USING EDGE EXCHANGES TO PROVE THE ERDŐS-JACOBSON-LEHEL CONJECTURE

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Abstract. We consider a problem in extremal graph theory as introduced by Erdős, Jacobson and Lehel in [3]. That is, given an n-term graphic degree sequence, for n sufficiently large, we prove that the minimum degree sum necessary to guarantee a realization containing a t-clique, $t \geq 2$, is $(t-2)(2n-t+1)+2$. The proof involves the notion of an edge exchange, which is well-known but has not been used in previous approaches to this problem. It is our hope that the proof will demonstrate the utility of this technique and inspire new approaches to similar problems.

1. Introduction

Let G be a simple undirected graph, and let $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. We let $\overline{G}$ denote the complement of G. Denote the complete graph on t vertices by K_t and let $N(v)$ and $d(v)$ denote the neighborhood and degree of a vertex v in a graph G. Furthermore, if H is a subgraph of G, let $N_H(v)$ denote those neighbors of a vertex v that lie in H. Given any two graphs G and H we will denote their join by $G+H$.

A sequence of nonincreasing, nonnegative integers $\pi = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a graph G of order n having degree sequence π. In this case, G is said to *realize* π, and we will write $\pi = \pi(G)$. If a sequence π consists of the terms $d_1, \ldots, d_t$ having multiplicities $m_1, \ldots, m_t$, we may write $\pi = (d_1^{m_1}, \ldots, d_t^{m_t})$. We generally use notation as given in [14] and refer the reader there for any undefined terms.

1.1. **Edge Exchanging.**

Let G be a realization of a graphic sequence π and let u, v, u' and v' be vertices in G such that $uv, u'v'$ are edges in G and

$u'u, v'v$ are nonedges in G. Removing the edges uv and $u'v'$ and replacing them with the nonedges $u'u$ and $v'v$ results in a realization G' of π that may or may not be isomorphic to G. This operation is frequently referred to as an *edge exchange*, a *2-switch* (see [14]) or *transfer* (see [1] and [13]). The following well-known theorem of S.L. Hakimi asserts that this operation is sufficient to navigate between the realizations of a graphic sequence.

Theorem 1.1. [6] *Let π be a graphic sequence, and let G and G' be realizations of π. Then there is a sequence of 2-switches, $S_1, \ldots, S_k$ such that the application of these switches to G in order will result in G'.*

A proof of this result is also given in [1] (pages 153-154) and [14] (page 47). Recently, an analogous result to that of Theorem 1.1 has been determined for 3-uniform hypergraphs, see [8].

More generally, let G be a graph of order n. A circuit $C = e_1 e_2, \ldots, e_{2\ell}$ in K_n, where e_i and e_{i+1} are incident is an *alternating circuit* if $e_i \in E(G)$ whenever i is even and $e_i \in E(\overline{G})$ whenever i is odd. In other words, the edges of C alternate being "in" and "out" of G. Removing the edges of C from G and adding back the edges of C from $\overline{G}$ results in a new graph G' that has the same degree sequence as G. We refer to this operation as *exchanging* the edges of the alternating circuit C, and we note that a 2-switch is simply the operation of exchanging the edges of an alternating circuit of length 4.

1.2. **Potentially H-graphic Sequences.** Let π be a graphic sequence and let H be a graph. Let $\sigma(\pi)$ denote the sum of the terms in π. We say that π is *potentially H-graphic* if there is some realization of π that contains H as a subgraph. Define $\sigma(H, n)$ to be the smallest even integer m so that every n-term graphic sequence π with $\sigma(\pi) \geq m$ is potentially H-graphic.

In [3], Erdős, Jacobson and Lehel conjectured that $\sigma(K_t, n) = (t-2)(2n-t+1)+2$. The conjecture rises from consideration of the graph $K_{(t-2)} + \overline{K}_{(n-t+2)}$. It is easy to observe that this graph contains no K_t, is the unique realization of its degree sequence and has degree sum $(t-2)(2n-t+1)$.

