Bounds for the Ramsey Number of a Disconnected Graph Versus Any Graph

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ABSTRACT

Bounds are determined for the Ramsey number of the union of graphs versus a fixed graph $H$, based on the Ramsey number of the components versus $H$. For certain unions of graphs, the exact Ramsey number is determined. From these formulas, some new Ramsey numbers are indicated. In particular, if

$$r(g_1, H) = (\sum_{i=1}^{k} |V(g_i)| - 1)[\chi(H) - 1] + t_1(H) + \beta_i,$$

$$G = \bigcup_{i=1}^{k} g_i,$$

$$p = \max_{1 \leq i \leq k} \left( (i - 1)[\chi(H) - 2] + \sum_{i \neq j} n_j k_j \right) + t_1(H) - 1,$$

where $k_i$ is the number of components of order $i$ and $t_1(H)$ is the minimum order of a color class over all critical colorings of the vertices of $H$, then

$$p \leq r(G, H) \leq p + \max_i \beta_i.$$

INTRODUCTION

All graphs in this article are without loops and multiple edges. If $G$ is a disconnected graph let $c(G)$ denote the maximum order of a component of $G$. A coloring of the vertices of $G$ with exactly $\chi(G)$ colors is called a critical coloring of $G$.
coloring. In any coloring of a graph, all vertices with the same color form a color class. Define \( r(G) \) to be the minimum number of vertices in any color class of any critical coloring of \( G \). Finally, the Ramsey number \( r(G_1, G_2) \) is the least positive integer \( p \) such that in any factorization of \( K_p = R \oplus B \) [i.e., \( V(K_p) = V(R) = V(B) \) and \( E(R) \cap E(B) = \emptyset \) and \( E(R) \cup E(B) = E(K_p) \)], either \( G_1 \subseteq R \) or \( G_2 \subseteq B \). Ramsey numbers have been studied extensively. Some results of interest include the following.

**Theorem A** (Burr [3]). If \( G \) is a connected graph of order \( n \geq t(H) \) then

\[
r(G, H) \geq (n - 1)[\chi(H) - 1] + t(H).
\]

**Theorem B** (Chvátal [5]). If \( T_m \) is a tree of order \( m \) and \( K_n \) a complete graph of order \( n \) then

\[
r(T_m, K_n) = (m - 1)(n - 1) + 1.
\]

**Theorem C** (Stahl [7]). If \( F \) is a forest then

\[
r(F, K_n) = \max_{1 \leq j \leq c(F)} \left( (j - 1)(n - 2) + \sum_{i=1}^{c(F)} i k_i \right),
\]

where \( k_i \) is the number of components of order \( i \).

**Theorem D** ([6]). If \( P_m \) is the path of order \( m, m \geq 4 \), and \( G \) is a graph of order \( n + 2 \), \( n \geq 3 \), with clique number \( n \), then

\[
r(P_m, G) = (m - 1)(n - 1) + 1.
\]

In this paper we present bounds related to that in Theorem A for \( G = \bigcup_{j=1}^{k} G_j \). We use these bounds and others to obtain a generalization of Theorem C, and from this, determine some new Ramsey numbers.

### 2. Upper and Lower Bounds

Let \( H \) be a graph, then define \( t_i(H) \) to be the minimum, over all critical colorings of the vertices of \( H \), of the order of the \( i \)th smallest color class. (Note that the first smallest is the smallest.) Also define \( \mathcal{G}(H) = \{ g \mid g \text{ is a connected graph and } r(g, H) = [|V(g)| - 1][\chi(H) - 1] + t_i(H) + \beta \}. \) This set is clearly well defined for all non-negative integers \( \beta \).

