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Pancyclic graphs and linear forests

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Abstract

Given integers k, s, t with $0 \leq s \leq t$ and $k \geq 0$, a (k, t, s)-linear forest F is a graph that is the vertex disjoint union of t paths with a total of k edges and with s of the paths being single vertices. If the number of single vertex paths is not critical, the forest F will simply be called a (k, t)-linear forest. A graph G of order $n \geq k + t$ is (k, t)-hamiltonian if for any (k, t)-linear forest F there is a hamiltonian cycle containing F. More generally, given integers m and n with $k + t \leq m \leq n$, a graph G of order n is (k, t, s, m)-pancyclic if for any (k, t, s)-linear forest F and for each integer r with $m \leq r \leq n$, there is a cycle of length r containing the linear forest F. Minimum degree conditions and minimum sum of degree conditions of nonadjacent vertices that imply that a graph is (k, t, s, m)-pancyclic (or just (k, t, m)-pancyclic) are proved.

Keywords: Pancyclic; Linear forest; Minimum degree

1. Introduction

We consider only finite graphs without loops or multiple edges. Notation will be standard, and generally follow that of Chartrand and Lesniak [2]. For a graph G we will use G to represent the vertex set $V(G)$ and the edge set $E(G)$ when the context is clear. Given a subset H of vertices of a graph G, the subgraph induced by H will also be denoted by H when its meaning is clear. Thus, for example, $G - H$ will denote a set of vertices in G not in H as well as a subgraph spanned by these vertices, depending on the context.

Various degree conditions have been investigated which imply that a graph has hamiltonian type properties. The most common is the minimum degree of a graph G denoted by $\delta(G)$. Another common degree condition is the sum of degrees of nonadjacent vertices. For a graph G, $\sigma_2(G) \geq p$ means that $d(u) + d(v) \geq p$ for each pair u and v of nonadjacent vertices in G. The classical results of Dirac [3] and Ore [7] give degree conditions that imply the existence of a spanning cycle, and so G is hamiltonian. Posa generalized this [8] by considering degree conditions that implied the existence of hamiltonian cycles that contain specified edges, or more generally specified vertex disjoint paths (linear forests). This leads to a series of definitions which will formalize this concept.

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Definition 1. Let $k \geq 0$, $t \geq 1$, and $0 \leq s \leq t$ be integers. A (k, t, s)-**linear forest** F is a vertex disjoint union of t paths with a total of k edges and with s of the paths just a single vertex. When the number of single vertex paths is not critical for F, it will be denoted as simply a (k, t)-linear forest. Further, a graph G is (k, t)-**hamiltonian** if for each (k, t)-linear forest F of G, there is a hamiltonian cycle of G containing F.

A σ_2 condition that implies a graph is $(k, t, 0)$-hamiltonian was proved by Posa [8] and Kronk [6]. The precise result will be stated in Section 2. In Section 3 we prove the following result, which sharpens the results of Posa and Kronk.

Theorem 1. *Let k, t and n be positive integers with $2 \leq k + t \leq n$, and let F be a (k, t)-linear forest. If*

(i) $\sigma_2(G) \geq n + k$ *when* $F = P_{k+1} \cup (t-1)K_1$, *and*
(ii) $\sigma_2(G) \geq n + k - \epsilon(k, n)$ *otherwise,*

then G is (k, t)-hamiltonian, where $\epsilon(n, k) = 1$ if $2|(n-k)$ and $\epsilon(n, k) = 0$ otherwise. Furthermore, the condition on $\sigma_2(G)$ is sharp.

For convenience, we will employ this use of $\epsilon(n, k)$ throughout this paper.

Next we give a consequence of Theorem 1.

Corollary 1. *Let k, t and n be positive integers with $2 \leq k + t \leq n$, and let F be a (k, t)-linear forest. If $\delta(G) \geq (n + k)/2$ then G is (k, t)-hamiltonian. The condition on $\delta(G)$ is sharp.*

Theorem 1 verifies that for some (k, t)-linear forests and for some integers n, $\sigma_2(G) \geq n + k - 1$ is sufficient to imply a graph G of order n is (k, t)-hamiltonian. However, $\sigma_2(G) \geq n + k$ is the universal bound.

A graph G of order n is called *pancyclic* whenever G contains a cycle of each length r for $3 \leq r \leq n$. Various generalizations of pancyclic graphs have been studied. For example in [1] Bondy considered *vertex pancyclic* graphs G, which require any fixed vertex of G be on a cycle of every length. A natural generalization of vertex pancyclic is to require that graphs have cycles that contain specified vertices and edges or more generally specified vertex disjoint paths (linear forests). This leads to the following:

Definition 2. Let $k \geq 0$, $s \geq 0$, and $t \geq 1$ be fixed integers with $s \leq t$ and G a graph of order n. For an integer m with $k + t \leq m \leq n$, a graph G is (k, t, s, m)-**pancyclic** (or just (k, t, m)-**pancyclic** if s is not critical) if for each (k, t, s)-linear forest ((k, t)-linear forest) F, there is a cycle C_r of length r in G containing F for each $m \leq r \leq n$.

In [4] generalizations of the classical results of Dirac [3], Ore [7] and Bondy [1] were proved that give both $\delta(G)$ and $\sigma_2(G)$ conditions that imply a graph is $(0, t, m)$-pancyclic. These results will be stated in Section 2.

In Section 4 we prove the following theorem on (k, t, m)-pancyclic graphs, which extends the results in [4] from cycles containing vertices to cycles containing linear forests.

Theorem 2. *Let k, t and n be positive integers with $2 \leq k+t \leq n$. If $\sigma_2(G) \geq n+k$, then G is $(k, t, 2t+k)$-pancyclic. The condition on $\sigma_2(G)$ is sharp for infinitely many n.*

An immediate consequence of Theorem 2 is the following:

Corollary 2. *Let k, t and n be positive integers with $2 \leq k + t \leq n$. If $\delta(G) \geq (n + k)/2$, then G is $(k, t, 2t + k)$-pancyclic. The condition on $\delta(G)$ is sharp for all n.*

For $m < 2t + k$, the $\sigma_2(G)$ bound that implies G is (k, t, m)-pancyclic is significantly larger than that of Theorem 2 and depends on m. In fact, if F is a (k, t)-linear forest, then $H = F + K_{n-t-k}$ will have no cycle containing F with less than $2t + k$ vertices, since any cycle containing F must contain at least one vertex of $H - F$ between each path of F. Since $\sigma_2(H) \geq 2n - 2t - 2k$, this gives a strict lower bound for σ_2 to imply that G is (k, t, m)-pancyclic for $m < 2t + k$. The following sharp bound on $\sigma_2(G)$ for linear forests will also be proved in Section 4. This settles the case when m is small (e.g. $t + k \leq m < 2t + k$).

