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Minimum degree and the minimum size of K_2^t-saturated graphs

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Abstract

A graph G is said to be F-saturated if G does not contain a copy of F as a subgraph and $G+e$ contains a copy of F as a subgraph for any edge e contained in the complement of G. Erdős et al. in [A problem in graph theory, Amer. Math. Monthly 71 (1964) 1107–1110.] determined the *minimum* number of edges, $\text{sat}(n, F)$, such that a graph G on n vertices must have when F is a t-clique. Later, Ollmann [$K_{2,2}$-saturated graphs with a minimal number of edges, in: Proceedings of the Third SouthEast Conference on Combinatorics, Graph Theory and Computing, 1972, pp. 367–392.] determined $\text{sat}(n, F)$ for $F=K_{2,2}$. Here we give an upper bound for $\text{sat}(n, F)$ when $F=K_t^2$ the complete t-partite graph with partite sets of size 2, and prove equality when G is of prescribed minimum degree.

Keywords: Saturated graphs; Minimum size; Minimum degree

1. Introduction

We let $G=(V, E)$ be a graph on $|V|=n$ vertices and $|E|=m$ edges. We denote the complete graph on t vertices by K^t, and the complete multipartite graph with t partite sets each of size s by K_s^t. Let $F=(V', E')$ be a graph on $|V'|\leqslant n$ vertices. The graph G is said to be F-saturated if G contains no copy of F as a subgraph, but for any edge e in the complement of G, the graph $G+(e)$ contains a copy of F, where $G+(e)$ denotes the graph $(V, E\cup e)$. The celebrated theorem of Turán determines the *maximum* number of edges in a graph that is K^t-saturated. This number, denoted $\text{ex}(n, K^t)$, arises from the consideration of the so-called Turán graph. In 1964 Erdős et al. [4] determined the *minimum* number of edges in a graph that is K^t-saturated. This number, denoted $\text{sat}(n, K^t)$, is $(t-2)(n-1)-\binom{t-2}{2}$ and arises from the split graph $K^{t-2}+\overline{K}^{n-t+2}$. Some years later Ollmann [6] determined the value $\text{sat}(n, K_{2,2})$. Tuza gave a shortened proof of this same result in [9]. Determining the exact value of this function for a given graph F has been quite difficult, and is known for relatively few graphs. Kászonyi and Tuza in [5] proved the best known general upper bound for $\text{sat}(n, F)$.

We will say $u\sim v$ (respectively, $u\not\sim v$) if $(uv)\in E(G)$ (respectively, $(uv)\notin E(G)$). For any undefined terms we refer the reader to [1].

Theorem 1 (*L. Kászonyi, Z. Tuza* [5]). *Let* $\mathscr{F}$ *be a family of non-empty graphs. Set*

$$u=\min\{|U| : F\in\mathscr{F}, U\subset V(F), F-U \textit{ is a star (or a star with isolated vertices)}\}$$

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and

$$s = \min\{|E(F-U)| : F \in \mathscr{F}, U \subset V(F), F-U \text{ is a star and } |U| = u\}.$$

Furthermore, let p be the minimal number of vertices in a graph $F \in \mathscr{F}$ for which the minimum s is attained. If $n \geqslant p$ then

$$\mathrm{sat}(n, \mathscr{F}) \leqslant \left(u + \frac{s-1}{2}\right)n - \frac{u(s+u)}{2}.$$

This result shows that $\mathrm{sat}(n, \mathscr{F}) = O(n)$ where $\mathscr{F}$ is a family of graphs. Pikhurko [7] generalized this result to a family, $\mathscr{F}'$, of k-uniform hypergraphs by showing that $\mathrm{sat}(n, \mathscr{F}') = O(n^{k-1})$. For a further summary of related results we refer the reader to [2].

