On $H$-Immersions

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Abstract: For a fixed multigraph $H$, possibly containing loops, with $V(H) = \{h_1, \ldots, h_k\}$, we say a graph $G$ is $H$-linked if for every choice of $k$ vertices $v_1, \ldots, v_k$ in $G$, there exists a subdivision of $H$ in it $G$ such that $v_i$ represents $h_i$ (for all $i$). An $H$-immersion in $G$ is similar except that the paths in $G$, playing the role of the edges of $H$, are only required to be edge disjoint. In this article, we extend the notion of an $H$-linked graph by determining minimum degree conditions for a graph $G$ to contain an $H$-immersion with a bounded number of vertex repetitions on any choice of $k$ vertices. In particular, we extend results found in [2,3,5].

Keywords: $k$-linked; $H$-linked; immersion

1. INTRODUCTION AND TERMINOLOGY

A graph is $k$-linked if for every sequence of $2k$ vertices, $v_1, \ldots, v_k, w_1, \ldots, w_k$, there are internally disjoint paths $P_1, \ldots, P_k$ such that $P_i$ joins $v_i$ and $w_i$. The class of $k$-linked graphs is interesting and important, hence the topic has drawn considerable attention (e.g., see [7]). In this article we wish to generalize this class.
Let $H$ be a multigraph, possibly containing loops, with the order of $H$ being $|H| = k$. For any graph $G$, let $\mathcal{P}(G)$ denote the set of paths in $G$.

**Definition 1.1.** An $H$-subdivision in $G$ is a pair of mappings $f : V(H) \to V(G)$ and $g : E(H) \to \mathcal{P}(G)$ such that:

(i) $f$ is injective;
(ii) for every edge $xy \in E(H)$, $g(xy)$ is an $f(x) - f(y)$ path in $G$ and distinct edges of $H$ map to internally disjoint paths in $G$.

A graph $G$ is $H$-linked if every injective map $f : V(H) \to V(G)$ can be extended to an $H$-subdivision. The notion of an $H$-linked graph is a generalization of the idea of a $k$-linked graph, since if $H$ is a matching with $k$ edges, $G$ is $k$-linked if and only if $G$ is $H$-linked.

Our result will depend on a parameter $\eta(H)$, defined as follows. If $H$ is connected, we define $\eta(H)$ to be the maximum size of an edge cut in $H$. If $H$ has several components, say $H_1, \ldots, H_c$, where at least one $H_i$ contains an even cycle, we define $\eta(H) = u(H) + \sum_{i=1}^{c} \eta(H_i)$, where $u(H)$ denotes the number of components of $H$ that contain no even cycles. For the remainder of this article, we will refer to components of this type as uneven components. If each component of $H$ is uneven, we will let $\eta(H) \cdot |V(H)|$. 1.

The following was shown [3].

**Theorem 1.2.** Let $H$ be a multigraph and $G$ be a simple graph such that $n = |G| \geq 10(|V(H)| + |E(H)|)$. If $\delta(G) \geq n + \eta(H) - 2$, then $G$ is $H$-linked.

This result (and earlier versions, found in [2,5] which were concerned solely with connected multigraphs) along with extension theorems (see [4,5]) established a new framework for viewing path and cycle problems as particular strong connectivity problems. This framework also supplied a way to generalize a number of results in this area at one time (see [4]).

We now extend the notion of an $H$-linked graph through the idea of an $H$-immersion, and thus broaden this new framework even more.

**Definition 1.3.** An $H$-immersion in $G$ is a pair of mappings $f : V(H) \to V(G)$ and $g : E(H) \to \mathcal{P}(G)$ such that:

(i) $f$ is injective;
(ii) for every edge $xy \in E(H)$, $g(xy)$ is an $f(x) - f(y)$ path in $G$ and distinct edges of $H$ map to edge-disjoint paths in $G$.

For an immersion $\mathcal{I}$ of $H$ in a graph $G$, we let $S = f(V(H))$. For a vertex $v \in G - S$, define the vertex repetition number, $r(v, \mathcal{I})$, to be one less than the number of paths in $g(E(H))$ containing $v$, if $v$ lies on a path of $\mathcal{I}$, and zero otherwise. We then define the vertex repetition number of an immersion $\mathcal{I}$, denoted $r(\mathcal{I})$, to be the sum of the vertex repetition numbers taken over the vertices of $G - S$. An $H$-immersion with vertex repetition number 0 is in fact an $H$-subdivision, establishing the connection between these two structures.