Theorem 3. *Let k, t, s, m and n be nonnegative integers with $1 \leq k + t \leq m < k + 2t \leq n$ and G a graph of sufficiently large order n. If*

(i) $\sigma_2(G) \geq 2n + 2k - 2m + 1 - \epsilon(m, k)$ *if* $t + k + 1 < m < 2t + k$, *and* $s = 0$,
(ii) $\sigma_2(G) \geq 2n + 2k - 2m + 1$ *if* $m = t + k$ *or* $t + k + 1$ *and* $s = 0$,
(iii) $\sigma_2(G) \geq 2n + 2t - 2m - 1$ *if* $k + t + 1 \leq m < k + 2t$ *and* $2 \leq s \leq 2k$,
(iv) $\sigma_2(G) \geq 2n + 2t - 2m$ *if* $k + t + 1 \leq m < k + 2t$ *and* $s = 1$,
(v) $\sigma_2(G) \geq 2n - t$ *if* $m = t + k$ *and* $1 \leq s \leq 2k$.
(vi) $\sigma_2(G) \geq 2n + 2t - 2m - 1$ *if* $2t - k \leq m < 2t + k$, *and* $s > 2k$,
(vii) $\sigma_2(G) \geq 2n + k - m$ *if* $k + t \leq m < 2t - k$, *and* $s > 2k$.

then G is (k, t, s, m)*-pancyclic. The condition on* $\sigma_2(G)$ *is sharp.*

An immediate corollary to Theorem 3 is the following:

Corollary 3. *Let* k, t, s, m *and* n *be positive integers with* $1 \leq k + t \leq m < k + 2t \leq n$ *and G a graph of sufficiently large order n. If*

(i) $\delta(G) \geq n + k - m + (1 - \epsilon(m, k))/2$ *if* $t + k + 1 < m < 2t + k$, *and* $s = 0$,
(ii) $\delta(G) \geq n + k - m + 1$ *if* $m = t + k$ *or* $t + k + 1$ *and* $s = 0$,
(iii) $\delta(G) \geq n + t - m$ *if* $k + t + 1 \leq m < k + 2t$ *and* $2 \leq s \leq k$,
(iv) $\delta(G) \geq n + t - m$ *if* $k + t + 1 \leq m < k + 2t$ *and* $s = 1$,
(v) $\delta(G) \geq n - t/2$ *if* $m = t + k$ *and* $1 \leq s \leq 2k$.
(vi) $\delta(G) \geq n + t - m$ *if* $2t - k \leq m < 2t + k$, *and* $s > 2k$,
(vii) $\delta(G) \geq n + (k - m)/2$ *if* $k + t + 1 < m < 2t - k$, *and* $s > 2k$

then G is (k, t, s, m)*-pancyclic. The condition on* $\delta(G)$ *is sharp.*

2. Known results

The following result in [4] gives sharp sum of degree conditions for a graph to be $(0, t, m)$-pancyclic, and generalizes the results of Ore [7] and Bondy [1].

Theorem 4 *([4]). Let* $1 \leq t \leq m \leq n$ *be integers, and G be a graph of order n. The graph G is* $(0, t, m)$*-pancyclic if* $\sigma_2(G)$ *satisfies any of the following conditions:*

(i) $\sigma_2(G) \geq n$ *when* $m = n$,
(ii) $\sigma_2(G) \geq \lfloor (4n + 1)/3 \rfloor$ *when* $t = 1$ *and* $m = 3$,
(iii) $\sigma_2(G) \geq 2n - 3$ *when* $t = 2$ *or 3 and* $m = 3$,
(iv) $\sigma_2(G) \geq 2n - m$ *when* $t = 3$ *and* $m = 4$ *or 5*,
(v) $\sigma_2(G) \geq 2n - 2\lceil (m - 1)/2 \rceil - 1$ *when* $4 \leq t \leq m < 2t$, $n > m$,
(vi) $\sigma_2(G) \geq n + 1$ *when* $t \geq 1$, $m \geq \max\{4, 2t\}$, *and* $n > m$.

Also, all of the conditions on $\sigma_2(G)$ *are sharp.*

The corresponding result for minimum degree implying that G is $(0, t, m)$-pancyclic follows directly from Theorem 4 with only a few observations in special cases. In most cases, but not all, the minimum degree condition is precisely half of the degree sum condition (e.g. $\delta(G) = \sigma_2(G)/2$).

Corollary 4 *([4]). Let* $(1 \leq k \leq m \leq n)$ *be positive integers, and let G be a graph of order n. The graph G is* $(0, t, m)$*-pancyclic if* $\delta(G)$ *satisfies any of the following conditions:*

(i) $\delta(G) \geq n/2$ *when* $m = n$,
(ii) $\delta(G) \geq (n + 1)/2$ *then* $t = 1$ *and* $m = 3$,
(iii) $\delta(G) \geq n - 1$ *when* $t = 2$ *or 3 and* $m = 3$,
(iv) $\delta(G) \geq n - 2$ *when* $t = 3$ *and* $m = 4$ *or 5*,
(v) $\delta(G) \geq n - (m/2)$ *when* $4 \leq t \leq m < 2k$, $n > m$,
(vi) $\delta(G) \geq (n + 1)/2$ *when* $t \geq 2$, $m \geq 2t$, *and* $n > m$.

Also, all of the conditions on $\delta(G)$ *are sharp.*

Degree conditions that imply the existence of hamiltonian cycles containing independent edges were investigated by Posa [8] and extended by Kronk [6]. The following sharp bound was proved for a general linear forest, but the bound is not sharp for a fixed linear forest.

Theorem 5 (*Posa [8]*). *Let $0 \le t \le k$ be integers and G a graph of order n. If $\sigma_2(G) \ge n + k$, then for any $(k, t, 0)$-linear forest F, there is a hamiltonian cycle of G that contains the linear forest F. Also, the $\sigma_2(G)$ bound is sharp with respect to general n and general $(k, t, 0)$-linear forests.*

3. Hamiltonian proofs

We begin this section by providing a graph that shows the sharpness of Theorem 5.

Example 1. Let $H_1 = K_{k+1} + (K_{\lfloor (n-k-1)/2 \rfloor} \cup K_{\lceil (n-k-1)/2 \rceil})$ with the linear forest F being a path P_{k+1} in the complete graph K_{k+1}. There is no hamiltonian cycle containing F and $\sigma_2(H_1) = n + k - 1$.

The linear forest in this case is a path with k edges, and so there is the possibility that for other fixed linear forests with k edges a different condition on $\sigma_2(G)$ could be optimal. This is in fact true, and is indicated by Theorem 1. In proving Theorem 1, the following result will be useful.