Here we further refine the idea of $\mathrm{sat}(n, F)$. To state the main result of this paper we define $\mathrm{sat}(n, F, \delta)$ to be the *minimum* number of edges in a graph on n vertices and minimum degree δ that is F-saturated. We show the following two results.

Theorem 2. *For integers $t \geqslant 3$, $n \geqslant 4t-4$,*

$$\mathrm{sat}(n, K_2^t, 2t-3) = \left\lceil \frac{(4t-5)n - 4t^2 + 6t - 1}{2} \right\rceil.$$

This immediately implies the following.

Theorem 3. *For integers $t \geqslant 3$, $n \geqslant 4t-4$,*

$$\mathrm{sat}(n, K_2^t) \leqslant \left\lceil \frac{(4t-5)n - 4t^2 + 6t - 1}{2} \right\rceil.$$

It is worth noting that the bound provided by Theorem 3 is a slight improvement over that provided by Theorem 1. We also make the following conjecture.

Conjecture 1. For integers $t \geqslant 3$, n sufficiently large, equality holds in Theorem 3.

2. General results

To prove Theorem 2 we will find the following results which are due to Tuza [9] to be useful.

Proposition 1 (*Tuza [9]*). (a) *If F is a k-vertex connected graph, other than the complete graph on k vertices, then every F-saturated graph G is $(k-1)$-vertex connected.* (b) *If F is a k-edge connected graph, then every F-saturated graph G is $(k-1)$-edge connected.*

Proposition 2 (*Tuza [9]*). (a) *Let F be a k-vertex connected graph, and let G be an F-saturated graph with a set X of $k-1$ vertices such that $G \backslash X$ is disconnected. Denote by $G_1, \ldots, G_l$ the connected components of $G \backslash X$. If X induces a clique, then*

(1) *$G \backslash G_i$ is F-saturated for $1 \leqslant i \leqslant l$;*
(2) *$G_i \cup X$ induces an F-saturated graph $1 \leqslant i \leqslant l$;*

(b) *Let F be a k-edge connected graph, and suppose that a graph G has a partition $V_1 \cup V_2 = V(G)$ such that there are just $k-1$ edges between V_1 and V_2. If G is F-saturated, then the subgraph induced by V_i $(i = 1, 2)$ is also F-saturated.*

Proposition 3. *If G is a K_2^t-saturated graph ($t \geqslant 2$) with cut-set X of order $2t-3$ and $G_1, G_2, \ldots, G_l$, are the components of $G \backslash X$, then all vertices belonging to X must belong to the K_2^t formed upon the addition of an edge $(v_i v_j)$ where $v_i \in G_i$, $v_j \in G_j (i \neq j)$. In other words, there exist three vertices outside the cutset belonging to any such K_2^t formed. Additionally, two of these three vertices are in the same component of $G \backslash X$.*

Proof. Let G be a K_2^t-saturated graph. Let v_i, v_j be in separate components of $G \backslash X$. Consider $G + (v_i v_j)$. Clearly, there exists a vertex $z \neq v_i, v_j$ in some G_k belonging to the K_2^t formed upon the addition of edge $(v_i v_j)$ to G. Vertex z cannot be in a component of $G \backslash X$ different from both v_i and v_j as then z would be non-adjacent to two vertices in the K_2^t-subgraph. Thus, without loss of generality z must be in say, G_i. Now suppose there exists another vertex w contained in the K_2^t in some G_k, $1 \leqslant k \leqslant l$. Similarly, w must be in either G_i or G_j. If $w \in G_i$ then as v_j is not adjacent to both z and w, a K_2^t cannot be formed, which is a contradiction. If $w \in G_j$ then as w is not adjacent to either v_i or z, again a K_2^t cannot be formed, a contradiction. Hence, there are at most three vertices outside X (and thus exactly three vertices) in any such K_2^t and of these three vertices, two of them are in the same component of $G \backslash X$. □