In proving the upper bound, the cases $t = 3, 4$ and 5 were handled separately (see respectively [3], [5] and [9], and [10]), and Li, Song and Luo [11] proved the conjecture true via linear algebraic techniques for $t \geq 6$ and $n \geq \binom{t}{2} + 3$. We prove the following (where the bound on n is not best possible).

Theorem 1.2. *Let $n \geq \frac{31}{2}t^2 + \frac{13}{2}t + 3$ and $t \geq 2$ be positive integers. If π is an n-term graphic sequence with $\sigma(\pi) \geq (t-2)(2n-t+1)+2$, then π is potentially K_t-graphic.*

To prove Theorem 1.2, we will choose a realization of π that is "close" to having a t-clique. Under the given conditions on π, we show that a sequence of edge exchanges is possible to move from this realization to one which indeed has a t-clique. This purely graph-theoretic technique of determining when a sequence is potentially H-graphic has been mostly abandoned in the literature since [13], although another recent example appears in [4]. In fact, Theorem 1.1 permits us to construct such a realization using only 2-switches, but it is generally less complicated to exchange the edges of a longer circuit instead.

The goal of this paper is to use this technique to give a new proof of the Erdős-Jacobson-Lehel conjecture. It is our hope that this proof will call greater attention to the technique of edge-exchanging, with the larger goal of facilitating general progress on the problem of determining $\sigma(H,n)$ for arbitrary choices of H.

2. A Proof of the Erdős-Jacobson-Lehel Conjecture Using Edge-Exchanges

2.1. **Preliminaries.** For the remainder of the paper, let $\pi = (d_1, \ldots, d_n)$ be a fixed nonincreasing n-term graphic sequence with $\sigma(\pi) \geq (t-2)(2n-t+1)+2$ and $n \geq \frac{31}{2}t^2 + \frac{13}{2}t + 3$. In constructing a realization of π that contains a copy of K_t, the following lemma from [5] will prove useful.

Lemma 2.1. *If S is a graphic sequence with a realization G containing H as a subgraph, then there is a realization G' of S containing H with the vertices of H having the $|V(H)|$ largest degrees of S.*

In seeking to prove Theorem 1.2, it is therefore logical to attempt to construct a copy of K_t on those vertices of degree $d_1, \ldots, d_t$. The next lemma, given in [7], follows from the well-known Erdős-Gallai [2] criterion for graphic sequences and serves to establish that π majorizes the degree sequence of K_t. For completeness, we give the proof of this next result.

Lemma 2.2. [7] *If $S = (d_1, d_2, \cdots, d_n)$ is a graphic sequence such that $\sigma(S) \geq (t-2)(2n-t+1)+2$ and $n \geq t$, then $d_t \geq t-1$.*

Proof: By way of contradiction, suppose that S is a graphic sequence with $\sigma(S) \geq (t-2)(2n-t+1)+2$ and that S has at most $t-1$ terms at least $t-1$. Then by applying the Erdős-Gallai criteria we obtain the following.

$$\begin{aligned}\sum_{i=1}^{n} d_i &= \sum_{i=1}^{t-1} d_i + \sum_{i=t}^{n} d_i \overbrace{\leq}^{E-G} ((t-1)(t-2) + \sum_{i=t}^{n} min\{t-1, d_i\}) + \sum_{i=st}^{n} d_i \\ &= t^2 - 3t + 2 + 2\sum_{i=t}^{n} d_i \\ &\leq t^2 - 3t + 2 + 2(n-t+1)(t-2) \\ &= (t-2)(2n-t+1).\end{aligned}$$

For all $n \geq t$, this contradicts the given degree sum and the result follows.□

Before we begin, we give a brief outline of the proof of Theorem 1.2. By induction we will show π contains a fairly large clique. Using Lemma 2.1, we then show that this clique can be situated on the vertices of highest degree. After this, we exchange the edges of alternating circuits to finish building the desired clique. The technical aspect of the proof is in proving that such edge exchanges are always possible.