Theorem 6. *Let G be a graph of order n and t a fixed integer with $2 \le t < n$. If $\sigma_2(G) \ge n - t - \epsilon(n, t)$, then G contains a spanning linear forest with t paths. The condition on $\sigma_2(G)$ is sharp.*

Proof. Let G be a graph of order n with $\sigma_2(G) \ge n - t - \epsilon(n, t)$, and assume that G contains no spanning linear forest with t paths. We can assume that G is edge maximal, so the addition of any edge will give the required spanning linear forest. Hence, there is a spanning linear forest F with $t+1$ paths, say $F = P_1 \cup P_2 \cup \cdots \cup P_{t+1}$. We can assume that $|P_1| \le |P_2| \le \cdots \le |P_{t+1}|$. If $t = n - 1$, the result is clear, so we can assume that $|P_{t+1}| > 1$, and that F is chosen to maximize the number of paths with just one vertex. For each i denote the first and last vertices of P_i by x_i and y_i respectively, with $x_i = y_i$ if the path has only one vertex. Clearly there are no adjacencies between x_i and any x_j or y_j for $j \ne i$.

Consider the vertices x_1 and x_2. If z is an adjacency of x_2 on a path P_j for $j \ge 2$, then x_1 is not adjacent to z^-, since this would imply a spanning linear forest with just t paths. Likewise if z is an adjacency of x_2 on the path P_1, then x_1 is not adjacent to z^+, since this would imply a spanning linear forest with just t paths. This implies that $d(x_1) \le n - (t + 1) - d(x_2)$, since x_1 is not adjacent to x_1 or any y_j for $j \ge 2$. Thus $d(x_1) + d(x_2) \le n - t - 1$, a contradiction when $\epsilon(n, t) = 0$. Hence we can assume that $\epsilon(n, t) = 1$ and so $2|(n - t)$. Furthermore, by repeating this count, it can be seen that neither $|P_{t+1}| = 2$ nor $x_{t+1}y_{t+1} \in G$.

If $|P_t| > 1$, then note that x_t is not adjacent to y^-_{t+1}, since this allows the two paths P_t and P_{t+1} to be replaced by two paths with one of the paths having only a single vertex, contradicting our choice of paths. These observations, along with the argument used on vertices x_1 and x_2 implies that $d(x_t) + d(x_{t+1}) \le n - t - 2$, a contradiction. Hence we can assume that $|P_t| = 1$, and so $|P_{t+1}| = n - t$. Neither x_1 nor x_2 can be adjacent to two consecutive vertices of P_{t+1} or x_{t+1} or y_{t+1}. Since $n - t$ is even, this implies that $d(x_1), d(x_2) \le (n - t - 2)/2$, which implies that $\sigma_2(G) \le n - t - 2$, a contradiction. The sharpness of the σ_2 condition is established by Example 2 which follows. This completes the proof of Theorem 6. □

The following edge analogue result follows directly from Theorem 6.

Theorem 7. *Let G be a graph of order n and p a fixed integer with $1 \le p \le n/2$. If $\sigma_2(G) \ge 2p - 1$, then G contains a spanning linear forest with $2p$ edges. The condition on $\sigma_2(G)$ is sharp.*

Proof. Let $r = n - (2p - 1)$, and note that $\epsilon(n, r) = 1$. Therefore, $\sigma_2(G) = n - r = n - (r - 1) - \epsilon(n, r)$. Thus, by Theorem 6, G has a spanning linear forest F with $r - 1$ paths. Hence, F has $n - (r - 1) = 2p$ edges. The sharpness of the $\sigma_2(G)$ condition is established by Example 2 which follows. This completes the proof of Theorem 7. □

We now describe an example providing sharpness for the $\sigma_2(G)$ condition in Theorems 1, 6 and 7.

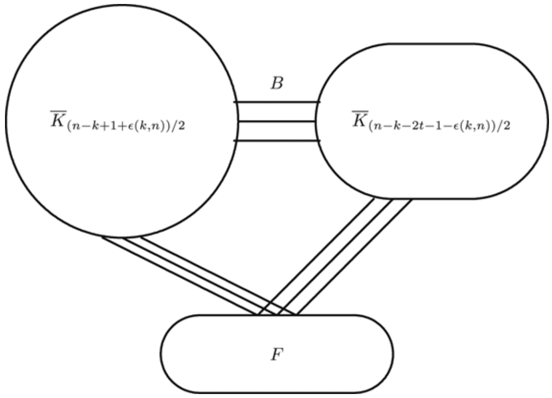

Fig. 1. $F = (k, t)$-linear forest.

Example 2. Consider the complete bipartite graph $B = K_{(n-2t-k-1-\epsilon(k,n))/2,(n-k+1+\epsilon(n,k))/2}$ when $n > 2t+k+1+\epsilon$. Then, $\sigma_2(B) = n - 2t - k - 1 - \epsilon(n, k)$ and any spanning linear forest of B will contain at least $t + 1 + \epsilon(n, k)$ paths. For any $(k, t)-$linear forest F, let $H(k, t, n) = F + B$ (see Fig. 1). There is no hamiltonian cycle of $H(k, t, n)$ containing F, since B has no spanning linear forest with t paths. Also, $\sigma_2(H(k, t, n)) = n + k - 1 - \epsilon(n, k)$, and $\delta(H(k, t, n)) = (n + k - 1 - \epsilon(n, k))/2$. The graph B with $n' = n - 2t$ vertices and $\sigma_2(B) = n' - k - 1 - \epsilon(n, k)$ implies the sharpness of the σ_2 condition in Theorems 6 and 7.

Proof of Theorem 1. The fact that the $\sigma_2(G)$ condition is sharp follows from Example 1 in the case when the linear forest F contains a P_{k+1} and from Example 2 for all of the other linear forests.

Clearly we need to only consider forests F with no isolated vertices, since all vertices of the graph are on a hamiltonian cycle. Let $F \subseteq G$ be such a linear forest. If the result fails, then we can assume that G is edge maximal, that is, the addition of any edge to G will result in the required hamiltonian cycle.

Choose x and y so that $e = xy \notin E(G)$. Hence, there is a hamiltonian cycle in $G + xy$ containing F and also xy. Thus, there is a hamiltonian path $P = (x = x_1, x_2, \ldots, x_n = y)$ of G containing the k edges of F. If x is adjacent to x_j, then y is not adjacent to x_j^-, unless $x_j^- x_j \in F$, since this would give the required hamiltonian cycle in G. Hence, $d(y) \le n - 1 - (d(x) - k)$. This implies that $d(x) + d(y) \le n + k - 1$, a contradiction unless F is not a path and $\epsilon(n, k) = 1$. Note that if x_1x_2 is an edge in F, then this implies that $d(y) \le n - 1 - (d(x) - (k - 1))$, giving $d(x) + d(y) \le n + k - 2$, a contradiction.

We now assume that each vertex in the graph $H = G - F$ is adjacent to each endvertex of every path in F. Also, H has $n-k-t$ vertices and $\sigma_2(H) \ge \sigma_2(G) - 2(k+t) \ge |H| - t - 1$. Since $\epsilon(n-k-t, t) = \epsilon(n, k) = 1$, Theorem 6 implies that there is a spanning forest of H with t paths. Thus, a hamiltonian cycle of G containing F can be formed using F, the linear forest of H with t paths, and the edges between H and the endvertices of F. This contradiction completes the proof of Theorem 1. □

4. Pancyclic proofs

Since for a graph G of order n, $\sigma_2(G) \ge n + k$ implies G is (k, t)-hamiltonian, it is reasonable to ask what other length cycles does G have that also contain F. The following example provides a lower bound on the order of a cycle containing F.