Proposition 4. *If G is a K_2^t-saturated graph ($t \geqslant 2$) with a cut-set X of order $2t-3$ then $X = \{x_1, x_2, \ldots, x_{2t-3}\}$ induces a clique in G.*

Proof. Let G be a K_2^t-saturated graph as above and denote the components of $G \backslash X$ by $G_1, \ldots, G_l$. Consider $G + (v_i v_j)$ where $v_i \in G_i,\ v_j \in G_j (i \neq j)$. By Proposition 3, the vertices of X are contained in the K_2^t formed upon inserting $(v_i v_j)$. Thus, on the vertices of X, a $K_2^{t-2} + x_k$ must be present in G. Now suppose there exists a pair of vertices x_i, x_j in X that are not adjacent in G. For any pair v_i, v_j as considered above, $G + (v_i v_j)$ contains a K_2^t where x_i and x_j must be in the same partite set. This implies that x_i, x_j are adjacent to all other vertices in the graph G. Thus $G \backslash \{x_i, x_j\}$ is K_2^{t-1}-saturated. Now consider $G + (x_i x_j)$. Upon the addition of edge $(x_i x_j)$ to G, a K_2^t is formed as a subgraph where x_i and x_j lie in different partite sets (as otherwise a K_2^t would have existed in G.) Thus, on $G \backslash \{x_i, x_j\}$ there exists a K_2^{t-1}, a contradiction. □

Proposition 5. *If G is a K_s^t-saturated graph with $t \geqslant 3$ ($t = 2$), then G has diameter at most 2 (respectively 3). Furthermore, if $t \geqslant 3$ then G contains $s(t-2)$ edge disjoint paths of length two between any two non-adjacent vertices.*

Proof. Consider any pair of non-adjacent vertices x, y. Since every edge of K_s^t, $t \geqslant 3$ ($t = 2$) is contained in $s(t-2)$ 3-cycles (resp. a 4-cycle) and $G + (xy)$ contains the subgraph K_s^t, the distance from x to y in G can be no more than 2 (respectively 3.). □

Proposition 6. *If G is a K_2^t saturated graph with cut set X of order $2t-3$, then all vertices not adjacent to all of X belong to the same component of $G \backslash X$. Additionally, this component contains at least three vertices.*

Proof. Consider vertices $v_i \in G_i,\ v_j \in G_j,\ i \neq j$ such that $v_i x_k \notin E(G)$ and $v_j x_l \notin E(G)$ for some $x_k, x_l \in X$ (note x_k may equal x_l). Now consider $G + (v_i v_j)$. By Proposition 3 there exists a vertex z in say G_i such that z is in the K_2^t formed upon the addition of edge $(v_i v_j)$ to G. But then v_j is not adjacent to both x_l and z, a contradiction. The same argument holds if z is in G_j. Thus v_i and v_j must be in the same component.

To see that this component has at least three vertices suppose that it did not. Then consider $G + (v_i x_k)$ and the K_2^t-subgraph formed. This copy of K_2^t must, by Proposition 2(2), lie entirely in X and this special component. But now we reach a contradiction, since X together with this component do not contain enough vertices. □

For convenience, from this point on we refer to the component described in Proposition 6 as G_1.

Proposition 7. *If G is a K_2^t-saturated graph with cut set X of order $2t-3$, then the components of $G \backslash X$ can be categorized as follows*: (i) *there is at most one component as described in Proposition* 6, (ii) *there is at most one component of order* 1, *and* (iii) *the remaining components are single edges.*

Proof. (i) Follows immediately from Proposition 6. To show (ii), consider two components of order 1, say $G_i = \{a\}$, $G_j = \{b\}$. The graph $G + (ab)$ must contain, by Proposition 3, a K_2^t on $X \cup \{a, b\}$. But this is impossible since

