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Graphic sequences with a realization containing a complete multipartite subgraph

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Abstract

A nonincreasing sequence of nonnegative integers $\pi = (d_1, d_2, \dots, d_n)$ is *graphic* if there is a (simple) graph G of order n having degree sequence π. In this case, G is said to *realize* π. For a given graph H, a graphic sequence π is *potentially H-graphic* if there is some realization of π containing H as a (weak) subgraph. Let $\sigma(\pi)$ denote the sum of the terms of π. For a graph H and $n \in \mathbb{Z}^+$, $\sigma(H, n)$ is defined as the smallest even integer m so that every n-term graphic sequence π with $\sigma(\pi) \geq m$ is potentially H-graphic. Let K_s^t denote the complete t partite graph such that each partite set has exactly s vertices. We show that $\sigma(K_s^t, n) = \sigma(K_{(t-2)s} + K_{s,s}, n)$ and obtain the exact value of $\sigma(K_j + K_{s,s}, n)$ for n sufficiently large. Consequently, we obtain the exact value of $\sigma(K_s^t, n)$ for n sufficiently large.

Keywords: Degree sequence; Potentially H-graphic sequence

1. Introduction

In this paper, G is a graph without loops or multiple edges. A good reference for any terms not defined here is [1]. For a vertex $v \in V(G)$ let $N(v)$ denote the set of neighbours (or neighbourhood) of v, and $d(v)$ the degree of v. Additionally, let $N[v] = N(v) \cup \{v\}$ and $\overline{G}$ denote the complement of G. Given any two graphs G and H, their join, denoted $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{gh \mid g \in V(G),\ h \in V(H)\}$.

We denote the complete t-partite graph with partition sizes $p_1, \cdots, p_t$ by $K_{p_1,\cdots,p_t}$. If $p_1 = p_2 = \cdots = p_t = s$, we denote this graph by K_s^t. Complete, balanced multipartite graphs are of interest in light of the following classical extremal result of Erdős and Stone [5].

Theorem 1 (*Erdős and Stone 1946*). *Let G be a graph with n vertices and at least*

$$\left(1 - \frac{1}{t} + \epsilon\right)\frac{n^2}{2}$$

edges. Then for n sufficiently large, G contains a copy of K_s^{t+1}.

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A nonincreasing sequence of nonnegative integers $\pi = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a (simple) graph G of order n having degree sequence π. In this case, G is said to *realize* π, and we write $\pi = \pi(G)$. There are numerous methods to check if a given sequence is graphic. One theorem that will prove useful is the Erdős–Gallai condition [3].

Theorem 2 (*Erdős and Gallai 1960*). *A nonincreasing sequence of nonnegative integers $\pi = (d_1, d_2, \ldots, d_n)$ $(n \geq 2)$ is graphic if, and only if, $\sum_{i=1}^{n} d_i$ is even and for each integer k, $1 \leq k \leq n-1$,*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

For a given graph H, a graphic sequence π is *potentially H-graphic* if there is some realization of π containing H as a subgraph.

We let $\sigma(\pi)$ denote the sum of the terms of π. For a graph H and $n \in \mathbb{Z}^+$, we define $\sigma(H, n)$ to be the smallest even integer m so that every n-term graphic sequence π with $\sigma(\pi) \geq m$ is potentially H-graphic. As $\sigma(\pi) = 2|E(G(\pi))|$, the problem of determining $\sigma(H, n)$ is a natural variant of the Turán problem.

In this paper, we will determine $\sigma(K_j + K_{s,s}, n)$ for $j \geq 1$, $s \geq 3$, and n sufficiently large, and, as a corollary, determine $\sigma(K_s^t, n)$. The following theorem from [6] is useful.

Theorem 3 (*Gould, Jacobson and Lehel 1999*). *Let π be a graphic sequence. If G is a realization of π that contains some subgraph H, then there is a realization G' of π that contains H on the $|V(H)|$ vertices of highest degree in G'.*

2. Construction of the (Degree–Sum) maximal counterexample

2.1. Cliques

In [4] Erdős, Jacobson and Lehel conjectured that $\sigma(K_t, n) = (t-2)(2n-t+1)+2$. The conjecture arises from consideration of the graph $K_{(t-2)} + \overline{K}_{(n-t+2)}$. It is easy to observe that this graph contains no K_t, is the unique realization of the sequence $((n-1)^{t-2}, (t-2)^{n-t+2})$, and has degree sum $(t-2)(2n-t+1)$. The cases $t = 3, 4$ and 5 were proved separately (see respectively [4,6,7], and [8]), and Li, Song and Luo [9] proved the conjecture true via linear algebraic techniques for $t \geq 6$ and $n \geq \binom{t}{2} + 3$. A purely graph-theoretical proof was given in [14] and the result is also given as a corollary to the main result in [2].