Example 3. For positive integers $k, t \ge 1$ and $n \ge k + 2t$, let F be a (k, t)-linear forest, and let $H = F + K_{n-k-t}$. Then $|H| = n$ and the smallest cycle containing F has order at least $k + 2t$, since any cycle containing F must have at least t vertices not in F. Also, $\sigma_2(H) \ge 2(n - k - t)$, and so $\sigma_2(H) \ge n + k$ if $n \ge 3k + 2t$.

Before proving Theorem 2, we consider the case of a linear forest with paths of length 0 or 1 only.

Theorem 8. *Let $1 \le k \le t$ be fixed integers, and let G be a graph of order $n \ge 3k+2t$. Let F be a $(k, t, t-k)$-linear forest of G. If $\sigma_2(G) \ge n+k$, then for each $r \ge \max\{4, k+2t\}$, G has a cycle of length r containing F. The condition on $\sigma_2(G)$ is sharp for general n.*

Proof. Since $\sigma_2(G) \geq n+k$, it follows that every pair of nonadjacent vertices have at least $k+2$ common neighbors. We show there is a cycle containing F of length at most $\max\{4, k+2t\}$. If $t = k = 1$, then consider an edge $e = xy$. If x and y have no common adjacency, then any vertex z adjacent to x that is nonadjacent to y has at least two common adjacencies with y, and so e is on a cycle of length 4. Now assume that $t \geq 2$.

Assume that G does not contain a cycle of length at most $k+2t$ containing F; that is, more specifically there is no cycle with at most one vertex between consecutive paths of F. Also assume G is edge maximal with respect to this property. Select some nonadjacent pair x and y such that x and y are endvertices of distinct paths of F, and let $e = xy$. Thus, in $G+e$ there is a cycle $D = (x = x_1, x_2, \ldots, x_p = y, x)$ of length $p \leq k+2t-1$. If x and y have a common adjacency in $H = G - D$, then the required cycle is in G, so we can assume that $d_H(x) + d_H(y) \leq n - p$. Note that if x is adjacent to x_i and y is adjacent to x_{i-1} with $x_{i-1}x_i \notin F$, then this gives the required cycle. Hence, $d_D(y) \leq p - 1 - (d_D(x) - k)$, and so $d(x) + d(y) \leq n + k - 1$, a contradiction. Thus, there is a cycle of length at most $k + 2t$ containing F.

Let $D' = (y_1, y_2, \ldots, y_p, y_1)$ be a cycle containing F of maximum length $p \leq k+2t$. We need $p = k+2t$ or $p = k+2t-1$. Assume that $p < k+2t-1$. With no loss of generality we can assume that y_1 has an adjacency, say z, in $H' = G - D'$ and that $y_1y_2 \notin F$. The maximality of p implies that y_2 and z are nonadjacent, and that y_2 and z have no common adjacency in H'. Therefore, $d_{H'}(y_2) + d_{H'}(z) \leq n - p - 1$. Also, just as in the previous argument, $d_{D'}(y_2) + d_{D'}(z) \leq p + k$. However, this implies that $d(y_2) + d(z) \leq n + k - 1$, a contradiction. Thus, $p = k + 2t$ or $k + 2t - 1$.

Select a cycle $B = (u_1, u_2, \ldots, u_m, u_1)$ of G containing F such that m is maximum with respect to the property that there are cycles in G containing F of each length r between p (where $p = k+2t-1$ or $k+2t$) and m. If $m = n$, the proof is complete, so assume that $m < n$, and there is no cycle of length $m+1$ containing F. We show this leads to a contradiction. Let $H = G - B$. Consider a vertex $u_j \in B$ having an adjacency v in H. We assume that $u_ju_{j+1} \notin F$. The maximality of B implies that $vu_{j+1} \notin G$. Also, if $vu_i \in G$, then $u_{j+1}u_{i+1} \notin G$ if $u_{i+1}u_i \notin F$. Thus, $d_B(v) + d_B(u_{j+1}) \leq m+k$, and so $d_H(v) + d_H(v_{j+1}) \geq n - m$. Hence, there is a vertex $v' \in H$ that is adjacent to both v and u_{j+1}, which implies the existence of a cycle of length $m+2$ containing F. By the connectivity of G there are several such vertices u_j and it follows that there are no chords of B that skip precisely one vertex of B not in F.

We claim there is a vertex $u_i \in B$ that has an adjacency in H with $u_{i+1} \notin F$ (or $u_{i-1} \notin F$). Assume this is not true. Choose a vertex $u_j \in B - F$. Note, it cannot have an adjacency in H, since this implies that u_{j-1} (and likewise u_{j+1}) has an adjacency in H by the argument above, which gives the claim. Also, neither u_{j+1} nor u_{j-1} can have adjacencies in H and $u_{j-1}u_{j+1} \notin G$. Thus, $d_B(u_{j-1}) + d_B(u_{j+1}) > n + k$, and so there exists $u_i, u_{i+1} \in B$ with $u_iu_{i+1} \notin F$, and with $u_{j+1}u_{i+1}, u_{j-1}u_i \in G$. Thus, using one of the previously mentioned cycles of length $m+2$ derived from B, there is a cycle of length $m+1$ containing F that avoids u_j. This gives a contradiction, and verifies the claim.

With no loss of generality we can assume that u_1 has an adjacency, say v_1, in H, and that u_2 is not in F. The maximality of m implies that u_2 is not adjacent to v_1 and in fact

$$d_H(u_1) + d_H(u_2) \leq n - m.$$

Also, as before, $d_B(v_1) + d_B(u_2) \leq m+k$. This implies $d_H(v_1) + d_H(u_2) \geq n - m$, and so v_1 and u_2 have a common adjacency in H, say v_2. Note that v_2 is not adjacent to either u_1 or u_3. As noted above, $u_1u_3 \notin E(G)$. Also, a repeat of a common previous argument, implies that

$$d_B(u_1) + d_B(u_3) \leq m + k.$$

Thus, u_1 and u_3 have a common adjacency in H, say v_3. To avoid a cycle of length $m+1$ avoiding u_2, we have that

$$d_H(v_3) + d_H(u_3) \leq n - m,$$

since v_3 and u_3 cannot have a common neighbor in H. Also, each of u_2 and v_3 cannot be adjacent to two consecutive vertices of B unless they are endvertices of an edge in F. Hence, each is adjacent to at most $(m+k)/2$ vertices of B. Thus,

$$d_B(u_2) + d_B(v_3) \leq m + k.$$

The four inequalities in the previous paragraph gives in the following inequality:

$$2(n+k) \leq d(u_1) + d(u_3) + d(u_2) + d(v_3) \leq 2(n-m) + 2(m+k).$$

Therefore these five inequalities are, in fact, equalities. It follows that

$$d_H(u_2) + d_H(v_3) = d_H(u_1) + d_H(u_3) = n - m.$$

Since the neighborhoods in H of u_1 and u_3 are disjoint from the neighborhoods of u_2 and v_3, the vertices of H are partitioned into two disjoint sets H_1 and H_2, where $H_1 = N(u_1) = N(u_3)$ and $H_2 = N(u_2) = N(v_3)$. The set H_1 is independent, and each vertex in H_1 has the same properties as v_3. Hence each vertex in H_1 is adjacent to precisely $(m+k)/2$ vertices of B, is adjacent to the endvertices of an edge of F, and is not adjacent to any other pair of consecutive vertices of B. Therefore each vertex in H_1 is nonadjacent to $(m-k)/2$ vertices of B, which we denote by B_1. Each of the vertices in B_1 will have the same properties as the vertices in H_1, since they can be interchanged with a vertex in H_1, and so the set $B_1 \cup H_1$ is an independent set with $(n-k)/2$ vertices, and each vertex in $B_1 \cup H_1$ is adjacent to the remaining $(n+k)/2$ vertices of G. With this structure it is easily seen that a cycle containing F of each length from $k+2t$ to n can be formed. To see this, recall that F has k edges and t paths, say $R_1, R_2, \ldots, R_t$, and hence $|V(F)| = k + t$. Now select vertices $x_1, \ldots, x_t$ in $H_1 \cup B_1$. As each x_i is adjacent to all of $G - (H_1 \cup B_1)$, we form the cycle $x_1, R_1, x_2, R_2, \ldots, R_t, x_t$ which contains F and has $k + 2t$ vertices. Extending this cycle in each of the remaining cases amounts to either including an extra edge from $G - (H_1 \cup B_1)$ adjacent to an endvertex of some R_i or replacing that edge with two vertices in the bipartite structure (one from $G - (H_1 \cup B_1)$ and one from $H_1 \cup B_1$).

The graph $H(k, t, n)$ described in Example 2 gives the sharpness of the σ_2 condition. This completes the proof of Theorem 8. □

Completing this, the proof of Theorem 2 now follows easily.

Proof. Let G be a graph of order n with $\sigma_2(G) \geq n + k$ containing a (k, t)-linear forest F. Consider the graph G' obtained from G by deleting all of the interior vertices, say $k - k'$ for some $k' \leq k$, of the paths in F and replacing each path by an edge. Thus, the forest F becomes a (k', t)-linear forest F' with only paths of length 0 or 1 (e.g. F' is a $(k', t, t-k')$-linear forest), G' has order $n' = n - k + k'$, and $\sigma_2(G') \geq n + k - 2(k - k') = n' + k'$. Thus, by Theorem 8, G' has cycles containing F' of each length from $\max\{4, k' + 2t\}$ to n'. Replacing each edge in F' with the corresponding path in each of the cycles of G' in F gives the required cycles in G. Again, the graph $H_2(k, t, n)$ of Example 2 gives the sharpness of the σ_2 condition. This completes the proof of Theorem 2. □

Before proving Theorem 3, we present some sharpness examples.

Example 4. (i) Let $k < r \leq 2k$ with $k \geq 2$ and $n \geq 2k$ be fixed integers, and let $F_r^* = (r/2)K_2$ when r is even and $F_r^* = ((r-1)/2)K_2 \cup K_1$ when r is odd. Define $H_{k,r} = F_r^* + K_{n-r}$. Then $H_{k,r}$ has order n and $\sigma_2(H_{k,r}) = 2n - 2r - \epsilon(r-1, 0)$. Consider a linear forest $F = kK_2$ in $H_{k,r}$ having r of its vertices and $\lfloor r/2 \rfloor$ of its edges in F_r^*. Any cycle of $H_{k,r}$ that contains F will have length at least $k + r$.

(ii) Modify the example in (i) by adding the positive integer t with $k+2 \leq t$, $n \geq k+t$, and $k < r \leq t+k$. For any r, let $F_r^* = ((r-t+k)/2)K_2 \cup (t-k)K_1$ when $r-t+k$ is even and $F_r^* = ((r-t+k-1)/2)K_2 \cup K_1 \cup (t-k)K_1$ when $r-t+k$ is odd. Define $H_{k,t,r} = F_r^* + K_{n-r}$. Then $H_{k,t,r}$ has order n and $\sigma_2(H_{k,t,r}) = 2n - 2r$. Consider a linear forest $F = kK_2 \cup (t-k)K_1$ in $H_{k,t,r}$ having r of its vertices and $\lfloor (r-t+k)/2 \rfloor$ of its edges in F_r^*. Any cycle of $H_{k,t,r}$ that contains F will have length at least $t + r$.

(iii) Modify the example in (ii) for the case when $t > 2k$, so that the number of independent vertices dominates the number of independent edges in the linear forest. Thus, $t > 2k$, $n \geq k+t$, and $t/2 \leq r \leq t-k$. For any r, let $F_r^* = rK_1$. Define $H_{k,t,r} = F_r^* + K_{n-r}$. Then $H_{k,t,r}$ has order n and $\sigma_2(H_{k,t,r}) = 2n - 2r$. Consider the linear forest $F = kK_2 \cup (t-k)K_1$ in $H_{k,t,r}$ having r vertices in F_r^*. Any cycle of $H_{k,t,r}$ containing F will have length at least $k + 2r$.

The next result concerns the special case of F being a matching.

Lemma 1. *Let k, m and n be integers with $2 \leq 2k \leq m \leq 3k - 1 \leq n$. If n is sufficiently large and*

(i) $\sigma_2(G) \geq 2n + 2k - 2m + 1 - \epsilon(m, k)$ *if* $2k + 1 < m < 3k$,
(ii) $\sigma_2(G) \geq 2n + 2k - 2m + 1$ *if* $m = 2k$ *or* $2k + 1$,

then G is $(k, k, 0, m)$-pancyclic. The condition on $\sigma_2(G)$ is sharp.

Proof. (i) Let M be a fixed set of k independent edges in G, and let H be the graph spanned by the vertices in M. Let $p = 3k - m$, and so condition (i) implies that $\sigma_2(H) \geq \sigma_2(G) - 2(n-2k) = 2p+1-\epsilon(3k-p,k) = 2p+1-\epsilon(p,0)$. To verify (i) it is sufficient to show that $\sigma_2(H) \geq 2p+1-\epsilon(p,0)$ implies that there is a linear forest F in H such that $M \subseteq F$ and F has $k+p$ edges. This can be seen from an induction on p, keeping in mind the degree conditions and the fact the p edges beyond those of M form a matching. The existence of a cycle of length m follows, since the degree condition for G implies that any pair of nonadjacent vertices has at least $n+2k-2m+2 = n-2k-2$ (when $m = 2k+2$, the smallest possible value) common neighbors. As n is sufficiently large, a cycle of length $3k - p = m$ containing the forest F is easily constructed as any pair of nonadjacent vertices have so many common adjacencies that there remain many choices, even after some of the paths of F have been linked together by these common neighbors. The existence of cycles of each length from m to n containing M also follows from the fact that the degree condition always implies the existence of a vertex not on a cycle that is adjacent to two consecutive vertices on the cycle that are not the endvertices of an edge of M.

