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SURVEY

Recent Advances on the Hamiltonian Problem: Survey III

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Abstract This article is intended as a survey, updating earlier surveys in the area. For completeness of the presentation of both particular questions and the general area, it also contains some material on closely related topics such as traceable, pancyclic and Hamiltonian connected graphs.

Keywords Cycle · Hamiltonian · Pancyclic · Hamiltonian-connected

1 Introduction

A graph G is *Hamiltonian* if it contains a cycle that spans the vertex set. The Hamiltonian problem is generally considered to be determining conditions under which a graph contains a spanning cycle. Named for Sir William Rowan Hamilton (and his Icosian game), this problem traces its origins to the 1850s. Today, however, the constant stream of results in this area continues to supply us with new and interesting theorems and still further questions.

To many, including myself, any path or cycle problem is really a part of this general area and it is difficult to separate many of these ideas. Thus, although I will concentrate on spanning cycles (the classic Hamiltonian problem), some other related results, both stronger (pancyclic and Hamiltonian connected) and weaker (traceable), will be presented in order to provide you with a better picture of the overall theory and problems as they exist today.

In doing this I shall generally restrict my attention to work done since 2003 when [102] appeared. This earlier article references a number of older or more specialized surveys and readers brand new to this area should begin in my first survey article [101], which appeared in 1991. I will also restrict my attention to graphs and hypergraphs. For digraphs, see [139].

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Throughout this article we consider finite simple graphs $G = (V, E)$, unless otherwise indicated. We reserve n to denote the order ($|V|$) and q the size ($|E|$) of G. We use $\delta(G)$ and $\Delta(G)$ for the minimum and maximum degrees of G respectively, and let $N(x)$ and $N(S)$ denote the neighborhood of the vertex x and set S, respectively. Further, let $c(G)$ denote the circumference of G, that is, the length of a longest cycle, and the girth $g(G)$, that is the length of a shortest cycle. Let

$$\sigma_k(G) = \min \{deg x_1 + \cdots + deg x_k \mid x_1, \ldots, x_k \text{ are independent in } G\}.$$

Graphs satisfying lower bounds on σ_k with $k = 2$ will often be called Ore-type graphs, while if $k = 1$, Dirac-type graphs. If G contains no induced subgraph isomorphic to any graph in the set $\mathcal{F} = \{H_1, \ldots, H_k\}$, we say G is $\mathcal{F}$-free, or H_1-free if $\mathcal{F}$ contains only H_1. The graph $G[S]$ is the graph induced by the set S in G. We denote the distance between vertices u and v by $dist(u, v)$. Let $\alpha(G)$ denote the (vertex) independence number of G, that is, the maximum number of vertices that are mutually nonadjacent. Let $\beta(G)$ denote the edge independence number of G.

Also, recall a graph is t-tough ($t > 0$) if for every vertex cut S, the number of components of $G - S$ is at most $|S|/t$. For terms not defined here see [104].

2 Generalizing the Classics

The founding results of Dirac [55] and Ore [171] established interest in Hamiltonian graphs. In fact, results based on minimum degree have come to be called Dirac-type results and those based on σ_2 have come to be called Ore-type results. It is both natural and exciting that new results that generalize their classic work continue to be found. In this section we will consider some of these new results.

Recall that a subdivision of a graph H is a graph obtained from H by subdividing some subset of the edges of H. We call such a graph an H-subdivision. In [14] the following was obtained.

Theorem 1 [14]

1. *Let H be a graph contained in the graph G such that every component of H is either a nontrivial tree or unicyclic. Let t be the number of tree components of H. If G has order n and $\delta(G) \geq \frac{n-t}{2}$, then G contains a spanning subdivision of H.*
2. *Let H be any graph of order n with k components, each of which is either a tree or a cycle. Let G be any graph with order at least n and with $\delta(G) \geq n - k$. Then G contains a subdivision of H where only the cycles are subdivided.*

Continuing along these lines in [176], a similar result was found.