$|X \cup \{a, b\}| = 2t - 1$. To show (iii) consider a component G_k where each vertex in G_k is adjacent to all of X and G_k contains at least three vertices. Note that in such a component there exists three vertices that induce at least two edges. This would imply the existence of a copy of K_2^t in G, which is a contradiction. Thus, these components have at most two vertices (and more than one) and therefore must be single edges. This proves (iii). □

Proposition 8. *If G is a K_2^t-saturated graph with cutset X of order $2t - 3$, then any vertex v in G_1 is adjacent to at least $2t - 4$ vertices of X.*

Proof. Let $v \in G_1$ such that $vx_i \notin E(G)$ for some $x_i \in X$. Let w be in a different component, say G_j of $G \backslash X$. By Proposition 3, $G + (vw)$ contains a K_2^t which uses all of X. Hence, v must be adjacent to all other vertices of X. □

2.1. Proof of main result

We are now ready to prove the main result.

Proof of Theorem 2. Let G be a K_2^t-saturated graph on $n \geqslant 4t - 4$ vertices with $\delta(G) = 2t - 3$.

We first note that in such a graph, $G + (v_1 v_2)$ contains a copy of K_2^t where v_1 and v_2 are in different partite sets of K_2^t, as otherwise a copy of K_2^t would have already existed in G. If v_1 is in a partite set of K_2^t we will refer to the other vertex in that partite set as v_1's *mate*. For convenience we will refer to v_1 as being in the first partite set, v_2 the second partite set. Also, as K_2^t is a $(2t - 2)$-connected graph, Proposition 1 implies that G is $(2t - 3)$-connected, thus the minimum degree of any K_2^t-saturated graph is at least $2t - 3$.

With reference to Proposition 7, we refer to a component of order 1 as a Type I component, a component of order 2 as a Type II component and a component of order 3 or more as a Type III component. Let y be a vertex of degree $2t - 3$ and set $N(y) = X$. Note that X is a cut-set of size $2t - 3$ and thus, by Proposition 4, the graph induced by X is complete. By Proposition 7 there is at most one component of Type III. Thus, there are two possibilities for the structure of G.

Case 1: Suppose G contains a component, G_1, of Type III.

We begin by setting the number of vertices in G_1 equal to $g_1 \geqslant 3$, and describe the structure of G_1 and the minimum number of edges it must contain. First note that the number of Type II components is $k = (n - 2t + 3 - 1 - g_1)/2$ (and thus n and g_1 have the same parity). Furthermore, by Proposition 2, $G_1 \cup X$ is a K_2^t-saturated graph. Denote by A the vertices of G_1 that are adjacent to all of X. Denote by X_1 the vertices of G_1 that are adjacent to $x_2, x_3, \ldots, x_{2t-3}$, but not x_1. Similarly, define X_i for $2 \leqslant i \leqslant 2t - 3$. Note by Proposition 8, there are no other vertices of G_1. First note that if A is non-empty then A induces a 1-regular graph in G, since for any vertex $a \in A$, the graph $G + (ya)$ contains a K_2^t, and thus a must be adjacent to a vertex in A which is y's mate. Further, there cannot exist two incident edges, say $(a_1 a_2)$ and $(a_2 a_3)$, in A as otherwise G would contain K_2^t as a subgraph. Namely a K_2^t would exist on $X \cup \{a_1, a_2, a_3\}$.

Furthermore, every vertex $v \in G_1 \backslash A$ is adjacent to exactly one vertex $a \in A$. To see this is true, first note that if $v \in G_1 \backslash A$ were adjacent to two vertices a_1, a_2 in A, then a K_2^t would be present in G, namely a K_2^t would exist on $X \cup \{v, a_1, a_2\}$. To see that v is adjacent to at least one vertex in A, note that $G + (vy)$ creates a K_2^t as a subgraph involving the $2t - 1$ vertices $v, y, x_1, x_2, \ldots, x_{2t-3}$. The remaining vertex in the K_2^t subgraph which is not adjacent to y (as y has no other adjacencies in $G + (vy)$) must be y's mate. Thus, this vertex must be adjacent to all others, which includes all of X, and thus this mate must be in A. This also shows that A cannot be empty. Together with the fact that A is 1-regular, this implies $|A| \geqslant 2$.