The existence of the desired forest F is shown by induction on p. If $p = 1$, then $\sigma_2(H) \geq 3$, and so there is an edge e in H not in M. This gives the required linear forest $F = M \cup \{e\}$. If $p = 2$, then $\sigma_2(H) \geq 4$ can easily be shown to imply the existence of two independent edges e_1 and e_2 in H that are disjoint from M and do not form a cycle with the edges of M. Thus, $M \cup \{e_1, e_2\}$ gives the required linear forest F with $k+2$ edges.

Now assume that $3 \leq p \leq k-2$, and the appropriate linear forest exists for any $j < p$. Partition the vertices of H into a minimum number of paths, say $P_1, P_2, \ldots, P_q$, which start and end in edges of M and contain all of M. Let $F = P_1 \cup P_2 \cup \cdots \cup P_q$. Let N be the edges in these paths that are not in M. Therefore, $|M| + |N| + q = 2k$. Let x_i and y_i be the endvertices of P_i for $1 \leq i \leq q$. Assume for some p that the required linear forest does not exist, and so by induction we can assume that $|N| = p - 1$. Note that $q \geq 3$, since $p \leq k-2$. It is possible that $x_i y_i$ is an edge of H. When this occurs and the path P_i has more than one edge, this implies that the vertices of P_i form a cycle. Otherwise, P_i is just an edge. Assume the first q' paths are either just an edge or induce a cycle, but this is not true for the remaining $q - q'$ paths. We also assume that the path system was chosen to maximize the number q' (possibly $q' = 0$).

First let $q' = q$ so that all of the paths induce cycles or are just edges. Select vertices x_1 and x_2 in P_1 and P_2 respectively. In this case, without loss of generality, we can assume that $|P_1| \leq |P_2| \leq \cdots \leq |P_q|$. Note that there are no edges between x_1 (or x_2) and any of the vertices of the other $q-1$ paths, since then two paths can be collapsed into one longer path, contradicting the choice of F. In this case $|P_1| + |P_2| \geq d(x_1) + d(x_2) + 2 \geq 2p + 2$. This implies that there are at least $p-1$ edges of N in $P_1 \cup P_2$ and at least one edge of N in P_3. This gives a contradiction, so we can assume that $q' < q$. Next consider the case when $q \geq q' + 2$, and consider the vertices x_{q-1} and x_q. There are no edges between x_{q-1} or x_q and any of the vertices in $P_1 \cup \cdots \cup Pq'$. Select an edge $e = z_1z_2 \in N$ that is on one of the paths P_j for $j > q'$. Note that if there are as many as 3 edges between $\{x_{q-1}, x_q\}$ and $\{z_1, z_2\}$, then in the case when the edge z_1z_2 is not on paths P_{q-1} or P_q, three paths can be collapsed into two paths with one more edge in N. If z_1z_2 is on either the path P_{q-1} or P_q, the two paths can be replaced by just one path or the two paths can be replaced by two paths with the same number of edges in N, but one of the paths is either an edge or induces a cycle. Each condition contradicts the choice of the forest F, and so there are at most two edges between $\{x_{q-1}, x_q\}$ and $\{z_1, z_2\}$. Therefore, the number of edges of N in the paths $P_{q'+1} \cup \cdots \cup P_q$ is at least $(d(x_{q-1}) + d(x_q))/2 \geq p$, a contradiction.

We are left with the case when $q' = q - 1$. In this final case, consider the vertices x_{q-1} and x_q and observe that the number of edges of N in the paths $P_{q-1} \cup P_q$ is at least $(d(x_{q-1}) + d(x_q))/2 \geq p$, a contradiction. The sharpness of the conditions of (i) follow from Example 4. This completes the proof of (i).

The proof of (ii) uses the same proof techniques as (i). By (i) we have that the maximal linear forest $F = P_1 \cup P_2$ will have two paths, and so $q = 2$. By assumption $\sigma_2(H) = 2k - 1$. There are three possibilities for the paths P_1 and P_2, both are cycles, precisely one is a cycle, or neither is a cycle. However, in each of the cases the previous arguments imply that $d(x_1) + d(x_2) \leq 2k - 2$, a contradiction. The sharpness of the condition of (ii) is a consequence of the graph $K_{n-2k} + (K_4 \cup K_{2k-4})$, where the forest $F \subseteq K_4 \cup K_{2k-4}$. This completes the proof of (ii) when $m = 2k + 1$. The proof of (ii) when $m = 2k$ follows directly from the results of Häggkvist in [5]. This competes the proof of Lemma 1. □

The next result concerns a more general case when the linear forest F has both edges and single vertices, but the number of edges dominates the number of single vertex paths.

Lemma 2. *Let $k < t < m < n$ be integers with $2 \le k+t < m \le k+2t-1 < n$ and $t \le 3k$. If n is sufficiently large and*

(i) $\sigma_2(G) \ge 2n + 2t - 2m - 1$ *if* $k + t < m < k + 2t$ *and* $t \ge k + 2$,
(ii) $\sigma_2(G) \ge 2n + 2t - 2m$ *if* $k + t < m < k + 2t$ *and* $t = k + 1$,
(iii) $\sigma_2(G) \ge 2n - t$ *if* $m = k + t$ *and* $k + 1 \le t \le 3k$,

then G is $(k, t, t-k, m)$-pancyclic. The condition on $\sigma_2(G)$ is sharp.

Proof. (i) Let M be a fixed set of k independent edges and $t-k$ independent vertices in G, and let H be the graph spanned by the vertices in M. Let $p = k + 2t - m$, and so condition (i) implies that $\sigma_2(H) \ge \sigma_2(G) - 2(n-k-t) = 2p - 1$. To verify (i) it is sufficient to show that $\sigma_2(H) \ge 2p - 1$ implies that there is a linear forest F in H such that $M \subseteq F$ and F has $k + p$ edges. The existence of a cycle of length m follows, since the degree condition for G is sufficiently large that any pair of nonadjacent vertices have many common adjacencies and so the linear forest can be extended to a cycle of length $k + 2t - p = m$. The existence of cycles of each length from m to n containing M also follows from the fact that the degree condition always implies the existence of a vertex not on a cycle that is adjacent to two consecutive vertices on the cycle that are not the endvertices of an edge of M.

The existence of the required forest F is proved by induction on p. If $p = 1$, then $\sigma_2(H) \ge 1$, and there is an edge e in H not in M, since M contains at least two independent vertices. This gives the required linear forest $F = M \cup \{e\}$. If $p = 2$, then $\sigma_2(H) \ge 3$ can easily be shown to imply the existence of two edges e_1 and e_2 in H that are disjoint from the edges of M and do not form a cycle with the edges of M. Thus $M \cup \{e_1, e_2\}$ gives the required linear forest F with $k + 2$ edges.