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In this article we determine a sharp minimum degree condition for a graph $G$ to contain an $H$-immersion $\mathcal{I}$ on any $|V(H)|$ vertices such that $r(\mathcal{I})$ is bounded from above.

For a graph $G$, let $\delta(G)$ denote the minimum degree of a vertex in $G$. For two subsets $A$ and $B$ of the vertex set of $G$, let $\delta(A, B) = \min_{v \in A} |N(v) \cap B|$, where $N(v)$ denotes the set of neighbors of $v$ in $G$. The subgraph induced by a set $A$ of vertices is denoted by $\langle A \rangle$. For terminology not defined here, the reader should consult [1].

2. IMMERSIONS AND MINIMUM DEGREE

As mentioned above, an $H$-subdivision in a graph $G$ is simply an $H$-immersion $\mathcal{I}$ with $r(\mathcal{I}) = 0$. Our goal is to prove the following extension of Theorem 1.2 to include immersions $\mathcal{I}$ with $r(\mathcal{I}) \geq 1$. In what follows, we say a graph is nontrivial if it contains at least one edge.

Theorem 2.1. Let $H$ be a loopless nontrivial multigraph of order $k$ and $G$ a simple graph such that $|G| = n \geq \max \{ 8|E(H)|^2 + 8|E(H)|k + 2k^2 - |E(H)| + 2, 34|E(H)| + 4k + 5 \}$. If $\lambda$ is an integer such that $0 \leq \lambda \leq \eta(H) - k + 1$, and $\delta(G) \geq \frac{n+\eta(H)-\lambda-2}{2}$, then any injective map $f : V(H) \rightarrow V(G)$ can be extended to an immersion $\mathcal{I}$ with $r(\mathcal{I}) \leq \lambda$.

To see that this minimum degree is required, suppose first that $H$ has maximum edge cut size $\eta(H)$, and that this cut determines a partition of $V(H)$ into sets $X$ and $Y$. Let $G$ be a graph formed from two complete graphs $G_1$ and $G_2$ of order $m$ that intersect on $\eta(H) - \lambda - 1$ vertices. If $S$, the image of $V(H)$ under $f$, is chosen such that the vertices of $f(X)$ lie in $G_1 - G_2$, and the vertices of $f(Y)$ lie in $G_2 - G_1$, then clearly $G_1 \cap G_2$ is not large enough to allow an immersion of $H$ with repetition, number at most $\lambda$.

In the case where $\eta(H)$ is not the size of a maximum edge-cut in $H$, specifically when $H$ is disconnected and has at least one uneven component, we proceed in a slightly different manner. Let $H_1, \ldots, H_t$ denote the components of $H$. Furthermore, let $G$ be defined in a manner identical to the previous case. If no component of $H$ contains an even cycle, choose any two adjacent vertices $v_1$ and $v_2$ in $H$ and represent $v_1$ by some vertex in $G_1 - G_2$ and $v_2$ by some vertex in $G_2 - G_1$. Then assign any $\eta(H) - \lambda - 1 = |V(H)| - \lambda - 2$ other vertices in $H$ to the vertices in $G_1 \cap G_2$ and assign the remaining vertices arbitrarily. This will preclude the possibility of an $H$-immersion on this choice of vertices.

Assume that there is some component of $H$, say $H_1$, that contains an even cycle. We will then embed as many of the vertices of the components that do not contain an even cycle in $G_1 \cap G_2$ and embed the rest of the vertices of these components arbitrarily, if necessary. For each other component $H_i$, let $X_i$ and $Y_i$ partition $V(H_i)$ such that $e(X_i, Y_i) = \eta(H_i)$, and embed each $X_i$ in $G_1$ and each $Y_i$ in $G_2$. As above, this precludes the possibility of an $H$-immersion on this choice of vertices.

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In each case, \( \delta(G) = m - 1 \) and \( n = |V(G)| = 2m - \eta(H) + \lambda + 1 \). We can then see that \( \delta(G) = \frac{n + \eta(H) - \lambda - 3}{2} \), proving the necessity of the stated degree condition.

In the proof of Theorem 2.1, let \( G \) be a graph on \( n \) vertices satisfying the stated conditions and let \( f : V(H) \rightarrow V(G) \) be an injection. As above, we will let \( S \) denote \( f(V(H)) \). The following well-known lemma is useful.