2.2. Complete bipartite graphs

We now turn our attention to the complete bipartite case, $K_{s,s}$. Trivially, $\sigma(K_{1,1}, n) = 2$ and $\sigma(K_{2,2}, n)$ was established in [6]. The case $s \geq 3$ was completed in [10], where following notation first appeared. Let $s = 8k+l$, where $k \geq 0$ and $0 \leq l \leq 7$. If $l \in \{0, 1, 4, 5\}$, let $F_l = \{(8k+l, n) : k \geq 0 \text{ and } n \geq 16k+2l\}$. If $l \in \{2, 3, 6, 7\}$ let $F_l' = \{(8k+l, n) : k \geq 0 \text{ and } n \geq 16k+2l \text{ and } n \text{ is odd}\}$ and $F_l'' = \{(8k+l, n) : k \geq 0 \text{ and } n \geq 16k+2l \text{ and } n \text{ is even}\}$. Denote by E_1, E_2, E_3, E_4 the set $F_0 \cup F_2' \cup F_6''$, $F_4 \cup F_2'' \cup F_6'$, $F_1 \cup F_3'' \cup F_7'$ and $F_5 \cup F_3' \cup F_7''$, respectively. The following summarizes these results.

Theorem 4 (*[6,10]*). *Let n and s be positive integers.*
(1) *If s is even and $n \geq 4s^2 - s - 6$, then*

$$\sigma(K_{s,s}, n) = \begin{cases} \left(\frac{5}{2}s - 2\right)n - \frac{11}{8}s^2 + \frac{5}{4}s + 2 & \text{if } (s,n) \in E_1 \\ \left(\frac{5}{2}s - 2\right)n - \frac{11}{8}s^2 + \frac{5}{4}s + 1 & \text{if } (s,n) \in E_2. \end{cases} \tag{1}$$

(2) *If s is odd and $n \geq 4s^2 + 3s - 8$, then*

$$\sigma(K_{s,s}, n) = \begin{cases} \left(\frac{5}{2}s - \frac{5}{2}\right)n - \frac{11}{8}s^2 + \frac{5}{2}s + \frac{7}{8} & \text{if } (s,n) \in E_3 \\ \left(\frac{5}{2}s - \frac{5}{2}\right)n - \frac{11}{8}s^2 + \frac{5}{2}s + \frac{15}{8} & \text{if } (s,n) \in E_4. \end{cases} \tag{2}$$

3. $\sigma(K_j + K_{s,s}, n)$

Let $j \geq 1$ be an integer and let H be any graph on $n - j$ vertices with degree sum $\sigma(K_{s,s}, n - j) - 2$ such that $\pi(H)$ is not potentially H-graphic.

Now consider the graph $G = K_j + H$ and let $\pi(n, s, j)$ denote $\pi(G)$. For any values of j and s, and for n sufficiently large, it is not difficult to verify that $\pi(n, s, j)$ has no realization containing $K_j + K_{s,s}$ as a subgraph. We can compute a lower bound for $\sigma(K_j + K_{s,s}, n)$ by noting that $\sigma(\pi(n, s, j)) = j(n-1) + j(n-j) + \sigma(K_{s,s}, n-j) - 2$.

Lemma 1. (i) *If s is even and $(s, n - j) \in E_1$, then*

$$\sigma(K_j + K_{s,s}, n) \geq n\left(2j + \frac{5}{2}s - 2\right) - j^2 + j - \frac{5}{2}sj - \frac{11}{8}s^2 + \frac{5}{4}s + 2. \tag{3}$$

(ii) *If s is even and $(s, n - j) \in E_2$, then*

$$\sigma(K_j + K_{s,s}, n) \geq n\left(2j + \frac{5}{2}s - 2\right) - j^2 + j - \frac{5}{2}sj - \frac{11}{8}s^2 + \frac{5}{4}s + 1. \tag{4}$$

(iii) *If s is odd, and $(s, n - j) \in E_3$, then*

$$\sigma(K_j + K_{s,s}, n) \geq n\left(2j + \frac{5}{2}s - \frac{5}{2}\right) - j^2 + \frac{3}{2}j - \frac{5}{2}sj - \frac{11}{8}s^2 + \frac{5}{2}s + \frac{7}{8}. \tag{5}$$

(iv) *If s is odd, and $(s, n - j) \in E_4$, then*

$$\sigma(K_j + K_{s,s}, n) \geq n\left(2j + \frac{5}{2}s - \frac{5}{2}\right) - j^2 + \frac{3}{2}j - \frac{5}{2}sj - \frac{11}{8}s^2 + \frac{5}{2}s + \frac{15}{8}. \tag{6}$$