We now assume that $3 \le p \le t - 1$, and the appropriate linear forest exists for any $j < p$. Partition the vertices of H into a minimum number of paths, say $P_1, P_2, \ldots, P_q$. Let $F = P_1 \cup P_2 \cup \cdots \cup P_q$, and let N be the edges in these paths that are not in M. Therefore, $|M| + |N| + q = k + t$. Let x_i and y_i be the endvertices of P_i for $1 \le i \le q$. Assume for some p that the required linear forest does not exist, and so by induction we can assume that $|N| = p - 1$.

Note that $q \ge 2$, since $p \le t - 1$. It is possible that $x_i = y_i$ and P_i is just a vertex. It is also possible that $x_i y_i$ is an edge of H. When this occurs and the path P_i has more than one edge, this implies that the vertices of P_i induce a cycle. Otherwise, P_i is just an edge. Assume that the first q' paths are either just a vertex, an edge, or induce a cycle (we call this collection generalized cycles), but this is not true for the remaining $q - q'$ paths. We will also assume that the path system was chosen to maximize the number q' (possibly $q' = 0$). We will also assume that the number of vertices in the q' generalized cycles is maximized relative to the other properties being maximized.

Case 1: $q' \ge 2$.

Select vertices x_1 and x_2 in P_1 and P_2 respectively. Each adjacency of x_1 and x_2 is an endvertex of an edge of N. If $e = z_1 z_2$ is an edge in N and also in P_i for $i \ge 3$, then the total number of adjacencies of x_1 and x_2 relative to e is a most two, for otherwise the three paths P_1, P_2, and P_i could be collapsed into just two paths, a contradiction. Also, since P_1 is a generalized cycle, x_2 cannot be adjacent to any vertex of P_1, and the same is true for x_1 relative to P_2. Therefore, the total number of adjacencies of x_1 and x_2 relative to any e of N is a most two. This implies $d(x_1) + d(x_2) \le 2|N| = 2p - 2$, which contradicts the $\sigma_2(G)$ condition. Thus, we can assume that $q' < 2$.

Case 2: $q' = 0$.

Again, select vertices x_1 and x_2 in P_1 and P_2 respectively. As before, the total number of adjacencies of x_1 and x_2 relative to any $e \in N$ on a path P_i for $i \ge 3$ is a most two. Therefore, there must be some edge $e = z_1 z_2$ in $P_1 \cup P_2$ such that the total number of adjacencies of x_1 and x_2 relative to e is at least three, for otherwise we have a contradiction to the σ_2 condition. With no loss of generality we assume that $e \in P_2$, and that z_1 proceeds z_2 on the path. Note that e cannot be the first edge of P_2, since each of x_1 and x_2 can have at most one adjacency to e. Consider the case when x_2 is adjacent to both z_1 and z_2. If x_1 is adjacent to z_1, then the two paths can be collapsed into a single path, and if x_1 is adjacent to z_2, then the two paths can be replaced by two paths with one of the paths being a generalized cycle. Each of these subcases gives a contradiction. A symmetric argument applies when x_1 is adjacent to both z_1 and z_2. This gives a contradiction that completes the proof of Case 2.

Case 3: $q' = 1$ and $|P_1| = 1$.

As before consider the vertices x_1 and x_2 in P_1 and P_2 respectively, and recall that there must be some edge $e = z_1 z_2$ in P_2 such that the total number of adjacencies of x_1 and x_2 relative to e is at least three. We can also assume that z_1 proceeds z_2 on the path P_2. Consider the case when x_2 is adjacent to both z_1 and z_2. Then if x_1 is adjacent to

z_1, then the two paths can be collapsed into a single path, and if x_1 is adjacent to z_2, then the two paths can be replaced by two paths with one of the paths being a generalized cycle larger than the single vertex generalized cycle. Each of these subcases gives a contradiction. When x_1 is adjacent to both z_1 and z_2, then x_1 can be inserted into P_2 and the two paths can be collapsed into one path. This gives a contradiction that completes the proof of Case 3.

Case 4: $q' = 1$ and $|P_1| \geq 2$.

Recall that for both pairs x_1 and x_2, as well as y_1 and x_2, there must be some edge $e = z_1z_2$ in $P_2 \cap N$ ($z_1 < z_2$) such that the total number of adjacencies of x_1 and x_2 (or y_1 and y_2) relative to e is at least three. Select e to be the last such pair of P_2 with this property starting with the vertex x_2. We can assume with no loss of generality that e is associated with the pair x_1 and x_2. Note that $z_1 \neq x_2$, since x_2 is not adjacent to either x_1 or y_2. Thus, $|P_2| \geq 3$.

Select z_1 and z_2 to be the last pair of vertices of P_2 starting with the vertex x_2. First consider the case when $x_1z_1, x_1z_2 \in G$. Observe that $x_2z_2 \notin G$, since then P_1 and P_2 can be collapsed into one path. Thus, $x_2z_1 \in G$. Note that neither y_1 nor y_2 can have any adjacencies in the interval of P_2 from x_1 to z_1, since then two paths could be collapsed into one path. Thus for the pair of vertices y_1 and x_2 the edge z_1z_2 is the edge in which y_1 and x_1 collectively have three adjacencies. Hence, $y_1z_1, y_1z_2 \in G$. This gives a contradiction, since then the path P_1 can be inserted into P_2 to form one path.

Therefore we assume that $x_2z_1, x_2z_2 \in G$. To avoid the paths P_1 and P_2 from collapsing into one path, $x_1z_2 \in G$ and $x_1z_1 \notin G$. Just as before, each of y_1 and y_2 have no adjacencies in the interval of P_2 from x_1 to z_1. Also, $y_1z_2 \in G$. Hence, z_1z_2 is the only edge of $P_2 \cap N$ where x_1 and x_2 collectively have three adjacencies. Likewise there is only one edge in $P_2 \cap N$ to which x_1 and y_2 collectively have three adjacencies. There can be no such edge of N between x_2 and z_1, since then there would be an edge of N in which y_1 and y_2 (and likewise x_1 and y_2) would have no adjacencies, which contradicts the sum of degrees of y_1 and y_2. This implies that $z_1 = x_2^+$ and $z_2 = x_2^{++}$.

By symmetry the same applies to x_1 and y_2 and so $y_2y_2^{--}, x_1y_2^{--}, y_1y_2^{--} \in G$, and $y_2^{--}y_2^-$ is the edge of N where x_1 and y_2 collectively have three adjacencies. Consider the three disjoint generalized cycles $C^* = (x_1, \ldots, y_1, x_2^{++}, \ldots, y_2^{--}, x_1)$, $x_2x_2^+$ and $y_2y_2^-$ that have the same vertices as $P_1 \cup P_2$. By the σ_2 condition x_2 and y_2 must collectively have three adjacencies into some edge $f = w_1w_2 \in N \cap C^*$, say x_2w_1, x_2w_2, y_2w_2. However this implies immediately that there is one path $(x_2^+, x_2, w_1, \ldots, w_2, y_2, y_2^-)$ that contains all of the vertices of $P_1 \cup P_2$. This gives a contradiction that completes the proof of Case 4.

