GRAPHS WITH PRESCRIBED DEGREE
SETS AND GIRTH

by

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Abstract

For a finite nonempty set \( \mathcal{S} \) of integers, each of which is at least two, and an integer \( n \geq 2 \), the number \( f(\mathcal{S}; n) \) is defined as the least order of a graph having degree set \( \mathcal{S} \) and girth \( n \). The number \( f(\mathcal{S}; n) \) is evaluated for several sets \( \mathcal{S} \) and integers \( n \). In particular, it is shown that \( f(\{2, 4\}; 2) = 18 \) and \( f(\{3, 4\}; 3) = 18 \).

For integers \( r \geq 2 \) and \( n \geq 3 \), the integer \( f(r, n) \) is defined as the smallest order of an \( r \)-regular graph having girth \( n \) (the girth being the length of the smallest cycle in the graph). Erdős and Sachs [1] have shown that \( f(r, n) \) exists for all integers \( r \geq 2 \) and \( n \geq 3 \). The problem of evaluating \( f(r, n) \) for various values of \( r \) and \( n \) has received considerable attention. The \( r \)-regular graphs having girth \( n \) and order \( f(r, n) \) are known as \( (r, n) \)-cages. The object of this paper is to extend the function \( f(r, n) \) and the \( (r, n) \)-cages.

The degree set \( \mathcal{D} = \{d_1, d_2, \ldots, d_k\} \) of a graph \( G \) is the set of degrees of the vertices of \( G \). We henceforth assume for \( \mathcal{D} = \{d_1, d_2, \ldots, d_k\} \) that \( d_1 < d_2 < \ldots < d_k \).

For a set \( \mathcal{S} = \{s_1, s_2, \ldots, s_k\} \) of integers with \( 2 \leq s_1 < s_2 < \ldots < s_k \) and for an integer \( n \geq 3 \), we define

\[
f(\mathcal{S}; n) = f(s_1, s_2, \ldots, s_k; n)
\]

to be the smallest order of a graph having girth \( n \) and degree set \( \mathcal{S} \). The existence of \( f(\mathcal{S}; n) \) is guaranteed by the above result of Erdős and Sachs. In particular, if \( H_i \) is an \( s_i \)-regular graph of girth \( n \), where \( V(H_i) \cap V(H_j) = \emptyset \) \( (i \neq j) \), then the graph \( G \) defined by

\[
V(G) = \bigcup_{i=1}^{k} V(H_i) \quad \text{and} \quad E(G) = \bigcup_{i=1}^{k} E(H_i)
\]

has degree set \( \mathcal{S} \) and girth \( n \). We shall refer to a graph \( G \) of order \( f(\mathcal{S}; n) \) having degree set \( \mathcal{S} = \{s_1, s_2, \ldots, s_k\} \) and girth \( n \) as an \( (\mathcal{S}; n) \)-cage or an \( (s_1, s_2, \ldots, s_k; n) \)-cage.

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In [2] Kapoor, Polimeni and Wall showed that for a given set \( \mathcal{D} = \{a_1, a_2, \ldots, a_k\} \) of positive integers (with \( a_1 < a_2 < \cdots < a_k \)), the minimum order of a graph \( G \) with degree set \( \mathcal{D} \) is \( 1 + a_k \). If \( v \) is a vertex of degree \( a_k \) in a graph \( G \) with degree set \( \mathcal{D} = \mathcal{D} \) containing no vertices of degree 1, then there must be two adjacent vertices which are themselves adjacent to \( v \), producing a 3-cycle. This gives the following observation.

**Theorem 1.** If \( \mathcal{D} = \{a_1, a_2, \ldots, a_k\} \) is a set of positive integers with \( 2 \leq a_1 < a_2 < \cdots < a_k \), then \( f(\mathcal{D}; 3) = 1 + a_k \).

The difficulty of evaluating \( f(\mathcal{D}; n) \) appears to increase sharply when \( n > 3 \). By placing restrictions on \( \mathcal{D} \), however, we are able to determine \( f(\mathcal{D}; n) \) in certain cases. In particular, if \( \mathcal{D} \) has cardinality two and \( a_1 = 2 \), the following result can be obtained.

**Theorem 2.** For \( m \geq 3, n \geq 3 \),

\[
f(2, m; n) = \begin{cases} 
\frac{m(n - 2) + 4}{2} & \text{if } n \text{ is even}, \\
\frac{m(n - 2) + 2}{2} & \text{if } n \text{ is odd}.
\end{cases}
\]

**Proof.** We observe that \( f(2, m; n) \leq 2 + m(n - 2)/2 \) for \( n \) even and \( (2, m; n) \leq 1 + m(n - 1)/2 \) if \( n \) is odd, since the graphs \( G_1 \) and \( G_2 \) of Fig. 1 have degree set \( \{2, m\} \), girth \( n \) and the appropriate orders.