(i) Suppose \( g_1, g_2, \ldots, g_k \in \mathcal{G}(H) \) with smallest graph of order \( m_0 \) and largest graph of order \( m_1 \). Let \( G = \bigcup_{j=1}^{k} g_j \) and let \( k_i \) be the number of components of order \( i \). Choose \( j_0 \) such that
\[(j_0 - 1)[\chi(H) - 2] + \sum_{i=j_0}^{m_1} i k_i\]

\[= \max_{m_0 \leq j \leq m_1} \left( (j - 1)[\chi(H) - 2] + \sum_{i=j}^{m_1} i k_i \right).\]

We prove the following.

**Theorem 1.** Suppose conditions (i) hold. If \(j_0 \geq t_1(H)\) then

\[r(G, H) \geq (j_0 - 1)[\chi(H) - 2] + \sum_{i=j_0}^{m_1} i k_i + t_1(H) - 1.\]

**Proof.** Let \(p = (j_0 - 1)[\chi(H) - 2] + p_0 + t_1(H) - 2\), where \(p_0 = \sum_{i=j_0}^{m_0} i k_i\). Consider the factorization of \(K_p = R \oplus B\), where \(R = K_{p_0-1} \cup [\chi(H) - 2] K_{p_0-1} \cup K_{t_1(H)-1}\). To show that \(G \not\subseteq R\) we will concentrate on the subgraph \(G_{j_0}\) of \(G\) which consists of all components of \(G\) which have \(j_0\) or more vertices. Clearly \(G_{j_0} \not\subseteq K_{m_0-1}\) since there are not enough vertices. Further, \(K_{m_0-1}\) is too small to contain any component of \(G_{j_0}\), and since \(j_0 \geq t_1(H)\) it is clear that \(K_{t_1(H)-1}\) is also too small to contain any component of \(G_{j_0}\). Thus \(G_{j_0} \not\subseteq R\); hence \(G \not\subseteq R\). Since \(B\) is a complete \(\chi(H)\)-partite graph with \(t_1(B) = t_1(H) - 1\) this implies that \(H \not\subseteq B\), and the theorem follows. \(\blacksquare\)

**Theorem 2.** Suppose conditions (i) hold. Then

\[r(G, H) \leq (j_0 - 1)[\chi(H) - 2] + \sum_{i=j_0}^{m_1} i k_i + t_1(H) + \beta - 1.\]

**Proof.** It will be convenient for \(m_0 \leq j \leq m_1\) to let \(G_j\) denote the subgraph of \(G\) consisting of all components of order at least \(j\), so that \(G_j\) has order \(p_j = \sum_{i=j}^{m_1} i k_i\). Let \(p = (j_0 - 1)[\chi(H) - 2] + p_0 + t_1(H) + \beta - 1\). Consider an arbitrary factorization of \(K_p = R \oplus B\) in which \(H \not\subseteq B\). We show that \(G \subseteq R\) by descending induction on \(j\).

First assume \(G = G_{m_1}\). By an easy induction on \(k\), the number of components of \(G\), we show \(G = G_{m_1} \subseteq R\). This is clear for \(k = 1\). If \(k > 1\) and \(g\) is an arbitrary component of \(G\), then the factorization of \(K_p = R \oplus B\) induces a factorization on \(K_p - V(g)\) with \(|V(K_p) - V(g)| = (m_1 - 1) \times [\chi(H) - 2] + m_1(k - 1) + t_1(H) + \beta - 1\). By induction \(G - V(g) \subseteq (K_p - V(g)) \cap R\) since \(H \not\subseteq B\). Therefore \(G = G_{m_1} \subseteq r\).

To complete the induction on \(j\) assume \(G_{j+1} \subseteq R\), \(m_0 \leq j \leq m_1\). Clearly \(G_j \subseteq R\) when \(G_j = G_{j+1}\), so that \(G_j - V(G_{j+1})\) consists of \(k_j\) components,
each of order \( j \). Again the factorization of \( K_p = R \oplus B \) induces a factorization on \( K_p - V(G_{j+1}) \) with

\[
| V(K_p) - V(G_{j+1}) | = p - \sum_{i=j+1}^{m} ik_i \geq (j - 1)[\chi(H) - 2] + jk_j + t_1(H) + \beta - 1.
\]

As in the argument of the preceding paragraph \( G_j = V(G_{j+1}) \subseteq (K_p - V(G_{j+1}) \cap R \). Therefore \( G_j \subseteq R \) and the induction is complete. □

We now note some useful special cases of Theorems 1 and 2.