In this paper, we prove the following:

Theorem 5. *Let $j \geq 1$ and $s \geq 3$ be integers and π be an n-term graphic sequence, where n is sufficiently large. If $\sigma(\pi) \geq \sigma(\pi(n, s, j)) + 2$ then π is potentially $K_j + K_{s,s}$-graphic.*

This graph is of interest as in [14] the following was shown:

Theorem 6 (*[14]*). *Let n, s and t be positive integers, with n sufficiently large. Then*

$$\sigma(K_s^t, n) \geq \sigma(\pi(n, s, st - 2s)) + 2.$$

Equality holds if $s \leq 2$.

As K_s^t is a subgraph of $K_{s(t-2)} + K_{s,s}$, we obtain the following as a corollary to: Theorems 5 and 6.

Theorem 7. *Let n, s and t be positive integers with n sufficiently large. Then*

$$\sigma(K_s^t, n) = \sigma(\pi(n, s, st - 2s)) + 2.$$

The remainder of this paper is dedicated to the proof of Theorem 5.

Proof. Let π be an n-term graphic sequence with $\sigma(\pi) \geq \sigma(\pi(n, s, j)) + 2$ for sufficiently large n. In the sections that follow, we will reduce π to a sequence π_{j+2s}, according to the parity of s and the first $j + 2s$ terms of π, and then show that π is potentially $K_j + K_{s,s}$-graphic if and only if π_{j+2s} is graphic. We then show that π_{j+2s} is, in fact, graphic. The following counting result will be useful as we proceed.

Lemma 2. *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence such that $\sigma(\pi) \geq \sigma(\pi(n, s, j)) + 2$. Then for sufficiently large n, $d_{j+2s} \geq j + s$ and $d_j \geq j + 2s$.*

Proof. Assume first that d_{j+2s} is strictly less than $j+s$ and note that

$$\sigma(\pi) = \sum_{i=1}^{n} d_i = \sum_{i=1}^{j+2s-1} d_i + \sum_{i=j+2s}^{n} d_i.$$

By the Erdős–Gallai condition, this is at most

$$(j+2s-1)(j+2s-2) + \sum_{i=j+2s}^{n} \min\{d_i, j+2s-1\} + \sum_{i=j+2s}^{n} d_i,$$

which equals

$$(j+2s-1)(j+2s-2) + 2\sum_{i=j+2s}^{n} d_i.$$

By assumption, d_{j+2s} is at most $j+s-1$. Therefore,

$$\sigma(\pi) \le (j+2s-1)(j+2s-2) + 2(n-j-2s+1)(j+s-1) = n(2j+2s-2+o(1)).$$

This is less than $\sigma(\pi(n,s,j))$ for n sufficiently large, implying that $d_{j+2s} \ge j+s$.

Next, assume that $d_j < j+2s$. Then

$$\sigma(\pi) \le (j-1)(n-1) + (n-j+1)(j+2s-1) = n(2j+2s-2+o(1)),$$

which is less than $\sigma(\pi(n,s,j))$ for n sufficiently large. □

3.1. $s \equiv 0 \pmod 2$

In this subsection, we prove Theorem 5 in the case where s is even.

Let $s \ge 4$ be an even integer and let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence with $\sigma(\pi) \ge \sigma(\pi(n,s,j)) + 2$. We prove several facts about the elements of π in this case.

Lemma 3. *Let n be a sufficiently large integer, and let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence such that $\sigma(\pi) \ge \sigma(\pi(n,s,j)) + 2$, where s is even. Then for any integer $0 \le i \le j+s-2$,*

$$d_{j+s-1-i} \ge j + \frac{3}{2}s + i.$$

Proof. Assume the conclusion is false. Then

$$\sigma(\pi) \le (j+s-i-2)(n-1) + (n-j-s+i+2)\left(j+\frac{3}{2}s+i-1\right) = n\left(2j+\frac{5}{2}s-3+o(1)\right),$$

which is less than $\sigma(\pi(n,s,j))$ for sufficiently large n. □

Lemma 4. *If n is sufficiently large and $\pi = (d_1, \ldots, d_n)$ is as given, then*

$$d_n \ge j + \frac{5}{4}s - 1.$$

Proof. Let $G = G_0$ be a realization of π. If $\delta(G_0) \ge j + \frac{5}{4}s - 1$ we are done, so assume there is some vertex v_0 in G_0 such that the degree of $v_0 = \delta(G_0) < j + \frac{5}{4}s - 1$. Then, as the degree of v_0 must be an integer, we have that

$$d(v_0) \le j + \frac{5}{4}s - \frac{3}{2}.$$

Define G_1 to be $G_0 - v_0$. If there is some vertex v_1 in G_1 such that $d(v_1) = \delta(G_1) < j + \frac{5}{4}s - 1$ then we define $G_2 = G_1 - v_1$. Proceed in this manner to obtain a sequence $G_0 = G, G_1, G_2, \ldots$ of induced subgraphs of G.