Theorem 2 [176] *Let G be a graph of order n with $\delta(G) \geq n/2$. Let H be a subgraph of G with each component of H being either a cycle or a cycle with a single chorded. Then G contains a spanning subgraph isomorphic to a subdivision of H in which the chords of cycles are not subdivided.*

Generalizing these ideas, given a graph H (possibly a multigraph), we say a graph G is H-linked if every injective map $f: V(H) \to V(G)$ can be extended to an H subdivision. This idea generalizes the well-known concept of k-linked graphs.

Subsequently, minimum degree thresholds were determined for a graph G to be H-linked that are dependent upon the graph H itself. Let

$$b(H) = \max_{\substack{A\cup B\cup C=V(H)\\ |E(A,B)|\geq 1}} |E(A,B)| + |C|.$$

If H is connected, it is not difficult to see that $b(H)$ is precisely the maximum number of edges in a bipartite subgraph of H. Ferrara et al. [84] proved the following result.

Theorem 3 [84] *For any connected multigraph H, possibly containing loops, if G is a graph of sufficiently large order n with $\delta(G) \geq \lceil \frac{1}{2}(n + b(H) - 2)\rceil$, then G is H-linked. Furthermore, every injection $f : V(H) \to V(G)$ can be extended to an H-subdivision in which each edge-path has at most two intermediate vertices.*

The values of n required for the proof of Theorem 3 are quite large, in the vicinity of $2^{|E(H)|}$. Independently, Kostochka and Yu [130] obtained a similar result for connected, loopless multigraphs H with minimum degree at least two; however, their result required much smaller values of n.

Theorem 4 [130] *Let H be a loopless connected multigraph of order k with $\delta(H) \geq 2$. If G is a graph of order $n \geq 7.5k$ with $\delta(G) \geq \left\lceil \frac{1}{2}(n + b(H) - 2)\right\rceil$, then G is H-linked.*

The stronger aspects of Theorems 3 and 4 were subsequently combined by Gould et al. [106]. We say a multigraph is uneven if it does not contain even cycles. Let $u(H)$ denote the number of uneven components of H.

Theorem 5 [106] *Let H be a multigraph of size ℓ, possibly containing loops, and let $k_1 = k_1(H) = \ell + u(H)$. If G is a graph of order $n \geq 9.5(k_1 + 1)$ with $\delta(G) \geq \left\lceil \frac{1}{2}(n + b(H) - 2)\right\rceil$, then G is H-linked. Furthermore, every injection $f : V(H) \to V(G)$ can be extended to an H-subdivision with at most $5k_1 + 2$ vertices.*

Note that $\lceil \frac{1}{2}(n + b(H) - 2)\rceil$ is the minimum degree threshold for a graph G to be $b(H)$-connected, which is necessary for the H-linked property. Also note that since H can be a disconnected graph, using H as kK_2 gives k-linked results.

Recently, Ferrara et al. [82] gave sharp σ_2 conditions that assure a graph G is H-linked for general H. Note that for arbitrary H, this σ_2 threshold for H-linkedness is not necessarily twice the minimum degree given in Theorem 5. Results of Faudree et al. [68] on k-ordered graphs demonstrate that this is not the case for $H = C_k$, when n is sufficiently large.

Let $a(H) = \max_{\substack{A\cup B=V(H)\\ |E(A,B)|\geq 1}} (|E(A,B)| + |B| - \Delta_B(A))$, where $\Delta_B(A)$ is the maximum degree of a vertex in B relative to the set A.

Theorem 6 [82] *Let H be a graph and G be a graph of order $n > 20|E(H)|$. If*

$$\sigma_2(G) \geq n + a(H) - 2$$

then G is H linked. This result is sharp.

These results, when combined with the following extension theorem, generalize a great many (Hamiltonian) cycle and path results. Given a multigraph H and a graph G, we say that G is H-extendable if whenever there exists an H-subdivision, on less than $|G|$ distinct vertices of G, then there exists a spanning H-subdivision with the same ground set, that is, the same set of vertices playing the role of $V(H)$ in G. Here $h_0(H)$ is the cardinality of the set of isolated vertices in H and $h_1(H)$ the cardinality of the set of vertices of degree 1 in H.