We now consider the maximum number of vertices $x \in V(G_1 \backslash A)$ such that $d_{G_1}(x) = 1$. Let $v, w \in G_1 \backslash A$ with $d_{G_1}(v) = d_{G_1}(w) = 1$. Then we consider the following two possibilities. Note that these conditions imply that $vw \notin E(G)$, as v's one edge in G_1 must be to A.

Subcase (i): Suppose $v, w \in X_i$ for some i, then the neighbors of v and w which are in A are adjacent.

Consider $G + (vw)$ and the K_2^t subgraph formed. The vertex x_i cannot be in the K_2^t formed as x_i is not adjacent to either v or w. This implies that v and w cannot share a single neighbor in A as then the joint neighborhood of v and w would contain only $2t - 3$ vertices and any two non-adjacent vertices in G must have a joint neighborhood of at least $2t - 2$ vertices. Thus suppose $v \sim a_1$, $w \sim a_2$ for some $a_1, a_2 \in A$. Additionally, $a_1 \sim a_2$ since the joint neighborhood is exactly $2t - 2$ vertices and these two vertices lie in the symmetric difference of the joint neighborhood of v and w. In other words, a_1 is the mate of w and a_2 is the mate of v and thus the edge $(a_1 a_2)$ must exist.

Subcase (ii): Suppose $v \in X_i$, $w \in X_j$, $i \neq j$, then v and w share a common neighbor in A.

Without loss of generality suppose $v \in X_1$, $w \in X_2$. Further, suppose $v \sim a_1$ and $w \sim a_2$ for some $a_1, a_2 \in A$, $a_1 \neq a_2$. Now consider $G + (vw)$. Considering v, we see that the K_2^t formed must contain $v, w, a_1, x_2, x_3, \ldots, x_{2t-3}$. However, x_2 and a_1 are not adjacent to w, a contradiction. Therefore v, w must share the same neighbor in A.

For $t \geqslant 3$, (i) and (ii) together imply that the maximum number of vertices $x \in G_1$ such that $d_{G_1}(x) = 1$ is $2t-3$. Furthermore, this occurs when the $2t-3$ vertices are each in different X_i.

Once again we count the edges of G, and noting that $g_1 := |A| + |\bigcup_{i=1}^{2t-3} X_i|$. We explain the equation below. Beginning with line (1), recall that X is complete. Next, note that in this case each vertex in $G_2, G_3, \ldots, G_l$ is adjacent to each vertex in X and that each of these Type II components contains one edge. Next line (2), each vertex in A is adjacent to all of X, and A induces a 1-factor. Next, each vertex in $\bigcup_{i=1}^{2t-3} X_i$ is adjacent to $2t-4$ vertices in X, and one vertex in A. Finally line (3), since there are at most $2t-3$ vertices, $\{u_1, u_2, \ldots, u_{2t-3}\} \in \bigcup_{i=1}^{2t-3} X_i$ with $d_{G_1}(u_i) = 1$ the remainder must have degree at least two. Thus,

$$|E(G)| \geqslant \binom{2t-3}{2} + (n-2t+3-g_1)(2t-3) + \frac{n-2t+3-1-g_1}{2} \tag{1}$$

$$+ |A|(2t-3) + \frac{|A|}{2} + \left(\left|\bigcup_{i=1}^{2t-3} X_i\right|\right)(2t-4) + \left(\left|\bigcup_{i=1}^{2t-3} X_i\right|\right) \tag{2}$$

$$+ \left\lceil \frac{(|\bigcup_{i=1}^{2t-3} X_i|) - \min\{(2t-3), |\bigcup_{i=1}^{2t-3} X_i|\}}{2} \right\rceil \tag{3}$$

$$= \left\lceil \frac{(4t-5)n - 4t^2 + 8t - 4 - \min\{(2t-3), |\bigcup_{i=1}^{2t-3} X_i|\}}{2} \right\rceil \tag{4}$$

and when $n \geqslant 4t-3$, the minimum is achieved when there exists at least $2t-3$ vertices in $\bigcup_{i=1}^{2t-3} X_i$. Thus,

$$|E(G)| \geqslant \left\lceil \frac{(4t-5)n - 4t^2 + 6t - 1}{2} \right\rceil. \tag{5}$$