![Fig. 1. The \((2, m; n)\)-cages for \( n \) even and for \( n \) odd](image)

The graphs \( G_1 \) and \( G_2 \) are examples of \((2, m; n)\)-cages, where \( G_1 \) has girth 4 and order \( 2m + 2 \) if \( n \) is even, or \( 2m + 1 \) if \( n \) is odd. \( G_2 \) has girth 6 and order \( 3m + 2 \) if \( n \) is even, or \( 3m + 1 \) if \( n \) is odd.
Now suppose \( n \geq 4 \) is an even integer and let \( v \) be a vertex of degree \( m \) in a graph \( G \) having degree set \( \{2, m\} \) and girth \( n \). Since \( n \geq 4 \), the vertices \( v_{n,1}, v_{n,2}, \ldots, v_{n,m} \) adjacent to \( v \) are distinct and pairwise non-adjacent; therefore, \( G \) contains more than \( m + 1 \) vertices, which gives the desired result for \( n = 4 \). Thus, we assume \( n \geq 6 \). Since \( \emptyset = \{2, m\} \), each vertex \( v_{n,i} \) (\( i = 1, 2, \ldots, m \)) is adjacent to at least one new vertex \( v_{n,1} \). Since \( n \geq 6 \), the vertices \( v_{n,1}, v_{n,2}, \ldots, v_{n,m} \) are distinct and pairwise non-adjacent, so that \( G \) has order at least \( 2m + 2 \), which gives the required result for \( n = 6 \).

If \( n \geq 8 \) we repeat the above process until the vertices

\[
v_{n-2,1}, v_{n-2,2}, \ldots, v_{n-2,m}, n
\]

have been added (see Fig. 1a). These vertices are distinct and pairwise non-adjacent, for otherwise, an \((n - 1)\)-cycle is produced. Thus, \( G \) has order at least \( 2 + m(n - 2)/2 \), i.e.,

\[
f(2, m; n) \geq 2 + m(n - 2)/2,
\]

which completes the proof of the theorem if \( n \) is even.

The argument if \( n \) is odd is similar and is omitted.

Another case in which \( f(\emptyset, n) \) can be evaluated rather easily occurs when \(|\emptyset| = 2\) and \( n = 4\).

**Theorem 3.** For \( 2 \leq r < s \),

\[
f(r, s; 4) = r + s.
\]

**Proof.** The complete bipartite graph \( K(r, s) \) has degree set \( \{r, s\} \) and girth four; hence \( f(r, s; 4) \leq r + s \).

In order to show that \( f(r, s; 4) \geq r + s \), let \( G \) be a graph with degree set \( \{r, s\} \) and girth four. Let \( u \in V(G) \) such that \( \deg u = s \). Let \( v_1, v_2, \ldots, v_s \) be the \( s \) vertices adjacent to \( u \). Since \( G \) has no 3-cycles, \( \langle v_1, v_2, \ldots, v_s \rangle \) contains no edges. Since the degree of \( v_i \) is at least \( r \) and \( v_i \) is not adjacent to any of \( v_1, v_2, \ldots, v_s \), at least \( r \) other vertices must be present in \( G \), i.e., \( |V(G)| \geq \geq r + s \). Hence \( f(r, s; 4) \geq r + s \), giving the desired result.

Since it is well known that \( f(r; 3) = 2r \), the above result could be extended to include the case \( r = s \).

Due to the difficulty of determining \( f(r, n) \) when \( n \geq 5 \), it is probably not surprising that the problem of evaluating \( f(\emptyset, n) \) when \(|\emptyset| = 2\) and \( n \geq 5 \) seems to be extremely difficult. We now consider this problem when \( \emptyset = \{3, 4\} \) and \( n = 5 \) or \( n = 6 \).
Theorem 4. \( f(3, 4; 5) = 13 \).

Proof. Let \( G \) be a graph with degree set \( \{3, 4\} \) and girth 5. Let \( v \) be a vertex of degree 4 in \( G \), and let \( v_0, v_1, v_2, v_3 \) be the vertices adjacent to \( v \). Since \( G \) contains no 3-cycles, no two of the vertices \( v_0, v_1, v_2, v_3 \) are adjacent to each other. Since every vertex of \( G \) has degree 3 or 4, the vertex \( v_i \) (\( i = 0, 1, 2, 3 \)) is adjacent to at least two vertices different from \( v \), say \( v_{i,1} \) and \( v_{i,2} \). Further, since \( G \) contains no 4-cycles, for \( i \neq j \) we have \( v_{i,1} \neq v_{j,1} \) when \( i, j \in \{0, 1, 2, 3\} \) and \( i, j \in \{1, 2\} \). Thus \( G \) contains at least 13 vertices so that \( f(3, 4; 5) \geq 13 \).