Theorem 7 [107] *If H is a multigraph and G is a simple max $\{\alpha(H), \beta(H)+1\}$-connected graph of order $n > 11|E(H)| + 7(|H| - h_1(H))$ such that*

$$\sigma_2(G) \geq n + |E(H)| - |H| + h_1(H) + 2h_0(H),$$

then G is H-extendable.

Using H as a loop, Theorem 5 and this result yields Ore's Theorem [171] as a trivial corollary, while using $H = C_k$ yields a k-ordered Hamiltonian result (see [68]) or a Dirac-type k-ordered result (see [128]) using Theorem 3 and Theorem 7. A number of other well-known results also become simple corollaries.

We say a graph G is pan-H-linked if every H-subdivision is 1-extendable, that is, with the same ground set as H, there is a subdivision H^* where $|V(H^*)| = |V(H)|+1$. In [86], for certain H a Dirac-type condition on a sufficiently large G is obtained so that G is pan-H-linked.

Another classic result that is still generating new work is that of Chvátal and Erdös [50].

Theorem 8 [50] *A κ-connected graph G is*

1. *Traceable if $\alpha(G) \leq \kappa(G) + 1$.*
2. *Hamiltonian if $\alpha(G) \leq \kappa(G)$.*
3. *1-Hamiltonian, 1-edge Hamiltonian and Hamiltonian connected if $\alpha(G) < \kappa(G)$.*

Ainouche et al. [6] showed that condition (2) can be significantly reduced if G is $K_{1,3}$-free (i.e. claw-free).

Theorem 9 [6] *A k-connected claw-free graph G $(k \geq 2)$ is Hamiltonian if $\alpha(G^2) \leq k$.*

The condition of being claw-free is needed as is shown by $G = K_{2,3}$. Let $J(a, b) = \{u \in N(a) \cap N(b) \mid N[u] \subseteq N[a] \cup N[b]\}$. We say a graph is quasi claw-free if $J(x, y) \neq \emptyset$ for every pair of vertices at distance 2. Ainouche extended the last result to the class of quasi claw-free graphs. This led to the definition of a partially square graph G^*. The partially square graph G^* of a given graph $G = (V, E)$, is the graph $G^* = (V, E \cup \{uv \mid dist(u, v) = 2, J(u, v) \neq \emptyset\}$.

Theorem 10 [7] *Let G be a k-connected graph, and G^* its partially square graph.*

1. *If $\alpha(G^*) \leq k$, $(k \geq 2)$ then G is Hamiltonian.*
2. *If $\alpha(G^*) \leq k + 1$, $(k \geq 1)$ then G is traceable.*

3. *If $\alpha(G^*) < k$, ($k \geq 3$) then G is 1-Hamiltonian and 1-edge-Hamiltonian.*
4. *If G is a $(k+1)$-connected graph with $k \geq 2$ and $\alpha(G^*) \leq k$, then G is Hamiltonian connected.*

This left the case when $\kappa(G) = 3$ open. Ainouche and Lapiquonne [8] completed this case.

Theorem 11 [8] *A k-connected graph ($k \geq 3$) G of order n is Hamiltonian connected if $\alpha(G^*) < k$.*

As a result they obtain the following corollary.

Corollary 12 *A k-connected ($k \geq 3$) quasi claw-free (or claw-free) graph G is Hamiltonian connected if $\alpha(G^2) < k$.*

Here, $R(a, b)$ stands for the standard graph Ramsey number.

Theorem 13 [88] *Let G be a k-connected graph with independence number α such that*

$$k > \alpha + (\alpha + 1)\ R(\alpha + 1, \alpha + 1)$$

Then G is pancyclic.

Let G be a connected graph of order $n \geq 3$ with degree sequence $d_1 = \delta \leq d_2 \leq \ldots \leq d_n$. The graph G belongs to the class $D(n, \theta)$ if $\theta \geq 1$ is the smallest integer for which $d_\theta \geq n/2$. Let $Y = \{x \in V(G) \mid deg\ x < n/2\}$. Let $X = \{x \in V(G) \mid deg\ x \geq n/2\}$. Clearly, $V(G) = X \cup Y$. Let H^* denote the following 1-tough, nonHamiltonian graph:

$$H^* = ((K_{n-4} \cup \overline{K}_3) + K_1) \cup \{y_i x_i \mid i = 1, 2, 3\}, \quad (y_i, x_i) \in Y \times X.$$

Let the set of dominating vertices of G be denoted Ω.