Case 2: Suppose G contains no component of Type III.

If $n-2t+3$ is even (thus n is odd) then we reach a contradiction as $(n-2t+2)/2$ (the number, k, of Type II components) must be an integer. Thus $n-2t+3$ is odd and $k = (n-2t+2)/2$. We now count the number of edges G must contain. First, recall that X is complete. Next, note that in this case each vertex in $G \backslash X$ is adjacent to each vertex in X. Finally, note that each of the Type II components contains one edge. Thus,

$$|E(G)| = \binom{2t-3}{2} + (n-2t+3)(2t-3) + \frac{n-2t+2}{2} \tag{6}$$

$$= \frac{(4t-5)n - 4t^2 + 8t - 4}{2}. \tag{7}$$

The number of edges obtained in the Case 1 is obviously less than in Case 2. We will now show that there exists a graph G that contains the number of edges as given by the lower bound in Case 1 and which is K_2^t-saturated.

It suffices to now describe the structure of G_1. The set A contains two adjacent vertices a_1, a_2, with a_1 adjacent to all of $\bigcup_{i=1}^{2t-3} X_i$. In the case that n is odd, each X_i contains a vertex u_i such that $d_{G_1}(u_i) = 1$. In the case that n is even, all but one of the X_i contain such a vertex. The remainder of the vertices in a given X_i induce a 1-factor. (That is we forbid edges $z_i z_j$ where $z_i \in X_i$, $z_j \in X_j$, $i \neq j$.) We have now completely described the structure of the graph G. Fig. 1 helps to illustrate this.

We will now show that the minimal graph obtained in this case is indeed K_2^t-saturated, and thus the result will be established.

Claim 1. *The graph G contains no copy of K_2^t.*

First note that as the degree of y is $2t-3$, it cannot be contained in a copy of K_2^t. The same is true for any $u_i \in \bigcup_{i=1}^{2t-3} X_i$ such that $d_{G_1}(u_i) = 1$. If the copy of K_2^t contained all the vertices of X it would need to contain three vertices at distance

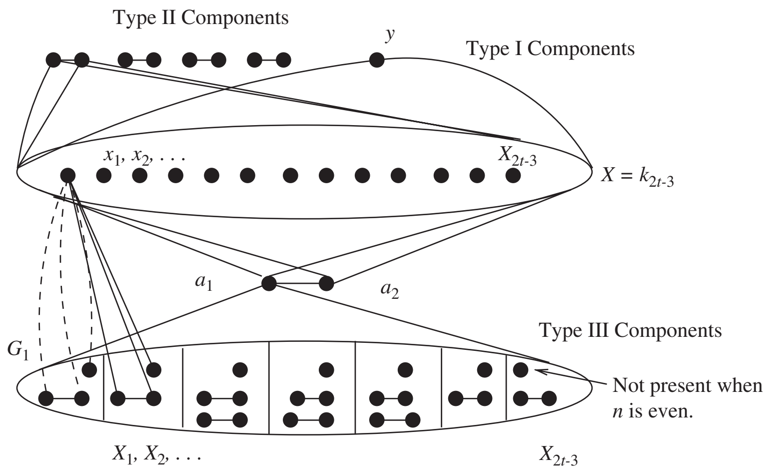

Fig. 1. K_2^t-saturated graph.