2.2. **The Proof.** The proof of Theorem 1.2 will proceed by induction on t. We first note that $\sigma(K_1, n) = 0$ and $\sigma(K_2, n) = 2$. Now assume the theorem true for all i, $2 \leq i \leq t-1$. As $\sigma(\pi) \geq \sigma(K_{t-2}, n)$ by induction (note that $\frac{31}{2}t^2 + \frac{13}{2}t + 3$ is a nondecreasing function) there exists a realization G of π, that contains a subgraph H isomorphic to K_{t-2}. If G contains a copy of K_t we are done, so we henceforth assume otherwise and let $V(G) = \{v_1, \ldots, v_n\}$ such that each v_i has degree d_i.

We will assume, in light of Lemma 2.1, that $V(H) = \{v_1, \ldots, v_{t-2}\}$ and also note that Lemma 2.2 assures that $d_t \geq t-1$. Additionally, amongst all realizations of π that contain a clique H on the vertices of degree $d_1, \ldots, d_{t-2}$ let G maximize the number of edges between H and the vertices of degree d_{t-1} and d_t. For convenience, we will let Z denote the set $\{v_{t-1}, v_t\}$. We now demonstrate that our assumption of maximality implies that all of the possible edges between H and Z are present in G.

Suppose, to the contrary, that there exists $v \in H, z \in Z$ such that $vz \notin E(G)$. Let $A = N_{G-H}(v) - N_{G-H}(z)$ and let $B = N_{G-H}(v) \cap N_{G-H}(z)$.

Claim 2.3. *If $x \in N_{G-H}(z)$ and $y \in N_{G-H}(v)$, then $xy \in E(G)$. Consequently each vertex in A is adjacent to every vertex in B and furthermore $|B| \leq t-2$.*

Suppose, to the contrary, that $x \in N_{G-H}(z)$ and $y \in N_{G-H}(v)$, and $xy \notin E(G)$. If we exchange the edges xz and yv with the nonedges xy and zv, the result will be a realization of π with more edges between H and Z than are present in G, contradicting the maximality of G.

The other assertions follow from the definitions of A and B and from the fact that the first statement implies that B must be complete. This establishes the claim.

Claim 2.4. *$d_{t-2} \leq 3t - 8 < 3t$.*

We will, in fact, show that $d(v) \leq 3t - 8$. As $d_{t-2} \leq d(v)$, the result will follow.

If A is empty, then v is adjacent to $t - 3$ vertices in H and at most $|B| + (|Z| - 1)$ vertices outside of H. Thus $d(v) \leq t - 3 + t - 2 + 1 = 2t - 4$.

Otherwise, there exists an $a \in A$. Suppose that x and y are nonadjacent vertices in $N_{G-H}(z)$. Then we could exchange the edges zx, zy and va in G and the nonedges xy, zv and za (which together form an alternating circuit of length 6) again contradicting the maximality of G. Thus we may assume that $N_{G-H}(z)$ is complete and hence has cardinality at most $t - 2$. As z is adjacent to at most $t - 3$ vertices in H, this implies that z has degree at most $t - 3 + t - 2 = 2t - 5$.

As z has degree at least $t - 1$, there is some vertex u in $N_{G-H}(z)$ that does not lie in Z. By Claim 2.3, u is adjacent to every vertex in both A and B in addition to z. Hence, as $d(u) \leq d(z) \leq 2t - 5$, we know that $|A| + |B| \leq 2t - 5$. This implies that $d(v) = (t - 3) + |A| + |B| \leq 3t - 8$. The result follows. This establishes the claim.

By assumption, v and z are nonadjacent and both have degree at least $t-1$. Thus there exist vertices $x \in N_{G-H-Z}(v)$ and $y \in N_{G-H-Z}(z)$ (note that x and y may be the same). If there exists an edge $x'y'$ in $G - H - Z$ such that xx' and yy' are both not in $E(G)$, then exchanging the edges $x'y', xv$ and yz for the nonedges vz, xx' and yy' would yield a contradiction to the maximality of G. We will guarantee the existence of such an edge $x'y'$ by bounding the number of edges incident to vertices in $N(x) \cup N(y)$.