Theorem 14 [4] *Let $G \in D(n, \theta)$, $\theta \leq \delta + 2$ be a 2-connected graph of order $n \geq 3$. Then G is nonHamiltonian if and only if $\theta - \delta \in \{1, 2\}$ and*

1. *either $H = H^*$*
2. *or $\omega(H - \Omega) > |\Omega|$.*

Ainouche also [4] generalized a number of results with the following theorem.

Theorem 15 [4] *Let G be any 2-connected graph in $D(n, \theta)$, $1 \leq \theta \leq \delta + 2$. Then G is $[3, c(G)]$-pancyclic with $c(G) \geq min\ \{n, n + \delta - \theta\}$, Hamiltonian bipartite or isomorphic to the complete bipartite graph $K_{\frac{n+1}{2}, \frac{n-1}{2}}$.*

Ainouche [4] also gave conditions under which $G \in D(n, \theta)$ would be nonHamiltonian.

A 2-connected graph G of order n belongs to class $B(n, \theta)$, $\theta > 0$ if $3(n - \theta) \geq 2(deg(x) + deg(y) + deg(z)) > 3(n - 1 - \theta)$ holds for all independent triples of vertices $\{x, y, z\}$. In [5], Ainouche gives a full characterization of graphs in $B(n, \theta)$, $\theta \leq 2$, in terms of their dual closure. Ainouche and Schiermeyer [9] showed, for many degree sum or neighborhood union conditions on 3 independent vertices sufficient to imply a graph is Hamiltonian, that these conditions also show the 0-dual closure is complete.

Recall that the lexicographic product of two graphs G and H is the graph $G \circ H$ with vertex set $V(G) \times V(H)$ and where vertices (u, x) and (v, y) are adjacent in the $G \circ H$ if and only if $u = v$ and x and y are adjacent in H or u and v are adjacent in G. Also recall that G is weakly pancyclic if it contains a cycle of each length from $g(G)$ to $c(G)$.

Theorem 16 [120]

1. *If G and H are graphs with nonempty edge sets then $G \circ H$ is weakly pancyclic.*
2. *If G is 4-tough, $|V(G)| \geq 2$ and moreover, the edge set of H is nonempty, then $G \circ H$ is pancyclic.*
3. *If H has order at least 2 and has no edges and G is bipartite with nonempty edge set, then $G \circ H$ contains cycles of all even lengths from 4 up to the circumference of the product.*

3 Forbidden Subgraphs

Given a family of graphs $\mathcal{F}$, we say a graph G is $\mathcal{F}$-free if G contains no induced subgraph isomorphic to a graph in $\mathcal{F}$. The graphs of $\mathcal{F}$ are called forbidden subgraphs.

The overriding question in this area has been, given a k-connected $(2 \leq k \leq 4)$ graph G, find the smallest nontrivial families $\mathcal{F}$ of connected graphs, such that G being $\mathcal{F}$-free implies G has some Hamiltonian type property. When $\kappa(G) \leq 3$, it is known that no single graph (except the trivial case of P_3) will suffice as $\mathcal{F}$.

Such questions have been studied for G being traceable, Hamiltonian, pancyclic and Hamiltonian connected. A great deal has been determined, but some very interesting questions remain open.

The crowning conjecture in this area is due to Matthews and Sumner [161].

Conjecture 1 [161] *Every 4-connected $K_{1,3}$-free graph is Hamiltonian.*

This simple to state conjecture is equivalent to a surprisingly large number of other conjectures (see [34]), including the following well-known conjecture due to Thomassen [201].

Conjecture 2 [201] *Every 4-connected line graph is Hamiltonian.*

Another recently found equivalent form is due to Ryjáček and Vrána [184].

Conjecture 3 *Every 4-connected claw-free graph is Hamiltonian connected.*

The strongest result to date was recently proven by Kaiser and Vrána [121].

Theorem 17 [121] *Every 5-connected line graph with minimum degree at least 6 is Hamiltonian.*

Further, this result can be strengthened in two directions. It extends to claw-free graphs by a standard application of results in [183] and it remains valid if Hamiltonian is replaced by Hamiltonian connected.