**Lemma 2.2.** If \( G \) is a graph such that \( \delta(G) \geq \frac{n + \kappa - 2}{2} \), then \( G \) is \( \kappa \)-connected.

If \( C \) is a cutset of \( G \) of minimum order, Lemma 2.2 implies that \( |C| \geq \eta - \lambda \geq k - 1 \). Using the following theorem, we can assume that \( |C| < 2|E(H)| + |H| \).

**Theorem 2.3** (Pfender [6]). If \( |G| = n \geq 2k \), \( G \) is \( 2k \)-connected, and \( \delta(G) \geq \frac{n}{3} + 10k \), then \( G \) is \( k \)-linked.

Suppose \( \kappa(G) = |C| \geq 2|E(H)| + |H| \). Then the graph \( G - S \) is \( 2|E(H)| \)-connected and

\[
\delta(G - S) \geq \frac{n + \eta - \lambda - 2}{2} - k \\
\geq \frac{(n - k) + \eta - \lambda - 2 - k}{2} \\
\geq \frac{n - k}{3} + 10|E(H)|
\]

as \( n \geq 60|E(H)| + 5k + 5 \). By Theorem 2.3, \( G - S \) is \( |E(H)| \)-linked, implying that \( G \) is \( H \)-linked.

Thus, \( \eta - \lambda \leq |C| < 2|E(H)| + |H| \). The minimum degree conditions imply that \( G - C \) consists of exactly two components, which we call \( A \) and \( B \). We now bound the number of adjacencies that each vertex of \( A \) and \( B \) must have to \( C \).

**Lemma 2.4.** \( \delta(A, C), \delta(B, C) \geq \eta - \lambda \).

**Proof.** The minimum degree conditions require that \( |B| + |C| \geq \frac{n + \eta - \lambda - 2}{2} + 1 \), which implies that \( |A| \leq \frac{n - \eta + \lambda}{2} \). Since the minimum degree of a vertex in \( A \) is \( n + \eta - \lambda - 2 \), this means that each vertex in \( A \) must be adjacent to at least \( \eta - \lambda \) vertices in \( |C| \). A similar argument proves the bound on \( \delta(B, C) \).

Suppose \( |C| = \eta - \lambda + t \) for some \( t \geq 0 \). Let \( S_A = S \cap A \), \( S_B = S \cap B \), and \( S_C = S \cap C \). We call a vertex in \( C \) bad to \( A \) (respectively, bad to \( B \)) if it has fewer than \( 2|E(H)| + k \) neighbors in \( A \) (resp. \( B \)). We call a vertex of \( C \) good if it is neither bad to \( A \) nor bad to \( B \). Let \( R_A \) and \( R_B \) denote those vertices in \( C \) that are bad to \( A \) and \( B \), respectively and let \( J \) denote the set of good vertices in \( S_C \). Then, if \( M_A = S_C \cap R_A \) and \( M_B = S_C \cap R_B \), we can write \( S_C = J \cup M_A \cup M_B \).

Note that if \( x \in R_A \cap R_B \), then \( d_A(x) \leq 2|E(H)| + k \) and the same bound holds for \( d_B(x) \). Furthermore, \( x \) has less than \( 2|E(H)| + k \) neighbors in \( C \). Thus, \( 6|E(H)| + 3k \geq d(x) \geq \delta(G) \). But then from the bound on \( n \) we see that \( R_A \) and \( R_B \) must be disjoint.
Lemma 2.5. \(|R_A| + |R_B| \leq t.\)

Proof. Suppose \(|R_A| + |R_B| > t.\) Then, as \(n \geq 8|E(H)|^2 + 8|E(H)|k + 2k^2 - \frac{n + \eta - \lambda}{2},\) there exist vertices \(v_A \in A\) and \(v_B \in B\) such that \(v_A\) has no neighbors in \(R_A\) and \(v_B\) has no neighbors in \(R_B.\)

Then

\[
d(v_A) + d(v_B) \leq |A| + |B| - 2 + 2|C| - (|R_A| + |R_B|)
= n + |C| - 2 - (|R_A| + |R_B|)
= n + (\eta - \lambda + t - (|R_A| + |R_B|) - 2 \\
< n + \eta - \lambda - 2 \\
= 2\delta,
\]

contradicting the minimum degree condition. \]

Before describing how we find the \(H\)-immersion in \(G,\) we prove a lemma about the parameter \(\eta(H)\) that will be useful.