Note that, for each $i \geq 1$,

$$\sigma(\pi(G_i)) = \sigma(\pi(G_i)) - 2d(v_i) \geq \sigma(\pi(G_{i-1})) - \left(2j + \frac{5}{2}s - 3\right).$$

Consequently, as $\sigma(\pi)$ is at least $n(2j + \frac{5}{2}s - 2) + g(s, j)$, where $g(s, j) = O(s^2 j^2)$, it follows that

$$\sigma(\pi(G_i)) \geq |G_i| \left(2j + \frac{5}{2}s - 2\right) + i + g(s, j)$$

for each G_i.

Either this process will terminate for some $i \leq \frac{3sn}{3s+1}$, or if n is sufficiently large, we may choose an i such that $\sigma(\pi(G_i)) > |G_i|(2j + 4s - 4 + o(1))$, implying that $\pi(G_i)$ is potentially K_{j+2s}-graphic, and hence that π is potentially $K_j + K_{s,s}$-graphic. If the process terminates, say with some G_i, then we redefine π to be $\pi(G_i)$ as $\sigma(\pi(G_i)) \geq \sigma(\pi(n - i, s, j))$ and $|G_i| = n - i \geq \frac{n}{3s+1}$. $\square$

Given π, we first obtain the sequence

$$\pi_1 = (d_2^{(1)}, \ldots, d_n^{(1)})$$

from π by deleting d_1 and subtracting 1 from the first d_1 remaining terms, then reordering $d_{j+2s+1} - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ to be nonincreasing.

We construct the sequences

$$\pi_i = (d_{i+1}^{(i)}, \ldots, d_n^{(i)})$$

for $2 \leq i \leq j$ from

$$\pi_{i-1} = (d_i^{(i-1)}, \ldots, d_n^{(i-1)})$$

by deleting $d_i^{(i-1)}$ from π_{i-1}, subtracting one from the first $d_i^{(i-1)}$ remaining positive terms, and then arranging the last $n - j - 2s$ terms in nonincreasing order.

We construct $\pi_{j+1}, \ldots, \pi_{j+2s}$ by considering $d_{j+1}^{(j)}, \ldots, d_{j+2s}^{(j)}$. We partition these $2s$ integers into two sets:

$$D^+ = \{d_i^{(j)} \mid j + 1 \leq i \leq j + 2s, \text{ and } d_i^{(j)} \geq 2s - 1\},$$

and

$$D^- = \{d_{j+1}^{(j)}, \ldots, d_{j+2s}^{(j)}\} \setminus D^+.$$

Note that Lemma 2 implies that all of the degrees in D^- are at least s. The construction of $\pi_{j+1} \ldots, \pi_{j+2s}$ depends on the size of D^- and falls into one of the following three cases:

Case 1. $|D^-| \leq s$.

In this case, we construct π_i, $j+1 \leq i \leq j+2s$, from π_{i-1} in a manner similar to that used to construct $\pi_1, \ldots, \pi_j$. First we delete $d_i^{(i-1)}$ from π_{i-1} and reduce the first $d_i^{(i-1)}$ remaining positive terms by 1. Next we arrange the first $j + 2s - i$ terms to be nonincreasing and then arrange the last $n - j - 2s$ degrees to be nonincreasing.

It remains to show that this process is feasible at each stage, that is there are at least d_i^{i-1} positive terms to be reduced at the point which we create π_i. To this end, we give a useful necessary condition for a sequence to be graphic. Specifically, we strengthen the Erdős–Gallai condition from Theorem 2 by noting that since π is graphic, for any $1 \leq k \leq n$ it must be the case that

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \min\{d_i, k - 1\} + \sum_{i=k+1}^{n} \min\{d_i, k\}.$$

Clearly, as a corollary to Theorem 2, this condition is also sufficient with the added constraint that $\sigma(\pi)$ is even. This strengthening of Theorem 2 along with the fact that we reorder the last terms in our residual sequence each time assure that a sufficient number of positive terms remain at each step in this process.