two from y. These three vertices would need to be in the same component (as they must induce at least two edges), thus must be in G_1. If two vertices from A were used then there must exist some $v \in \bigcup_{i=1}^{2t-3} X_i$ that is adjacent to both of them as v is nonadjacent to some $x_i \in X$. However, v has only one edge to A. If one vertex of A were used, then the two remaining vertices, v, w cannot come from the same X_i as $v, w \not\sim x_i$, and thus $v \in X_i$, $w \in X_j$, $i \neq j$. However, $v \not\sim x_i, w$ by construction. Thus all three vertices must come from $\bigcup_{i=1}^{2t-3} X_i$. Each would need to be in a different X_i, and thus must induce a triangle. However, this is forbidden from happening by our construction.

Thus, any copy of K_2^t would contain at most $2t-4$ vertices of X. Then at least three vertices of K_2^t must come from $G \backslash X$, and must be in the same component and thus lie in G_1. Furthermore, any four vertices of K_2^t contain a $K_{2,2}$ and a careful consideration of G_1 shows that no such $K_{2,2}$ exists. This proves the claim. □

Claim 2. *For any edge e in the complement of* G, $G+e$ *contains a copy of* K_2^t.

For convenience, let $a_1, a_2 \in A$, $z_{i,1}, z_{i,2} \in X_i$, $z_{j,1} \in X_j$, $v_j, w_j \in G_j$, $v_k \in G_k$ $(j, k \neq 1)$. We may assume that $d_{G_1}(z_{i,1}) = 2$ and will denote its neighbor in X_i by $z_{i,3}$. Also recall that for all $x \in \bigcup_{i=1}^{2t-3} X_i$ we have x adjacent to a_1.

To prove the claim we will show that for any edge e, the graph $G+e$ contains a copy of K_2^t and explicitly give each of the partite sets and their elements.

First we consider edges between components.

Case: Let $e = v_j v_k$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{w_j, v_k\}, \{v_j, x_1\}, \{x_2, x_3\}, \ldots, \{x_{2t-4}, x_{2t-3}\}\}$.

Case: Let $e = v_k a_1$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{a_2, v_k\}, \{a_1, x_1\}, \{x_2, x_3\}, \ldots, \{x_{2t-4}, x_{2t-3}\}\}$.

Case: Let $e = v_k a_2$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{a_1, v_k\}, \{a_2, x_1\}, \{x_2, x_3\}, \ldots, \{x_{2t-4}, x_{2t-3}\}\}$.

Case: Let $e = v_k z_{i,1}$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{a_1, v_k\}, \{z_{i,1}, x_i\}, \{x_1, x_2\}, \ldots, \{x_{2t-4}, x_{2t-3}\}\}$.

Next we consider edges from the cut-set to G_1.

Case: Let $e = x_i z_{i,2}$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{z_{i,2}, a_2\}, \{x_i, a_1\}, \overbrace{\{x_1, x_2\}, \ldots, \{x_{2t-4} x_{2t-3}\}}^{\text{omits } x_i}\}$.

This leaves us to consider edges within G_1.

Case: Let $e = a_2 z_{i,2}$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{z_{i,2}, x_i\}, \{a_1, a_2\}, \overbrace{\{x_1, x_2\}, \ldots, \{x_{2t-4}, x_{2t-3}\}}^{\text{omits } x_i}\}$.

Case: Let $e = z_{i,1}z_{i,2}$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{z_{i,1}, a_1\}, \{z_{i,2}, z_{i,3}\}, \overbrace{\{x_1, x_2\}, \ldots, \{x_{2t-4}, x_{2t-3}\}}^{\text{omits } x_i}\}$.

Case: Let $e = z_{i,1}, z_{j,1}$, then K_2^t is contained in the subgraph induced by the following partite sets $\{\{z_{i,1}, x_i\}, \{z_{j,1}, x_j\}, \{a_1, x_1\}, \overbrace{\{x_2, x_3\}, \ldots, \{x_{2t-4}, x_{2t-3}\}}^{\text{omits } x_i, x_j, x_1}\}$.