Lemma 2.6. Let \(M\) be a graph and let \(W, Z\) be a vertex partition of \(V(M).\) Then for any \(F \subseteq V(M),\)

\[e(W - F, Z - F) \leq \eta(M) - |F|.\]

Proof. It suffices to show that the result holds for \(F = \{v\}.\) We assume, without loss of generality, that \(v \in W\) and let \(W'\) denote \(W \setminus \{v\}.

Assume that the theorem is false, that is, that \(e(W', Z) \geq \eta(M).\) Note as well, that \(\eta(M)\) is always at least the size of a maximum edge cut in \(M,\) so it must be that \(e(W', Z) = \eta(M)\). If \(v\) is an isolated vertex, then by definition \(\eta(M)\) exceeds the size of a maximum edge-cut in \(M,\) contradicting the above assumption. Therefore, assume that \(v\) has at least one neighbor \(x\) in \(M.\) If \(x\) lies in \(Z,\) then \(e(W' \cup \{v\}, Z) > \eta(M),\) a contradiction. Similarly, if \(x\) lies in \(W',\) then \(e(W', Z \cup \{v\}) > \eta(M),\) again, a contradiction. \]

One should note that \(\langle A \rangle\) and \(\langle B \rangle\) are extremely dense and hence satisfy the conditions of Theorem 1.2. We will exploit this fact.

Let \(e = xy\) be an edge of \(H.\) The construction of the path in \(G\) that we use to represent this edge depends heavily on the location of the images of \(x\) and \(y\) under \(f.\) In many cases, we will select neighbors of \(f(x)\) and \(f(y)\) in \(A\) or \(B\) to serve as “proxies” for these vertices, as we will be able to use Theorem 1.2 to construct a path between the proxies, and thus between \(f(x)\) and \(f(y).\) Several cases are required.

Case 1. If \(f(x) \in (S_C \setminus R_A) (f(y) \in (S_C \setminus R_B),\) respectively) and \(f(y) \in (S_A),\)
\(f(x) \in (S_B)\) then we choose some vertex \(a_{e,x}^1\) in \(A \setminus S_A\) adjacent to \(f(x)\) \((b_{e,y}^1\) in \(B \setminus S_B\) adjacent to \(f(y)).\)
In this case, we wish to link a vertex in $S_A$ (or $S_B$) to a vertex in $S_C$ that is good to that particular set. This is not difficult as the vertex in $S_C$ will have a large number of neighbors to choose from in $A$ or $B$, respectively.

**Case 2.** If $f(x)$, $f(y) \in (S_C \setminus R_A)$, $(S_C \setminus R_B)$, respectively then we choose vertices $a'_e$ and $a''_e$ ($b'_e$ and $b''_e$) in $A \setminus S_A (B \setminus S_B)$ adjacent to $f(x)$ and $f(y)$, respectively.

Here, we would like to find a path between two vertices in $S_C$ that have a large number of neighbors in the same set ($A$ or $B$). We cannot assure that $x$ and $y$ are adjacent or have a common neighbor, but the density of $A$ and $B$ will make it relatively simple to link their proxies.

To this point, we have not been forced to make any repetitions among the vertices we have chosen. This is because we have been able to select vertices from $A$ and $B$. In the remaining cases, in order to represent the remaining edges with paths in an immersion, we must construct paths that cross from $A$ to $B$ through $C$. In the procedure below, some of the vertices labeled $v_e$, which lie in $C$, may be repeated. We will choose these vertices such that the number of repetitions is minimized.

**Case 3.** If $f(x) \in (M_B)$ and $f(y) \in (M_A)$ then we choose some vertex $v_e$ in $C \setminus (R_A \cup R_B \cup S_C)$ having neighbors $a'_{e,x}$ in $A \setminus S_A$ and $b'_{e,y}$ in $B \setminus S_B$, and vertices $a''_{e,x}$ in $A \setminus S_A$ and $b''_{e,y}$ in $B \setminus S_B$, adjacent to $f(x)$ and $f(y)$, respectively (Fig. 1).

Here, we would like to construct a path between two vertices in $S_C$. However, we may not be able to choose proxies for these vertices in the same sets. Therefore, we will choose proxies for $x$ and $y$ in the sets where they have many neighbors. Our goal is to then construct paths from $a'_{e,x}$ to $a''_{e,x}$ and $b'_{e,y}$ to $b''_{e,y}$, which would complete a path from $x$ to $y$.