Claim 2.4 implies that $|N(x) \cup N(y)| \leq 6t$. This, and Claim 2.4 again, implies there are at most $(t-3)(n-1)+3t(5t+3)$ edges incident to vertices in $N(x) \cup N(y)$. However, $E(G) = \frac{1}{2}\sigma(\pi) > (t-3)(n-1)+3t(5t+3)$ provided $n > \frac{31}{2}t^2 + \frac{13}{2}t + 3$. For n in this range, this implies that it is possible to find an appropriate edge $x'y'$. This establishes that for all $v \in H$ and $z \in Z$ the edge $vz \in E(G)$.

As we assumed that G does not contain a K_t, the only pair of nonadjacent vertices in $\{v_1, \ldots, v_t\}$ is v_{t-1}, v_t. We now show that an edge exchange is possible in G to create a copy of K_t on $\{v_1, \ldots, v_t\}$. Let $N_{G-H}(v_{t-1}) = N_1$ and $N_{G-H}(v_t) = N_2$. Since both v_{t-1} and v_t have degree at least $t - 1$, neither of these sets is empty so let $x \in N_1$ and $y \in N_2$. If $xy \notin E(G \setminus H)$, then we may exchange the edges $v_{t-1}x, v_ty$ for the nonedges $v_{t-1}v_t, xy$ constructing the desired K_t. Otherwise, $xy \in E(G)$ and so $N_1 \cap N_2$ is complete, and hence has cardinality at most $t-2$. Our next step is to again show that there exists some $v \in H$ such that $d(v) < 3t$.

Case 1: Assume that $N_1 \subseteq N_2$. In this case, N_1 induces a complete graph, implying that $|N_1| \leq t-2$ and that $d(v_{t-1}) = |N_1| + |N_H(v_{t-1})| \leq t-2+t-2 = 2t-4$. By assumption $d(x) \leq d(v_{t-1})$, which implies that there is a vertex v in H such that xv is not in $E(G)$. Let a be any neighbor of v that lies outside of H and Z. If $xa \notin E(G)$ then we could exchange the edges $v_{t-1}x, v_tx$ and va for the nonedges xv, xa and $v_{t-1}v_t$, constructing the desired K_t. Otherwise, $xa \in E(G)$. Thus, if $d_{G-H-Z}(v) \geq 2t-5$, it must be the case that $d(x) \geq 2t-3 > d(v_{t-1})$, a contradiction. Hence, in this case, $d(v) \leq d_H(v) + (2t-6) + 2 = 3t-7 < 3t$. The case in which $N_2 \subseteq N_1$ is identical.

Case 2: Assume then that $N_1 - N_2$ and $N_2 - N_1$ are both nonempty. We first show that $N_1 \cup N_2$ is complete. Let x_1 and x_2 be in N_1. If $x_1x_2 \notin E(G)$ then we may exchange the edges $v_{t-1}x_1, v_{t-1}x_2$ and v_ty for the nonedges $x_1x_2, v_{t-1}v_t$ and $v_{t-1}y$, where y is any vertex in $N_2 - N_1$, constructing the desired K_t. Otherwise, $x_1x_2 \in E(G)$. A similar argument yields that any y_1 and y_2 in N_2 must be adjacent, and together with the previous observation that any vertex in N_1 is adjacent to each vertex in N_2 yields that $N_1 \cup N_2$ is complete. In particular, both v_{t-1} and v_t have degree at most $2t-4$, as in the previous case. Let x and y be in $N_1 - N_2$ and $N_2 - N_1$, respectively. There is some v in H such that $yv \notin E(G)$; otherwise $H \cup \{v_t, y\}$ is a t-clique. Let a be any neighbor of v that lies outside of H and Z. If $xa \notin E(G)$, then we could exchange the edges $v_{t-1}x, v_ty$ and va for the nonedges yv, xa and $v_{t-1}v_t$, completing the desired K_t. Otherwise, $xa \in E(G)$. Thus, if $d_{G-H-Z}(v) \geq 2t-5$ then $d(y) \geq d_{G-H-Z}(v) + 1 \geq 2t-3 > d(v_t)$, a contradiction. Hence $d_{G-H-Z}(v) \leq 2t-6$ and $d(v) \leq 3t-7 < 3t$.