Other recent developments in this area include the following special case results. Let the hourglass be two disjoint triangles with one vertex from each identified. Let T be a chain of three triangles, i.e. the graph obtained by identifying a vertex of one triangle with a vertex of degree 2 in an hourglass. The following result generalizes a claw and hourglass-free result from [32].

Theorem 18 [173] *Every 4-connected $\{K_{1,3}, T\}$-free graph is Hamiltonian.*

Theorem 19 [143] *Every 4-connected line graph of a quasi-claw-free graph is Hamiltonian.*

We also have the following.

Theorem 20 [131]

1. *Every 4-connected line graph of a claw-free graph is Hamiltonian connected.*
2. *Every 4-connected hourglass-free line graph is Hamiltonian connected.*

This was extended as follows.

Theorem 21 [153] *Every 4-connected $\{$claw, $(P_6)^2$, hourglass $\}$-free graph is Hamiltonian connected.*

The core of a graph G, denoted G_0, is obtained by deleting all vertices of degree 1 and contracting exactly one edge xy or yz for each path xyz in G with $deg_G\ y = 2$. An almost claw-free graph G has the centers of all claws as an independent set and the neighborhoods of each of the centers of each claw contain 2 vertices adjacent to other neighbors of that center. Theorem 20 was also extended as follows:

Theorem 22 [141] *Let G be a connected graph with $|E(G)| \geq 4$. If every 3-edge-cut of the core G_0 has at least one edge lying in a cycle of length at most 3 in G_0, and if $\kappa(L(G)) \geq 3$, then $L(G)$ is Hamiltonian connected.*

Corollary 23 [141]

1. *Let G be a graph with $|V(G)| \geq 4$. Suppose that $L(G)$ is hourglass-free in which every 3-cut of $L(G)$ is not an independent set. If $\kappa(L(G)) \geq 3$, then $L(G)$ is Hamiltonian connected.*
2. *Every 4-connected line graph of an almost claw-free graph is Hamiltonian connected.*

In attempting to build towards a solution to the Matthews–Sumner Conjecture and for the completeness of the general theory, it is natural to consider the 3-connected case. Here a number of new results have appeared.

A graph G is said to be distance claw-free if for each vertex $v \in V(G)$, the independence number of the subgraph of G induced by the set of vertices at distance i from v is at most 2 for each $i \geq 0$.

Theorem 24 [152] *Every 3-connected distance claw-free graph is Hamiltonian connected.*

Theorem 25 [151] *Every 3-connected quasi-claw-free graph G of order n with $\delta(G) \geq (n+5)/5$ is Hamiltonian.*

Theorem 26 [150] *Every 3-connected claw-free graph with minimum degree δ and order at most $6\delta - 7$ is Hamiltonian.*

Theorem 27 [155] *Every 3-connected claw-free graph with minimum degree δ and order $n \leq 5\delta - 8$ is Hamiltonian connected.*

Srong evidence of forbidden pairs for 3-connected graphs was given in the next two results.

Theorem 28 [157] *Every 3-connected claw and P_{11}-free graph is Hamiltonian. Further, this result is sharp in the sense that P_{11} cannot be replaced by P_{12}.*

A slightly weaker result is:

Theorem 29 [144] *Every 3-connected claw and Z_8-free graph is Hamiltonian.* (*Here Z_8 is the graph obtained by identifying the end vertex if a P_9 with one vertex of a triangle.*)

Let $V^* = \{v \in V(G) \mid deg\ v = 6\}$.

Theorem 30 [117] *If G is a 6-connected line graph and if $|V^*| \leq 29$ or $G[V^*]$ contains at most 5 vertex disjoint K_4's, then G is Hamiltonian connected. Hence, every 8-connected claw-free graph is Hamiltonian connected.*

Answering a conjecture of Kuipers and Veldman (see [140]), the following was shown in [142].