The following lemma is useful in the next case:

Lemma 5. *If π is a graphic sequence with $\sigma(\pi) \geq \sigma(\pi(n, s, j)) + 2$ and $d_j = n - 1$ then π is potentially $K_j + K_{s,s}$ graphic.*

Proof. Simply note that this implies $d_1, \ldots, d_j$ would all be equal to $n - 1$, and hence any realization of π would be a copy of K_j joined to a graph that is potentially $K_{s,s}$-graphic due to its degree sum. □

Case 2. $|D^-| = s + 1$.

In this case, for $j + 1 \leq i \leq j + s$, we construct π_i from π_{i-1} as follows:

(1) Remove $d_i^{(i-1)}$ from π_{i-1}.
(2) Subtract 1 from the first $d_i^{(i-1)}$ remaining positive degrees except for $d_{j+s}^{(i-1)}$. As shown above, $d_n \geq j + \frac{5}{4}s - 1$ which implies that $d_n^{(j+s-1)}$ is at least 1. Thus, as each degree in D^- is at most $2s - 2$, a sufficiently large choice of n assures us that there will be a sufficient number of positive terms remaining to be reduced.
(3) Reorder *only* the final $n - j - 2s$ terms so that they are nonincreasing.

The reader should note that we do not reorder the first collection of terms in this process. We can be assured that step (2) is possible by considering Lemma 4, which assures us that $d_n^{(j)}$ is at least $d_n - j \geq \frac{5}{4}(s - 1)$, and Lemma 5, which assures us that $d_i^{(i-1)}$ is at most $n - i - 1$.

For each $j + s + 1 \leq i \leq j + 2s$, we construct π_i from π_{i-1} by removing $d_i^{(i-1)}$ from π_{i-1}, reducing the first $d_i^{(i-1)}$ remaining terms by one, and reordering the initial and terminal subsequences, as above.

The remaining case will require the following lemma:

Lemma 6. *If n is sufficiently large and $d_{j+s-1-i}^{(j)}$ is in D^- for $0 \leq i \leq s - 2$ then*

$$d_{j+2s} \geq j + \frac{3}{2}s + i.$$

Proof. Assume that the lemma is false and note that if $d_{j+s-1-i}^{(j)}$ is in D^- then it follows by the definition of D^- that $d_{j+s-1-i} \leq j + 2s - 2$. This implies that

$$\begin{aligned}\sigma(\pi) &\leq (j + s - i - 2)(n - 1) + (s + i + 1)(j + 2s - 2) + (n - j - 2s + 1)\left(j + \frac{3}{2}s + i - 1\right)\\ &= n\left(2j + \frac{5}{2}s - 3 + o(1)\right),\end{aligned}$$

which is less than $\sigma(\pi(n, s, j))$ for n sufficiently large. □

Case 3. $|D^-| \geq s + 2$.

Note that this implies that $d_{j+s-1}^{(j)}$ is in D^-. Hence, Lemma 6 gives that $d_{j+2s}^{(j)} \geq \frac{3}{2}s$.

We partition D^+ into sets A^+ and B^+ and D^- into sets A^- and B^- such that the following hold:

(a) $|A^+| + |A^-| = |B^+| + |B^-| = s$,
(b) $|A^+|$ and $|B^+|$ differ by at most 1, and
(c) $|A^+|$ consists of the largest integers in D^+.

The sets $A = A^+ \cup A^-$ and $B = B^+ \cup B^-$ will contain the degrees of the vertices of each partite set of $K_{s,s}$.

We now reorder $d_{j+1}^{(j)}, \ldots, d_{j+2s}^{(j)}$ so that $d_{j+1}^{(j)}, \ldots, d_{j+|A^+|}^{(j)}$ are the elements of A^+ in nonincreasing order, $d_{j+|A^+|+1}^{(j)}, \ldots, d_{j+s}^{(j)}$ are the elements of A^- in nonincreasing order, $d_{j+s+1}^{(j)}, \ldots, d_{j+s+|B^+|}^{(j)}$ are the elements of B^+ in nonincreasing order, and $d_{j+s+|B^+|+1}^{(j)}, \ldots, d_{j+2s}^{(j)}$ are the elements of B^- in nonincreasing order.

We construct π_i, $j + 1 \leq i \leq j + 2s$, from π_{i-1} based on which of the four sets A^+, A^-, B^+ or B^- contains $d_i^{(j)}$.

If $d_i^{(j)}$ is in A^+, B^+ or B^-, construct π_i by deleting $d_i^{(i-1)}$ from π_{i-1}, reducing the next $d_i^{(i-1)}$ positive terms by one, and reordering *only* the last $n - j - 2s$ degrees to be nonincreasing.