Having bounded the degree of some vertex v in H, we now complete the proof of Theorem 1.2. Let x and y be in N_1 and N_2, respectively. Suppose there exists an edge $x'y'$ lying outside of $H \cup Z$ such that $x'x$ and $y'y$ are not edges in G. We may then exchange the edges $v_{t-1}x, v_ty$ and $x'y'$ for the nonedges $x'x, y'y$ and $v_{t-1}v_t$, completing the desired K_t. As we have bounded the degree of some vertex v in H by $3t$, we can assure the existence of such an edge by bounding the number of edges incident to the vertices in $N(x) \cup N(y)$. This completes the proof. $\square$

3. Conclusion

The purpose of this paper is to demonstrate the utility of the technique of edge exchanging by giving a new, short proof of the Erdős-Jacobson-Lehel conjecture. It is our hope that this will serve to broaden the collection of available techniques that can be used to approach problems pertaining to potentially H-graphic sequences, with the additional hope that new and general progress may be made in the area.

We would like to note that the bound on n given in our proof of Theorem 1.2 could be improved with a more detailed analysis. This however would make the proof considerably longer, and detract from our stated purpose of focusing on the technique of edge exchanging.

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References

[1] Berge, C., *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1973.

[2] Erdős, P. & Gallai, T., Graphs with prescribed degrees (in Hungarian), *Matemoutiki Lapor* **11** (1960), 264-274.

[3] Erdős, P., Jacobson, M.S., Lehel, J., Graphs Realizing the Same Degree Sequence and their Respective Clique Numbers, *Graph Theory, Combinatorics and Applications*, Vol. I, (1991), ed. Alavi, Chartrand, Oellerman and Schwenk, 439-449.

[4] Ferrara, M. and Schmitt, J., A Lower Bound for Potentially H-graphic Sequences, submitted.

[5] Gould, R.J., Jacobson, M.S., Lehel, J., Potentially G-graphic degree sequences, *Combinatorics, Graph Theory, and Algorithms* (eds. Alavi, Lick and Schwenk), Vol. I, New York: Wiley & Sons, Inc., (1999), 387-400.

[6] Hakimi, S.L., On realizability of a set of integers as degrees of the vertices of a linear graph, II. Uniqueness, *J. Soc. Indust. Appl. Math* **11** (1963), 135-147.

[7] Kézdy, A.E., Lehel, J., Degree Sequences of graphs with prescribed clique size, *Combinatorics, Graph Theory, and Algorithms* (eds. Y. Alavi *et al.*), Vol. 2, New Issues Press, Kalamazoo (1999), 535-544.

[8] Kocay, W., Li, P. C., On 3-hypergraphs with equal degree sequences, *Ars Combin.* **82** (2007), 145-157.

[9] Li, J., Song, Z., An extremal problem on the potentially P_k-graphic sequences, *The International Symposium on Combinatorics and Applications*, June 28-30, 1996 (W.Y.C. Chen et. al., eds.) Tanjin, Nankai University (1996), 269-276.

[10] Li, J., Song, Z., The smallest degree sum that yields potentially P_k-graphic sequences, *J. Graph Theory* **29** (1998), no.2, 63-72.

[11] Li, J., Song, Z., Luo, R., The Erdős-Jacobson-Lehel conjecture on potentially P_k-graphic sequences is true (in Chinese), *Science in China, Ser. A*, **41**, (1998), (5):510-520.

[12] Rao, A.R., The clique number of a graph with a given degree sequence, *Proc. Symposium on Graph Theory* (A.R. Rao ed.), MacMillan and Co. India Ltd., I.S.I. Lecture Notes Series 4 (1979), 251-267.

[13] Rao, S.B., A survey of the theory of potentially P-graphic and forcibly P-graphic degree sequences, *Lecture Notes in Math.* No. 885, Springer-Verlag, (1981), 417-440.

[14] West, D.B., *Introduction to Graph Theory, 2e*, Prentice Hall, Upper Saddle River, NJ, 2001.