Theorem 31 [142] *If G is a 3-connected claw-free graph of order $n \geq 196$ and minimum degree $\delta(G) \geq (n+5)/10$, then either G is Hamiltonian or $\delta(G) = (n+5)/10$ and the closure of G is isomorphic to the line graph of the graph obtained from the Petersen graph by adding pendent edges to its vertices.*

For the 2-connected situation, all possible forbidden pairs were shown for all graphs by Bedrosian [20] and for all sufficiently large graphs by Faudree and Gould [70]. Thus, different questions were asked. In particular, forbidden triples were considered and it was asked: do all such forbidden triples contain a claw?

Brousek [36] characterized all triples of connected nontrivial graphs C, X, Y where C is the claw, such that every 2-connected CXY-free graph G is Hamiltonian. In [72] the triples that do not include the claw were determined for all graphs. Forbidden triples for sufficiently large graphs were studied in [73]. In [71], forbidden triples of graphs, no one of which is a generalized claw ($K_{1,s}$, $s \geq 4$.) were characterized. It is clear from these works that considering sets of 4 or more forbidden graphs is not useful, as the number of such sets would grow unmanageably large.

The second major open question is to determine the set of forbidden pairs $\mathcal{F}$, such that every 3-connected $\mathcal{F}$-free graph is Hamiltonian connected.

Shepherd [191] showed the first significant result. A net is a triangle with three additional vertices, where each additional vertex is adjacent to a distinct vertex of the triangle.

Theorem 32 [191] *If G is a 3-connected claw and net-free graph, then G is Hamiltonian connected.*

The graph L_1 is formed from two disjoint triangles joined by an edge.

Theorem 33 [33] *If G is a 3-connected claw-free and L_1-free graph, then G is Hamiltonian connected.*

In an effort to build a minimal nontrivial forbidden family for Hamiltonian connected graphs, properties of the graph paired with the claw (known to be one of the two graphs) were investigated. Beginning with [70] and continuing with [46] and [33], the collection of properties evolved to the following: (here $N(i, j, k)$ is the generalized net, a triangle with paths of length i, j and k respectively, attached to distinct vertices of the triangle, and L_k is two distinct triangles joined by a path with k edges.)

1. P_k with $k \leq 9$.
2. $N(i, j, k)$ with some restrictions on how large i, j, k can be.
3. L_k $(k \neq 2)$.
4. L_k $(k \neq 2)$ with tree components attached to either of the triangles.

Item (4) has been shown to not be in the family (see [25]). For item (3), the values of k are known to be odd and at most 5. Several results on item (2) have appeared in this survey. It is known that $i + j + k \leq 7$ see [25]. For item (1), in [69] it was shown that P_8 works. This was improved recently to:

Theorem 34 [25] *If G is a claw-free and P_9-free 3-connected graph, then G is Hamiltonian connected.*

Pancyclicity of 3-connected graphs with forbidden pairs was characterized in [105].

Theorem 35 [105] *If X and Y are connected graphs of order at least 3 with $X, Y \neq P_3$ and $Y \neq K_{1,3}$, then a 3-connected XY-free graph G is pancyclic if and only if $X = K_{1,3}$ and Y is a subgraph of a member of the family $\{P_7, L_1, N(4, 0, 0), N(3, 1, 0), N(2, 2, 0), N(2, 1, 1)\}$.*

It is interesting that the family contains a path and generalized nets, where the path is only one longer than in the 2-connected Hamiltonian case, and the generalized nets here all have $i + j + k = 4$ while those in the 2-connected Hamiltonian case have $i + j + k = 3$. The only new graph here is L_1 which was defined earlier.

A graph G is said to be pancyclic mod k if for all natural numbers s, it has a cycle whose length is congruent to s mod k.

Theorem 36 [202] *A 2-connected $K_{1,4}$-free graph G such that $\delta(G) \geq k + 1$ is pancyclic mod k.*

A graph G on $n \geq 3$ vertices is called claw-heavy if every induced claw of G has a pair of nonadjacent vertices with degree sum at least n. Recall that H is the hourglass.