To show that \( f(3, 4; 5) = 13 \), it now suffices to verify the existence of a graph of order 13 having girth 5 and degree set \( \{3, 4\} \). To the graph partially constructed above, add the edges

\[ v_{i,1} v_{i+1,1}, v_{i,1} v_{i+2,1}, v_{i,1} v_{i+3,1}, v_{i,2} v_{i+1,1}, \text{ and } v_{i,3} v_{i+3,1} \]

for \( i = 0, 1, 2, 3 \), where \( i + 1, i + 2 \) and \( i + 3 \) are expressed as 0, 1, 2 or 3 modulo 4. The graph \( H \) so described is shown in Figure 2. The graph \( H \) has order 13 and \( \mathcal{B}_2 = \{3, 4\} \). Also \( v, v_0, v_{0,1}, v_{0,2}, v_0, v \) is a 5-cycle of \( H \). It remains only to show that \( H \) contains no 3-cycles or 4-cycles. It is straightforward to see that \( H \) has no 3-cycle or 4-cycle containing any vertex in the set \( U = \{v, v_0, v_1, v_2, v_3\} \). If \( H \) contains a 3-cycle or 4-cycle, all vertices of such a cycle must belong to the set \( V(H) - U \). Such a cycle \( C \) must contain a vertex \( v_{i,2} \) for \( i = 0, 1, 2 \) or 3. Thus, \( C \) must contain the path

\[ v_{i,1}, v_{i+1,2}, v_{i+2,1}, v_{i+3,1}, v_{i+4,2} \]

or the path

\[ v_{i,1}, v_{i+1,2}, v_{i+2,1}, v_{i+3,1}, v_{i+4,2} \]

which cannot occur if \( G \) has length 3 or 4. Thus \( G \) has girth 5.
Theorem 5. \( f(3, 4; 6) = 18 \).

Proof: Let \( G \) be a graph with degree set \( \{3, 4\} \) and girth six. Let \( v \) be a vertex of degree 4 in \( G \) and let \( u_0, u_1, u_2, u_3 \) be the vertices adjacent to \( v \). Since \( G \) contains no 3-cycles, no two of the vertices \( u_0, u_1, u_2, u_3 \) are adjacent. Since every vertex of \( G \) has degree 3 or 4, the vertex \( v_i (i = 0, 1, 2, 3) \) is adjacent to at least two vertices different from \( v \), say \( v_{i,j} \) and \( v_{i,j'} \), for \( i \neq j, j' \). Further, since \( G \) contains no 4-cycles, for \( i \neq j \), we have \( v_{i,j} \neq v_{j,i} \) where \( i, j \in \{0, 1, 2, 3\} \) and \( i \neq j \).

Again, each \( v_{i,j} \) (\( i = 0, 1, 2, 3; \ j = 1, 2 \)) has degree at least three. Thus only these 17 vertices, then each \( u_k (k = 0, 1, 2, 3) \) must have degree 4 and be adjacent to exactly four of the vertices \( v_{i,j} \) (\( i = 0, 1, 2, 3; \ j = 1, 2 \)). If \( G \) has \( \text{adjacent to both } v_{i,j} \text{ and } v_{i,j'} \text{ a } 4\text{-cycle is produced. Thus each } u_k \text{ must be adjacent to exactly one of } v_{i,j} \text{ and } v_{i,j'} \text{ (}i = 0, 1, 2, 3; \ j = 1, 2\), thereby producing a 4-cycle. Thus , the existence of a graph of order 18 having girth 6 and degree set \( \{3, 4\} \). To the graph partially constructed above, add the edges

\[ u_i v_0, u_i v_{i+1,2}, u_i v_{i+3,2}, u_i v_{i+1,3}, u_i v_{i+2,3}, u_i v_{i+3,4} \]

where the subscripts \( i + 1 \) and \( i + 2 \) are expressed as 0, 1, 2, or 3 modulo 4.

Observe that \( v, v_0, v_{0,1}, v_1, v_{1,2}, v_2, v_{3,2} \) is a 6-cycle. It remains only to show that \( G \) contains no \( r \)-cycle for \( 3 \leq r \leq 5 \). It is straightforward to see that \( G \)

![Fig. 3. Two drawings of a (3, 4; 6)-cage](image-url)
contains no such cycle containing any vertex in the set $M = \{v, v_o, v_1, v_2, v_3\}$. If $G$ contains a cycle of length five or less, all vertices of such a cycle must belong to the set $V(G) - M$.

Such a cycle $C$ must contain a vertex $u_i$ ($i = 0, 1, 2, 3$). Thus $C$ must contain one of the following paths:

1. $u_i, v_{i,3}, u_i, v_{i,3}, u_{i+2}$
2. $u_i, v_{i,3}, u_{i+1}, v_{i,3}, u_{i+2}$
3. $u_i, v_{i,3}, u_{i+1}, v_{i,3}, u_{i+2}$, $u_i$
4. $u_i, v_{i,3}, u_{i+1}, v_{i,3}, u_{i+2}$
5. $u_i, v_{i,3}, u_{i+1}, v_{i,3}, u_{i+2}$

where $i = 0, 1, 2, 3$ and $k = 1, 2, 3$ and all subscripts are expressed modulo 4. Since these paths do not extend to a cycle of length less than six, the graph $G$ has girth six. Also $\Delta G = \{3, 4\}$. Thus $f(3, 4; 6) = 18$. 

REFERENCES


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