If $d_i^{(j)}$ is in A^-, construct π_i by removing $d_i^{(i-1)}$ from π_{i-1}, reducing the next $d_i^{(i-1)}$ positive degrees *starting with* $d_{j+s+1}^{(i-1)}$ by one, and reordering the last $n-j-2s$ degrees to be nonincreasing. We can assure that there are a sufficient number of positive terms remaining using a similar argument to the one given in the last case.

Claim 1. *If π_{j+2s} is graphic, then π is potentially $K_j + K_{s,s}$-graphic.*

Proof. The key to this claim is to note that in constructing π_i from π_{i-1}, it is possible to keep track of the degrees that were reduced by one. Therefore, if π_i is graphic with realization G_i, it is possible to construct a realization of π_{i-1} by adding a vertex to G_i and connecting it to the appropriate vertices in G_i so as to make π_{i-1} the degree sequence of this new graph. This implies that if π_{j+2s} is graphic, so too are $\pi_1, \ldots, \pi_{j+2s-1}$.

Lemma 2 gives us that $d_1, \ldots, d_j$ are all at least $j + 2s$. Thus, in constructing $\pi_1, \ldots, \pi_j$, we are reducing the degrees of the remaining members of $d_1, \ldots, d_{j+2s}$ by one each time. Hence, if π_{j+2s} is graphic, the rebuilding process will yield a realization of π that contains a copy of K_j joined to some graph on $2s$ vertices.

While examining the various methods used to construct $\pi_{j+1}, \ldots, \pi_{j+s}$, the reader should note that in each case, after removing the first remaining degree, we reduce (amongst others),

$$d_{j+s+1}^{(i)}, \ldots, d_{j+2s}^{(i)}$$

by one each time. Additionally, it is important to note that we do not reorder the first subsequence during these constructions. Hence, when constructing our realization of π, it will contain the join of two sets of s vertices, forming a $K_{s,s}$. □

3.2. $s \equiv 1 \pmod 2$

Let $s \geq 3$ be an odd integer and let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence with $\sigma(\pi) \geq \sigma(\pi(n, s, j)) + 2$. We proceed in a manner similar to the above.

Lemma 7. *Let n be a sufficiently large integer, and let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence such that $\sigma(\pi) \geq \sigma(\pi(n, s, j)) + 2$, where $s \geq 3$ is odd. Then for any integer $0 \leq i \leq j + s - 2$,*

$$d_{j+s-i-1} \geq j + \frac{3s-1}{2} + i.$$

Proof. Assume otherwise. Then, as in Lemma 3,

$$\sigma(\pi) \leq (n-1)(j+s-2) + \left(j + \frac{3s-3}{2}\right)(n-j-s+2) = n\left(2j + \frac{5}{2}s - \frac{7}{2} + o(1)\right).$$

However, for n sufficiently large, this is less than $\sigma(\pi(n, s, j))$. □

We may also make an assumption about d_n similar to Lemma 4.

Lemma 8. *If n is sufficiently large and $\pi = (d_1, \ldots, d_n)$ is as given, then*

$$d_n \geq j + \frac{5}{4}s - \frac{5}{4}.$$

Proof. Let $G_0 = G$. If $\delta(G_0) \geq j + \frac{5}{4}s - \frac{5}{4}$, we are done soso assume there is some vertex $v_0 \in G_0$ such that $d(v_0) = \delta(G_0) < j + \frac{5}{4}s - \frac{5}{4}$. Then, as the degree of v_0 must be an integer, we have that

$$d(v_0) \leq j + \frac{5}{4}s - \frac{7}{4}.$$

Define G_1 to be $G_0 - v_0$. If there is some vertex v_1 in G_1 such that $d(v_1) = \delta(G_1) < j + \frac{5}{4}s - 1$ then we define $G_2 = G_1 - v_1$. If necessary, we proceed in this manner and form G_3, and so on.

Note that, for each $i \geq 1$,

$$\sigma(\pi(G_i)) = \sigma(\pi(G_i)) - 2d(v_i) \geq \sigma(\pi(G_{i-1})) - 2\left(j + \frac{5}{4}s - \frac{7}{4}\right)$$

and hence

$$\sigma(\pi(G_i)) \geq |G_i| \left(2j + \frac{5}{2}s - \frac{5}{2}\right) + i + g(s, j)$$

for each G_i we have constructed, where $g(s, j) = O(s^2 j^2)$.

As in Lemma 4, either this process terminates, or if n is sufficiently large, we may choose an i such that $\sigma(G_i) > |G_i|(2j + 4s - 4 + o(1))$, implying that $\pi(G_i)$ is potentially K_{j+2s}-graphic, and hence that π is potentially $K_j + K_{s,s}$-graphic. If the process terminates, say with some G_i, then we redefine π to be $\pi(G_i)$, as $\sigma(\pi(G_i)) \geq \sigma(\pi(n - i, s, j))$. □

The remainder of the proof will resemble that of the case when s is even. We construct $\pi_1, \ldots, \pi_j$ and define D^+ and D^- in the same manner as above. Again we proceed by considering the number of elements in D^-.