Theorem 37 [42] *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is claw-heavy and moreover, P_7-free and $N(2, 2, 0)$-free, or P_7-free and H-free, then G is Hamiltonian.*

Theorem 38 [96] *Let G be a 2-connected claw-heavy graph on $n \geq 3$ vertices. If each induced $M = K_{1,3} + e$ has an end vertex x satisfying $deg\ x \geq (n-2)/3$ or a pair of vertices y and z satisfying $deg\ y + deg\ z \geq n$, then G is Hamiltonian.*

4 Multiple Hamiltonian Cycles

The hunt for more than one Hamiltonian cycle has a long and interesting history. An early result due to Nash-Williams [166] proved the fundamental result that the conditions of Dirac's Theorem ($\delta(G) \geq n/2$) actually guaranteed the existence of many edge disjoint Hamiltonian cycles, in fact, at least $\lfloor 5n/224 \rfloor$ of them. Nash-Williams asked if this number could be improved and this has been a question of interest ever since. Nash-Williams [166] gave an example of a graph on $n = 4m$ vertices with minimum degree $2m$ having at most $\lfloor (n+4)/8 \rfloor$ edge disjoint Hamiltonian cycles. Here is a similar example given in [48].

Let A be an empty graph on $2m$ vertices, B a graph consisting of $m+1$ disjoint edges and let G be the graph obtained from the disjoint union of A and B along with all edges between the two sets. Thus, G is a graph on $4m+2$ vertices with minimum degree $2m+1$. Further, observe that any Hamiltonian cycle in G must use at least two edges from B and thus, G has at most $\lfloor (m+1)/2 \rfloor$ edge disjoint Hamiltonian cycles. This was shown by Christofides et al. [48] to be asymptotically best possible.

Theorem 39 [48] *For every $\alpha > 0$, there is an integer n_0 so that every graph on $n \geq n_0$ vertices of minimum degree at least $(1/2 + \alpha)n$ contains at least $n/8$ edge disjoint Hamiltonian cycles.*

Nash-Williams [166] noted that the construction given above depends on the graph being non-regular. He conjectured [166] the following, which is best possible, and was also conjectured independently by Jackson [118].

Conjecture 4 *Let G be a d-regular graph on at most $2d$ vertices. Then G contains $\lfloor n/2 \rfloor$ edge disjoint Hamiltonian cycles.*

Note: At the time of submission, D. Osthus (and a group of others) has announced a solution of the conjecture, although no paper was yet available.

For complete graphs the conjecture is true by the famed construction of Walecki (see e.g [11]). However, the best result concerning this conjecture was due to Jackson [118] who showed that a d-regular graph on $14 \leq n \leq 2d+1$ vertices contains $\lfloor (3d-n+1)/6 \rfloor$ edge disjoint Hamiltonian cycles. Recently in [48] the following approximate version of the Conjecture was shown.

Theorem 40 [48] *For every $\alpha > 0$ there is an integer n_0 so that every d-regular graph on $n \geq n_0$ vertices with $d \geq (1/2 + \alpha)n$ contains at least $(d - \alpha n)/2$ edge disjoint Hamiltonian cycles.*

They further showed:

Theorem 41 [48] *There exists $\alpha_0 > 0$ so that for every $0 < \alpha \leq \alpha_0$ there is an integer n_0 for which every graph on $n \geq n_0$ vertices with minimum degree $\delta \geq (1/2 + \alpha)n$ and maximum degree $\Delta \leq \delta + \alpha^2 n/5$ contains at least $(\delta - \alpha n)/2$ edge disjoint Hamiltonian cycles.*

Finally, they provided a result that approximately describes how the number of edge disjoint Hamiltonian cycles in G gradually approaches $\delta(G)/2$ as $\delta(G)$ approaches $n - 1$.

Theorem 42 [48]

1. *For all positive integers δ, n with $n/2 < \delta < n$, there is a graph G on n vertices with minimum degree δ such that G contains at most*

$$\frac{\delta + 2 + \sqrt{n(2\delta - n)}}{4}$$

edge disjoint Hamiltonian cycles.

2. *For every $\alpha > 0$, there is a positive integer n_0 so that every graph on $n \geq n_0$ vertices of minimum degree $\delta \geq (1/2 + \alpha)n$ contains at least*

$$\frac{\delta - \alpha n + \sqrt{n(2\delta - n)}}{4}$$

edge disjoint Hamiltonian cycles.

