, c, we will denote it by $S_w(a, b, c)$. Note that there may be other leaves in S_w not specified.

The subgraph H of G is said to be a *2-factor of* G if H spans G and for every $v \in V(H)$, $\deg_H : v = 2$. A trivial consequence of the definition is that every 2-factor of a graph G consists of a spanning collection of vertex disjoint cycles. In particular, a hamiltonian cycle is an example of a 2-factor consisting of precisely one cycle.

Early studies of 2-factors centered on the question of existence, often of simply a hamiltonian cycle. More recently the focus in the area of 2-factors has shifted from the problem of showing the existence of a 2-factor to that of showing the existence of 2-factors with specific structural features. In 1978 Sauer and Spencer [8] made the following conjecture along those lines.

Conjecture 1 *Let H be any graph on n vertices with maximum degree $\Delta \leq 2$. If G is a graph on n vertices with minimum degree $\delta(G) > 2n/3$ then G contains an isomorphic copy of H.*

In 1993 Aigner and Brandt [1] settled Conjecture 1 with a slight improvement.

Theorem 1 *Let G be a graph of order n with $\delta(G) \geq (2n-1)/3$, then G contains any graph H of order at most n with $\Delta(H) \leq 2$.*

In the above result the minimum degree must be very high to guarantee that a graph contains all possible 2-factors or 2-factors with a particular structure. Thus, a more relaxed question would be: is there a lesser degree condition that will imply the existence of 2-factors with k cycles for a range of k.

The following was shown in [2].

Theorem 2 *Let k be a positive integer and let G be a graph of order n. If $deg(x)+deg(y) \geq n$ for all $x, y \in V(G)$ such that $xy \notin E(G)$, then G contains a 2-factor with k cycles for all k, $1 \leq k \leq \lfloor n/4 \rfloor$.*

Note that Theorem 2 is a generalization of the classic hamiltonian result of Ore [7] for the case when $n \geq 4k$. The complete bipartite graph $K_{n/2,n/2}$ shows that this result is best possible.

This type result naturally leads to the question of whether or not other hamiltonian results can be extended in a similar manner.

The following is the well-known result of Harary and Nash-Williams [6] characterizing graphs with hamiltonian line graphs.

Theorem 3 *Let G be a graph without isolated vertices. Then $L(G)$ is hamiltonian if, and only if, $G \simeq K_{1,n}$, for some $n \geq 3$, or G contains a dominating circuit.*

Given a graph G, we say that G contains a *k-system that dominates* if G contains a collection of k edge disjoint circuits and stars, $(K_{1,n_i}, n_i \geq 3)$, such that each edge of G is either contained in one of the circuits or stars, or is adjacent to one of the circuits.

We will use a generalization of Theorem 3 that allows us to characterize those graphs whose line graphs contain a 2-factor with exactly $k (k \geq 1)$ cycles.

Theorem 4 (Gould, Hynds[5]) *Let G be a graph with no isolated vertices. The graph $L(G)$ contains a 2-factor with k $(k \geq 1)$ cycles if, and only if, G contains a k-system that dominates.*

The following result gives specific conditions on a graph G that imply that the line graph $L(G)$ is hamiltonian. Our goal is to generalize this result.

Theorem 5 (Brualdi, Shanny[3]) *Let G be a graph with $n \geq 4$ vertices and at least one edge. Suppose that for each edge $xy \in E(G)$, $deg(x) + deg(y) \geq n$, then $L(G)$ is hamiltonian.*

2 Extension

We now show that this same condition actually implies much more.

Theorem 6 *Let G be a graph with $n \geq 4$ vertices and at least one edge. Suppose that for each edge $xy \in E(G)$, $deg(x) + deg(y) \geq n$. Then $L(G)$ has a 2-factor with k cycles for $k = 1, ..., \lfloor \frac{n-2}{4} \rfloor$.*

Proof: Let G be as in the theorem. We know from Theorem 5 that $L(G)$ is hamiltonian, thus the result holds when $k = 1$. We will proceed by induction on k. Suppose that $L(G)$ has a 2-factor with $k - 1$ cycles for $k \leq \lfloor \frac{n-2}{4} \rfloor$. We want to show that $L(G)$ then also has a 2-factor with k cycles. Suppose, by way of contradiction, that $L(G)$ does not have a 2-factor with k cycles. We know by Theorem 4 that G does have a dominating $(k - 1)$-system, but does not have a dominating k-system. Let $xy \in E(G)$, and consider a dominating $(k - 1)$-system of G. We will let i be the number of stars in this system and thus $k - i - 1$ is the number of circuits.

Claim 1 *All stars in this system have at most 5 edges.*

Proof: Suppose there is a star with six or more edges. Then we can separate the star into two smaller stars with at least 3 edges each. This gives us a dominating k-system in G and a contradiction. □

Claim 2 *The circuits in this system must be cycles.*

Proof: Suppose there is a circuit in the system that is not a cycle. Then we can separate the circuit into 2 edge disjoint circuits which again gives us a dominating k-system and a contradiction. □

Now consider a vertex $v \in V(G)$. If v is the center of a star in our system then that star contributes at least 3 to the degree of v. If the star actually consists of 4 (or 5) edges then we will choose 1 (or 2) of those edges and say they are moveable. We say they are moveable because if v appears elsewhere in our system, as the center of another star or as a vertex on a cycle, we can move the edge(s) to that location of v without changing the basic structure of our system. By this we mean that after moving the edge we still have a $(k - 1)$-system with i stars and $k - 1 - i$ cycles. If v is incident to an edge that is dominated by a cycle in our system, we will call that edge moveable as well.

Claim 3 *A vertex v in our $(k - 1)$-system can be adjacent to at most 2 moveable edges.*

Proof: Suppose we have a vertex v that is adjacent to 3 or more moveable edges. We can use those edges to form a new star, centered at v, which when added to the $(k - 1)$-system that remains gives us a dominating k-system and a contradiction. □

Now we will use the results of these 3 claims to establish upper bounds for $deg(x)$ and $deg(y)$. Let l be the number of stars in our system that have x as the center and m the number of stars in our system that have y as the center. Thus we have $i - l - m$ stars in our system that have neither x nor y as the center. For both x and y we will remove the moveable edges and count them separately. We now consider the maximum degree x can have. Each star with x at the center contributes a total of 3 to $deg(x)$. The entire collection of stars with y at the center contributes at most 1 to $deg(x)$. And the remaining stars, with neither x nor y at the center, each contribute at most 1 to $deg(x)$. Each cycle contributes at most 2 to $deg(x)$ and finally there are at most 2 moveable edges incident with x. Therefore, $deg(x) \leq 3l+1+1(i-m-l)+2(k-i-1)+2$. Similarly, $deg(y) \leq 3m+1+1(i-m-l)+2(k-i-1)+2$.

Now, the edge xy appears only once in the system so it cannot be the case that y is found on a star with center x and x is found on a star with center y. Hence, we may subtract one from our degree sum. It follows then that

$$\begin{aligned} deg(x) + deg(y) &\leq 3l + 3m + 1 + 2(i - m - l) + 4(k - i - 1) + 4 \\ &= l + m - 2i + 4k + 1 \\ &\leq i - 2i + 4k + 1 \\ &\leq 4k + 1. \end{aligned}$$

But $n \leq deg(x) + deg(y)$ which implies that $n \leq 4k + 1$ and thus $k \geq \frac{n-1}{4}$. But this contradicts our original assumption that $k \leq \lfloor \frac{n-2}{4} \rfloor$ which means that $L(G)$ does have a 2-factor with k cycles. □

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