Case 1. $|D^-| \leq s$ and *Case* 2. $|D^-| = s + 1$.

These cases are resolved in a manner identical to that used when s is even. The reader should simply note that Lemma 5 still holds and can therefore be utilized in Case 2.

The remaining case will require the following lemma.

Lemma 9. *If n is sufficiently large and $d^{(j)}_{j+s-1-i}$ is in D^- for $0 \leq i \leq s - 2$ then*

$$d_{j+2s} \geq j + \frac{3s - 1}{2} + i.$$

Proof. Assume that the lemma is false and note that if $d^{(j)}_{j+s-1-i}$ is in D^- then $d_{j+s-1-i} \leq j + 2s - 2$. This implies that

$$\begin{aligned}\sigma(\pi) &\leq (j + s - i - 2)(n - 1) + (s + i + 1)(j + 2s - 2) + (n - j - 2s + 1)\left(j + \frac{3s - 3}{2}s + i\right)\\ &= n\left(2j + \frac{5}{2}s - \frac{7}{2} + o(1)\right),\end{aligned}$$

which is less than $\sigma(\pi(n, s, j))$ for n sufficiently large. □

Case 3. $|D^-| \geq s + 2$.

Note that this implies that $d^{(j)}_{j+s-1}$ is in D^-. Hence, Lemma 9 gives that $d^{(j)}_{j+2s} \geq \frac{3s-1}{2}$.

We partition D^+ into sets A^+ and B^+ and D^- into sets A^- and B^- such that the following hold:

(a) $|A^+| + |A^-| = |B^+| + |B^-| = s$,
(b) $|A^+|$ and $|B^+|$ differ by at most 1, and
(c) $|A^+|$ consists of the largest degrees in D^+.

We now reorder $d^{(j)}_{j+1}, \ldots, d^{(j)}_{j+2s}$ so that $d^{(j)}_{j+1}, \ldots, d^{(j)}_{j+|A^+|}$ are the elements of A^+ in nonincreasing order, $d^{(j)}_{j+|A^+|+1}, \ldots, d^{(j)}_{j+s}$ are the elements of A^- in nonincreasing order, $d^{(j)}_{j+s+1}, \ldots, d^{(j)}_{j+s+|B^+|}$ are the elements of B^+ in nonincreasing order, and $d^{(j)}_{j+s+|B^+|+1}, \ldots, d^{(j)}_{j+2s}$ are the elements of B^- in nonincreasing order.

We construct π_i, $j + 1 \leq i \leq j + 2s$, from π_{i-1} based on which of the four sets A^+, A^-, B^+ or B^- contains $d^{(j)}_i$.

If $d^{(j)}_i$ is in A^+, B^+ or B^-, construct π_i by deleting $d^{(i-1)}_i$ from π_{i-1}, reducing the next $d^{(i-1)}_i$ positive terms by one, and reordering the last $n - j - 2s$ degrees to be nonincreasing.

If $d^{(j)}_i$ is in A^-, construct π_i by removing $d^{(i-1)}_i$ from π_{i-1}, reducing the next $d^{(i-1)}_i$ positive degrees starting with $d^{(i-1)}_{j+s+1}$ by one, and reordering the last $n - j - 2s$ degrees to be nonincreasing.

Again, we may claim the following.

Claim 2. *If π_{j+2s} is graphic, then π is potentially $K_j + K_{s,s}$-graphic.*

4. Completing the proof

Next, we show that π_{j+2s} is, in fact, graphic for any value of s. We use the following two theorems. The first can be found in [10,11].

Theorem 8 (*Li and Yin 2002*). *Let $\pi = (d_1, d_2, \dots, d_n)$ be a nonincreasing sequence of nonnegative integers, where $d_1 = m$ and $\sigma(\pi)$ is even. If there exists an integer $n_1 \le n$ such that $d_{n_1} \ge h \ge 1$ and $n_1 \ge \frac{1}{h}[\frac{(m+h+1)^2}{4}]$, then π is graphic.*

The next theorem can be found in [12].