**Case 4.** If $f(x) \in (S_A)$ and $f(y) \in (S_B)$, then we choose some vertex $v_e$ in $C \setminus (R_A \cup R_B \cup S_C)$ having neighbors $a'_{e,x}$ in $A \setminus S_A$ and $b'_{e,y}$ in $B \setminus S_B$.

In this case, we simply wish to link a vertex in $S_A$ to a vertex in $S_B$ using a path through the vertex $v_e$.

**Case 5.** If $f(x) \in (M_B)$ and $f(y) \in (S_B)$ ($f(x) \in (S_A)$ and $f(y) \in (M_A)$, respectively) then we choose some vertex $v_e$ in $C \setminus (R_A \cup R_B \cup S_C)$ having neighbors $a'_{e,x}$ in $A \setminus S_A$ and $b'_{e,y}$ in $B \setminus S_B$, and a vertex $a''_{e,x}$ in $A \setminus S_A$ adjacent to $f(x)$ ($b''_{e,y}$ in $B \setminus S_B$ adjacent to $f(y)$) (Fig. 2).

Here, we would like to link a vertex in $S_A$ or $S_B$ to a vertex in $S_C$ that is bad to $A$ or $B$, respectively. We must then choose a proxy for the vertex in $S_C$ in the other set, and construct a path through the vertex $v_e$.

As mentioned above, we have chosen the vertices $v_e$ in such a way to minimize the number of times they are repeated. We will also choose the vertices $a'_{e,x}$, $a''_{e,x}$, $a'$, and $a''$ (resp. $b'_{e,y}$, $b''_{e,y}$, $b'$, and $b''$) to be distinct, which is not difficult as $n$ is sufficiently large. It remains to show that we have no more than $\lambda$ total repetitions, as desired.

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Lemma 2.7. In the procedure above, there are at most $\lambda$ repetitions of the vertices $v_e$.

Proof. Let $Q$ be the set of good vertices of $C$ that are not in $S_C$. We must show that $Q$ is large enough. Note that

$$|C| = |S_C| + |Q| + |R_A \setminus M_A| + |R_B \setminus M_B|$$

$$= \eta(H) - \lambda + t.$$

By Lemma 2.5, we have that

$$|C \setminus (R_A \cup R_B)| \geq \eta(H) - \lambda$$

$$|J| + |Q| \geq \eta(H) - \lambda,$$

which implies that $|Q| \geq \eta(H) - \lambda - |J|$.
In constructing this immersion of $H$, none of the good vertices in $S_C$ are ever involved in paths containing a $v_e$. Consequently, if we apply Lemma 2.6 to the partition of $H - f^{-1}(S_C)$ defined by $f^{-1}(S_A)$ and $f^{-1}(S_B)$, we can see that there are only $\eta - |J|$ vertices $v_e$ that must be chosen. Thus, the selection of the $v_e$ can be carried out with at most $\lambda$ repetitions.

Let $H_A$ be a graph with vertex set $V(H_A) = \cup_{e \in H}(a_1^{e,x} \cup a_2^{e,x} \cup a'_e \cup a''_e) \cup S_A$. The edges of $H_A$ are determined as follows:

1. If $w, z \in H_A \cap S_A$, then $wz \in E(H_A)$ if and only if $f^{-1}(w)f^{-1}(z) \in E(H)$.
2. If $w \in H_A \cap S_A$, and $f^{-1}(w)$ is an endvertex of an edge $e = f^{-1}(w)x \in E(H)$, then $w$ is adjacent to $a_1^{e,x}$ in $H_A$.
3. The vertex $a_1^{e,x}$ is adjacent to $a_2^{e,x}$.
4. The vertex $a'_e$ is adjacent to $a''_e$.

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Similarly, let $H_B$ be a graph with vertex set $V(H_B) = \cup_{e \in B}(b_{e,y}^1 \cup b_{e,y}^2 \cup b_e^I \cup b_e^\prime \cup b_e^\prime\prime) \cup S_B$. The edges of $H_B$ are determined as follows:

(1) If $w, z \in H_B \cap S_B$, then $wz \in E(H_B)$ if and only if $f^{-1}(w)f^{-1}(z) \in E(H)$.
(2) If $w \in H_B \cap S_B$, and $f^{-1}(w)$ is an endvertex of an edge $e = f^{-1}(w)y \in E(H)$, then $w$ is adjacent to $b_{e,y}^1$ in $H_B$.
(3) The vertex $b_{e,y}^1$ is adjacent to $b_{e,y}^2$.
(4) The vertex $b_e^I$ is adjacent to $b_e^\prime$.