**Corollary 3.** Suppose \( g_1, g_2, \ldots, g_k \in \mathcal{A}(H) \) and \( | V(g_i) | = m \) (i = 1, 2, \ldots, k). Let \( G = \bigcup_{i=1}^{k} g_i \). If \( m \geq t_1(H) \) then \( r(G, H) \geq (m - 1)[\chi(H) - 2] + mk + t_1(H) - 1 \).

**Corollary 4.** Suppose \( G = \bigcup_{i=1}^{k} g_i \), where \( g_i \in \mathcal{A}(H) \) and \( | V(g_i) | = m \). Then

\[
r(G, H) \leq (m - 1)[\chi(H) - 2] + mk + t_1(H) + \beta - 1.
\]

We now note that Theorems 1 and 2 yield a generalization of Theorem C.

**Corollary 5.** If \( g_1, g_2, \ldots, g_k \in \mathcal{A}(H) \) and \( G = \bigcup_{i=1}^{k} g_i \), then

\[
r(G, H) = \max_{1 \leq j \leq v(G)} \left( (j - 1)[\chi(H) - 2] + \sum_{i=j}^{v(G)} ik_i \right) + t_1(H) - 1.
\]

We also note that \( T_m \in \mathcal{A}(K_n) \) from Theorem B, so Theorem C now follows as a corollary to Theorem B and Corollary 5.

3. APPLICATIONS AND CONCLUSION

In [1] it was shown that \( C_n \in \mathcal{A}(K_n) \) when \( m > n^2 - 2 \). Since \( T_m \in \mathcal{A}(K_n) \) as well, we may use Corollary 5 to obtain the Ramsey number for \( G = (\bigcup_{i=1}^{t} T_{m_i}) \cup (\bigcup_{i=1}^{t} C_{m_i}) \), where each \( m_i > n^2 - 2 \), versus \( K_n \). Similarly, \( C_m \in \mathcal{A}(K_n) \cup (K_{n^2}) \) denotes the complete \( n \)-partite graph \( K_{n,\ldots,n} \), with \( n \) subscripts) when \( s, n \), and \( m \) are sufficiently large (see [1]). Thus Corollary 5 may be applied to unions of cycles (sufficiently large) versus \( K_{n^2} \) as well.

In fact, Burr and Erdős [4] have shown that any sufficiently large graph homeomorphic to a connected graph is in \( \mathcal{A}(K_n) \). They further conjecture that "large" connected graphs with "small" edge density are in \( \mathcal{A}(K_n) \).
As a final application of Corollary 5 to Theorem D we can produce the Ramsey numbers for stripes (unions of paths) with smallest stripe of order 4 versus any graph $G$ with order $n + 2$ having clique number $n$ ($n \geq 3$).

Clearly the list of applications is far from exhausted. We merely mention a few to point out possible applications of Corollary 5.

Finally, we state a theorem bounding the Ramsey number and allowing one to vary the $\mathcal{F}_\beta$ classes.

**Theorem 6.** If $g_i \subseteq \mathcal{F}_\beta(H)$ and $G = \cup_{i=1}^k g_i$, let

$$p = \max_{1 \leq j \leq \chi(G)} \left( j - 1 \right) \left( \chi(H) - 2 \right) + \sum_{i=0}^{\chi(G)} ik_i \right) + \ell_i(H) - 1;$$

then

$$p \leq r(G, H) \leq p + \max_i (\beta_i).$$

The proof of Theorem 6 would be exactly the same as that for Theorems 1 and 2 with $\max (\beta_i)$ substituted for $\beta$.

We also feel that investigation of $\ell_i(H)$ for $i > 1$ may result in new and improved bounds. We direct the reader to [2].

**References**