Theorem 9 (*Li and Yin 2005*). *Let $n \ge 2r+2$ and $\pi = (d_1, d_2, \dots, d_n)$ be graphic with $d_{r+1} \ge r$. If $d_{2r+2} \ge r-1$, then π is potentially K_{r+1}-graphic.*

We proceed with the following two lemmas:

Lemma 10. *If $n \ge 2j+4s$ and $d_{2j+4s} \ge j+2s-2$ then π is potentially $K_j + K_{s,s}$-graphic.*

Proof. Note that if $d_{j+2s} \ge j+2s-1$, then π is potentially K_{j+2s}-graphic by Theorem 9.

Thus, we may assume that $d_{j+2s} \le j+2s-2$. Along with the hypothesis, this implies that

$$d_{j+2s} = d_{j+2s+1} = \cdots = d_{2j+4s} = j+2s-2.$$

Therefore, for each $1 \le i \le j+2s$ the values $d^{(i)}_{j+2s+1}, \dots, d^{(i)}_{2j+4s}$ differ by at most one from our construction. It follows that π_{j+2s} satisfies

$$j+2s-2 \ge m = d^{(j+2s)}_{j+2s+1} \ge \cdots \ge d^{(j+2s)}_{2j+4s} \ge m-1.$$

If $m = 1$, π_{j+2s} must be graphic as $\sigma(\pi_{j+2s})$ is even. If $m \ge 2$, then

$$\frac{1}{m-1}\left[\frac{(m+(m-1)+1)^2}{4}\right] \le m+2 \le j+2s.$$

We apply Theorem 8 with $m = m$, $h = m-1$ and $n_1 = j+2s$ to see that π_{j+2s} is graphic, and hence by Claim 1, π is potentially $K_j + K_{s,s}$-graphic. □

Lemma 11. *Let $s \ge 3$ be an integer and let $\pi = (d_1, \dots, d_n)$ be a graphic sequence with n sufficiently large and $\sigma(\pi) \ge \sigma(\pi(n, s, j)) + 2$. If $d_{2j+4s} \le j+2s-3$, then π is potentially $K_j + K_{s,s}$-graphic.*

Proof. First, note that $d_1 > 2j+4s-2$, otherwise $\sigma(\pi) < (2j+4s-1)(4s-3) + (n-2j-4s+1)(j+2s-3) < \sigma(\pi(n, s, j))$, a contradiction.

We proceed by induction on j. If $j = 0$, then by Theorem 4, π is potentially $K_{s,s}$-graphic.

Therefore, assume that $j \ge 1$. If $d_1 = n-1$, then $\sigma(\pi_1) \ge \sigma(\pi(n-1, s, j-1))$ and the induction hypothesis and Theorem 3 imply that there is a realization of π_1 containing a copy of $K_{j-1} + K_{s,s}$. Hence, as d_1 has degree $n-1$ π is potentially $K_j + K_{s,s}$-graphic.

Suppose then, that $d_1 < n-1$, and there exists an integer r, with $j+2s \le r \le d_1+1$ such that $d_{r+1} < d_r$. By induction, π_1 is potentially $K_{j-1} + K_{s,s}$-graphic and by Theorem 3, there is a realization of π_1 that contains $K_{j-1} + K_{s,s}$ on the highest $j+2s-1$ degrees. By the definition of r, these degrees correspond to $d_2, \dots, d_{j+2s}$ in π. Therefore, π is potentially $K_j + K_{s,s}$-graphic.

If d_1 is not equal to $n-1$ and no such r exists, then

$$n-2 \ge d_1 \ge d_2 \ge \cdots \ge d_{j+2s} = d_{j+2s+1} = \cdots = d_{2j+4s} = \cdots = d_{d_1+2}.$$

As above, we may then assume that there is some m such that

$$j+2s-2 \ge m = d^{(j+2s)}_{j+2s+1} \ge \cdots \ge d^{(j+2s)}_{2j+4s} \ge m-1.$$

We may then complete the proof as in Lemma 10. □

Lemmas 10 and 11 together imply that π is potentially $K_j + K_{s,s}$-graphic, completing the proof of Theorem 5. □

As noted above, this also completes the proof of Theorem 7.

5. Conclusion

At the time of revision, we were made aware that the results in this paper were concurrently obtained in [13]. We invite the interested reader to compare the approaches used in both papers. While there are similarities, this serves to underscore the versatility of these residual subsequence techniques when applied to problems of this type.

As the astute reader will realize, our result also implies the value of $\sigma(K_{r_1,r_2,\ldots,r_l,s,s}, n)$, where $r_1 \leq r_2 \leq \ldots \leq r_l \leq s$ since the function $\sigma(F, n)$ is monotone with respect to subgraph inclusion (that is $\sigma(F', n) \leq \sigma(F, n)$ whenever $F' \subseteq F$). Our result falls short of the result by Li and Yin only in the fact that we require the two largest partite sets to be of equal size, while they do not.

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