Claim 2.8. If $\langle A \rangle$ is $H_A$-linked and $\langle B \rangle$ is $H_B$-linked, then $f$ can be extended to an immersion $I$ with $r(I) \leq \lambda$.

Proof. For every edge $e = xy$ in $H$, we must show that there is a path from $f(x)$ to $f(y)$, and that there are at most $\lambda$ repetitions of internal vertices. Let $F_A$ be a subdivision of $H_A$ in $\langle A \rangle$ and let $F_B$ be a subdivision of $H_B$ in $\langle B \rangle$. We consider several cases.

Case 1. Suppose $f(x), f(y) \in S_A$ ($S_B$, respectively). Then $F_A$ ($F_B$) contains an $f(x) - f(y)$ path.

Case 2. Suppose $f(x) \in S_A$, $f(y) \in S_B$. Then $F_A$ contains an $f(x) - a_{e,x}^1$ path, $F_B$ contains an $f(y) - b_{e,y}^1$ path, and these paths are joined by the edges $a_{e,x}^1, v_e$ and $v_e b_{e,y}^1$.

Case 3. Suppose $f(x) \in S_C \setminus R_A$, $f(y) \in S_A$ ($f(x) \in S_B$, $f(y) \in S_C \setminus R_B$, respectively). Then $F_A$ ($F_B$) contains an $f(y) - a_{e,x}^1$ path (a $f(x) - b_{e,y}^1$ path), which combines with the edge $a_{e,x}^1 f(x) (b_{e,y}^1 f(y))$ to form the desired path.

Case 4. Suppose $f(x) \in S_A$, $f(y) \in M_A$ ($f(x) \in M_B$, $f(y) \in S_B$, respectively). Then $F_A$ ($F_B$) contains an $f(x) - a_{e,x}^1$ path ($b_{e,y}^1 - f(y)$ path), $F_B$ ($F_A$) contains a $b_{e,y}^1 - b_{e,y}^2$ path (a $a_{e,x}^2 - a_{e,x}^1$ path), and these combine with the edges $a_{e,x}^2, v_e, v_e b_{e,y}^1$, and $b_{e,y}^1 f(y)$ to form the required path.

Case 5. Suppose $f(x) \in M_B$, $f(y) \in M_A$. Then $F_A$ contains an $a_{e,x}^2 - a_{e,x}^1$ path, $F_B$ contains a $b_{e,y}^1 - b_{e,y}^2$ path, and these combine with the edges $f(x) a_{e,x}^2, a_{e,x}^1, v_e, v_e b_{e,y}^1, y, b_{e,y}^2 f(y)$ to form the required path.

Case 6. Suppose $f(x), f(y) \in S_C \setminus R_A$ ($f(x), f(y) \in M_A$, respectively). Then $F_A$ ($F_B$) contains an $a_{e,x}^I - a_{e,x}^\prime$ path (a $b_{e,y}^I - b_{e,y}^\prime$ path), which combines with the edges $f(x) a_{e,x}^\prime$ and $a_{e,x}^I f(y)$ ($b_{e,y}^I$ and $b_{e,y}^I f(y)$) to form the required path.

Note that these paths form an $H$-immersion on the required vertices in $G$. The only vertices that are possibly repeated are the $v_e$’s, but by Lemma 2.7, these vertices could be chosen such that at most $\lambda$ repetitions are used.

To complete the proof, we must now show that $\langle A \rangle$ is $H_A$-linked and $\langle B \rangle$ is $H_B$-linked. To do this, we wish to apply Theorem 2.3 and Lemma 2.2 to $\langle A \rangle$ and $\langle B \rangle$. 

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Note that a vertex in $A$ can be adjacent to every vertex in $C$, so we want:

$$\delta(\langle A \rangle) - k \geq \frac{|A|}{3} + 10|E(H)|.$$ 

But, since $|C| \leq 2|E(H)| + k$, this implies that

$$\frac{n + \eta(H) + \lambda - 2}{2} - (2|E(H)| + k) - k \geq \frac{n}{3}$$

$$- \frac{n + \eta(H) + \lambda - 2}{6} - \frac{1}{3} + 10|E(H)|$$

and hence that

$$n \geq 34|E(H)| + 4k + 5.$$ 

Note that a similar argument applies to $\langle B \rangle$. Thus, $\langle A \rangle$ is $H_A$-linked and $\langle B \rangle$ is $H_B$-linked, which completes the proof.

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