This completes the proof of Claim 2, and the proof of Theorem 2. □

We now give further evidence to support Conjecture 1. To do this we begin by generalizing a theorem used by Duffus and Hanson in [3].

Theorem 4. *For integers* $t \geqslant 3$, $s \geqslant 1$, $\delta \geqslant s(t-1) - 1$, $n \geqslant st$,

$$\mathrm{sat}(n, K_s^t, \delta) \geqslant \frac{\delta + s(t-2)}{2}(n - \delta - 1) + \delta + s^2\binom{t-2}{2} + s(s-1)(t-2). \tag{8}$$

Proof. Let y be a vertex of minimum degree δ and X the set of δ vertices adjacent to y. Let Z denote the remaining $n - \delta - 1$ vertices, which are at distance two (by Proposition 5) from y. First, X contains a copy of $K_s^{t-2} + \overline{K}_{s-1}$ since $G + (yv)$ contains a K_s^t, $v \in Z$, for any $v \not\sim y$. Next, each $v \in Z$ must be adjacent to all of the vertices of a K_s^{t-2} in X since $G + (yv)$ creates a copy of K_s^t. Therefore, by summing the degrees of the vertices in each set we obtain,

$$\begin{aligned}\Sigma_{x \in G} d(x) \geqslant \delta &+ \{\delta + s(t-2)(n - \delta - 1) + s(t-2)[s(t-3) + (s-1)] + (s-1)[s(t-2)]\} \\ &+ \{(n - \delta - 1)\delta\}.\end{aligned}$$

The lower bound thus follows. □

We now use Theorem 4 in support of Conjecture 1. Evaluating Eq. (8) for $s = 2$ and $\delta \geqslant 2t$ we find that the coefficient in n is at least $(4t-4)/2$ which is greater than the coefficient in n given by Theorem 2, which is $(4t-5)/2$. Thus for n sufficiently large the number of edges in an K_2^t-saturated graph with minimum degree $\delta \geqslant 2t$ is strictly greater than the number of edges in an K_2^t-saturated graph with minimum degree $2t - 3$.

This leads to another conjecture (which generalizes one given by Bollobás in [2]), the proof of which would settle Conjecture 1.

Conjecture 2. Given a fixed graph F, for n sufficiently large the function $\mathrm{sat}(n, F, \delta)$ is monotonically increasing in δ.

We note that the word “monotonically” cannot be replaced by “strictly.” One can see this by examining the extremal graphs for $K_{2,2}$ provided by Ollmann [6].

References

[1] B. Bollobás, Extremal Graph Theory, Academic Press, New York, 1978.

[2] B. Bollobás, Extremal graph theory, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, North-Holland, Amsterdam, 1995, pp. 1231–1292.

[3] D.A. Duffus, D. Hanson, Minimal k-saturated and color critical graphs of prescribed minimum degree, J. Graph Theory 10 (1986) 55–67.

[4] P. Erdős, A. Hajnal, J.W. Moon, A problem in graph theory, Amer. Math. Monthly 71 (1964) 1107–1110.

[5] L. Kászonyi, Z. Tuza, Saturated graphs with minimal number of edges, J. Graph Theory 10 (1986) 203–210.

[6] L.T. Ollmann, $K_{2,2}$-saturated graphs with a minimal number of edges, in: Proceedings of the Third SouthEast Conference on Combinatorics, Graph Theory and Computing, 1972, pp. 367–392.

[7] O. Pikhurko, The minimum size of saturated hypergraphs, Combinatorics Probab. Comput. 8 (1999) 483–492.

[9] Z. Tuza, C_4-saturated graphs of minimum size, Acta Univ. Carolin.—Math. Phys. 30 (2) (1989) 161–167.