The proof of (ii) is essentially identical to that of (i). Using the same notation as in (i), note that in the case of $p = 1$ the stronger condition that $\sigma_2(H) \geq 2$ is needed to get the linear forest with $k + 1$ edges, but from that point forward the proof is identical. This completes the proof of (ii).

Note that the linear forest F has k edges and $t - k$ vertices. As we want $m = k + t$, then the graph F^* induced by $V(F)$ must be hamiltonian. As $\sigma_2(G) \geq 2n - t$ and $|V(F^*)| = k + t$, then we see that $\sigma_2(F^*) \geq 2n - t - 2(n - k - t) > k + t$, hence F^* is hamiltonian. The remaining cycle lengths are easily constructed since the degree sum condition is so large. This completes the proof of Lemma 2. □

The next lemma deals with the case when the linear forest has both edges and single vertices, but the number of vertices dominates the number of edges.

Lemma 3. *Let $k < t < m < n$ be integers with $2 \leq k + t < m \leq k + 2t - 1 < n$ and $t > 2k$. If n is sufficiently large and*

(i) $\sigma_2(G) \geq 2n + 2t - 2m - 1$ *if* $2t - k \leq m < 2t + k$,
(ii) $\sigma_2(G) \geq 2n + k - m$ *if* $k + t \leq m < 2t - k$,

then G is $(k, t, t - k, m)$-pancyclic. The condition on $\sigma_2(G)$ is sharp.

Proof. (i) The proof of this case is identical to that of Lemma 2(i), and so will not be repeated here.

(ii) Let M be a fixed set of k independent edges and $t - k$ independent vertices in G, and let H be the graph spanned by the vertices in M. Let $p = k + 2t - m$, and so condition (ii) implies that $\sigma_2(H) \geq \sigma_2(G) - 2(n - k - t) = 2k + p$. To verify (ii) it is sufficient to show that $\sigma_2(H) \geq 2k + p$ implies that there is a linear forest F in H such that $M \subseteq F$ and F has $2k + p$ edges. The existence of a cycle of length m follows, since the degree condition for G is sufficiently large that any pair of nonadjacent vertices have many common adjacencies and so the linear forest can be extended to a cycle of length $k + 2t - p = m$. The existence of cycles of each length from m to n containing M also follows from the fact that the degree condition always implies the existence of a vertex not on a cycle that is adjacent to two consecutive vertices on the cycle that are not the endvertices of an edge of M.

The remainder of the proof is by induction on p, where in this case $2k < p < t$. The case when $p = t$ follows directly from Theorem 1. The case $p = 2k$ is a consequence of (i), and the induction can be started with $p = 2k$. We now assume that $2k < p \leq t - 1$, and the appropriate linear forest exists for any $j < p$. Partition the vertices of H into a minimum number of paths, say $P_1, P_2, \ldots, P_q$. Let $F = P_1 \cup P_2 \cup \cdots \cup P_q$, and let N be the edges in these paths that are not in M. Therefore, $|M| + |N| + q = k + t$. Let x_i and y_i be the endvertices of P_i for $1 \leq i \leq q$. Assume for some p that the required linear forest does not exist, and so by induction we can assume that $|N| = p - 1$.

Observe that if $x_1 z \in H$ for a vertex $z \in P_1$, then $x_2 z^- \notin H$ unless $z^- z \in M$, since this would allow the paths P_1 and P_2 to be collapsed into one path with an additional edge in N, a contradiction. Also, if $x_1 z \in H$ for a vertex $z \in P_i$ for $i \geq 2$, then $x_2 z^+ \notin H$ unless $z z^+ \in M$, since this would allow the paths P_1 and P_2 and P_i if $i > 2$ to be collapsed into one path (or two paths if $i > 2$) with an additional edge in N, a contradiction. This implies that $d_H(x_2) \leq k + t - (d_H(x_1) - k) - q$, since there are no adjacencies between endvertices of distinct paths P_i and P_j. Thus,

$$d_H(x_1) + d_H(x_2) \leq t + 2k - q = 2k + p - 1,$$

a contradiction. This completes the proof for $2k < p < t$, which completes the proof of Lemma 3. $\square$

Theorem 3 is a corollary of Lemmas 1–3.

Proof of Theorem 3. (i) Let L be a $(k, t, 0)$-linear forest in a graph G of order n such that $\sigma_2(G) \geq 2n + 2k - 2m + 1 - \epsilon(m, k)$. Let G' be the graph obtained from G by deleting the $k - t$ interior vertices of the paths in F and replacing the path by an edge. Thus, the forest L becomes a $(t, t, 0)$-linear forest L' with only paths of length 1 and G' has order $n' = n - k + t$. Let $m' = m - k + t$. Then $\sigma_2(G') \geq 2n + 2k - 2m + 1 - \epsilon(m, k) - 2(k - t) = 2n' + 2t - 2m' + 1 - \epsilon(m', t)$. Thus, by Lemma 1(i), G' has cycles containing L' of each length from m' to n'. Replacing each edge in L' with the corresponding path in L for each of the cycles in G' gives the required cycles in G. The sharpness of the condition for (i) is a result of the graphs obtained from Example 4(i) by starting with the forest $F = tK_2$ instead of $F = kK_2$ and replacing each edge in F with the corresponding path in L (with interior vertices of the paths adjacent to all other vertices) to obtain L and a graph H. This completes the proof of (i).

(ii) The proof of this case is identical to (i) in that each path is reduced to an edge and Lemma 2(ii) is applied to a graph G' of order $n' = n - k + t$ with $m' = m - k + t$ and with $\sigma_2(G') \geq 2n + 2k - 2m + 1 - 2(k - t) = 2n' + 2t - 2m' + 1$. Also, the sharpness comes from expanding the edges in the sharpness example in Theorem 1(ii) to the corresponding length paths in L. This completes the proof of (ii).

(iii) and (vii) The proof of the last five cases have precisely the same form as the first two cases. Each of the $t - s$ paths with edges are reduced to a single edge, which decreases the number of vertices in G by $k - t + s$, and reduces the $\sigma_2(G)$ bound by $2(k - t + s)$ in the new graph G'. Then, Lemmas 2 and 3 are applied to complete the proof of these four cases. The sharpness examples come from Example 4(ii) and (iii), and so this completes the proof of Theorem 3. $\square$

5. Concluding remark

The bounds in Theorem 1 are sharper when k and n have the same parity, so it is natural to ask if these sharper bounds are also valid for Theorem 2. More specifically, there is the following question.

Question 1. *Let k, t and n be positive integers with $1 \leq k + t \leq n$, and let F be a (k, t)-linear forest. If*

(i) *$\sigma_2(G) \geq n + k$ when $P_{k+1} \subseteq F$ and*
(ii) *$\sigma_2(G) \geq n + k - \epsilon(k, n)$ otherwise,*

then is $G(k, t, 2t + k)$-pancyclic?

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