The proofs use the Regularity Lemma, so the values of n are accordingly large. A similar result was obtained by Hartke and Seacrest [113] that does not rely on the Regularity Lemma.

Theorem 43 [113] *If G is a graph with n vertices and minimum degree $\delta \geq n/2 + 5n^{3/4}\ ln\ n$, then G contains at least*

$$\frac{\delta + \sqrt{2\delta n - n^2}}{2} - 5n^{7/8}\ ln\ n$$

edge disjoint Hamiltonian cycles.

The above questions and results extend naturally to graphs with even larger minimum degree (see [136]). This question becomes:

Question 1 [136] *How many edge disjoint Hamiltonian cycles can one guarantee in a graph on n vertices with minimum degree δ, $n/2 \leq \delta \leq n - 1$?*

As pointed out in [136] a natural bound is provided as follows: Let $reg_{\text{even}}(G)$ be the largest degree of an even regular spanning subgraph of G. Then let

$$reg_{\text{even}}(n, \delta) := min\{reg_{\text{even}}(G) \mid |G| = n, \delta(G) = \delta\}.$$

Clearly, we cannot expect more than $reg_{\text{even}}(n, \delta)/2$ edge disjoint Hamiltonian cycles in a graph of order n and minimum degree δ. In fact, in [136] Kühn et al. conjecture the bound can be obtained.

Conjecture 5 [136] *Suppose G is a graph on n vertices with minimum degree $\delta \geq n/2$. Then G contains at least $reg_{\text{even}}(n, \delta)/2$ edge disjoint Hamiltonian cycles.*

They are able to nearly show their conjecture with the following:

Theorem 44 [136] *For every $\epsilon > 0$, there exists an integer $n_0 = n_0(\epsilon)$ such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1/2 + \epsilon)n$ contains at least $reg_{\text{even}}(n, \delta(G))/2$ edge disjoint Hamiltonian cycles.*

Later, Csaba, Kühn, Lo, Osthus and Treglown (see [136]) proved the conjecture for large n by solving the case when δ is allowed to be close to $n/2$.

If we are not concerned about edge disjoint cycles, the number of possible Hamiltonian cycles grows much larger. Let G be a graph of order n and minimum degree $\delta(G) \geq n/2$. The number of Hamiltonian cycles in $G = K_{\lfloor n/2\rfloor, \lfloor n/2\rfloor}$ is $\lfloor n/2\rfloor!(\lfloor n/2\rfloor - 1)!$ and adding a vertex v_n for odd n raises this number to $(n-1)\lfloor n/2\rfloor!(\lfloor n/2\rfloor - 1)!$. Let $h(n, \delta)$ denote the number of Hamiltonian cycles in a graph of order n with the specified minimum degree δ. Using the Regularity Lemma, the following was shown in [188] for $\delta = \lceil n/2\rceil$:

Theorem 45 [188] *There is a constant $c > 0$ such that for n sufficiently large*

$$(cn)^n \leq h(n, \lceil n/2\rceil) \leq (n-1)\lfloor n/2\rfloor!(\lfloor n/2\rfloor - 1)!.$$

This was followed by:

Theorem 46 [170] *If G is a graph on n vertices and $C\binom{n}{2}$ edges where $3/4 < C \leq 1$, then G contains at least $(C_1 n)^{C_2 n}$ Hamiltonian cycles, where the constants C_1, C_2 depend upon C.*

For an edge weighting $x : E \to R^+$ let $h(x) = \Sigma_e\, x_e log(1/x_e)$. Let $e \ni v$ denote that the edge e is incident to vertex v. Call an edge weighting proper if $\Sigma_{e \ni v}\, x_e = 1$ for each $v \in V(G)$ (such an x is sometimes called a perfect fractional matching). Let $h(G)$ be the maximum of $h(x)$ over proper edge weightings of x. Finally, let $\Psi(G)$ denote the number of Hamiltonian cycles in G. Motivated by the last result, and using the above definitions, Cuckler and Kahn [52] showed the following:

Theorem 47 [52] *For any n vertex graph with minimum degree $\delta \geq n/2$,*

1. $\Psi(G) \geq \frac{n!}{(2+o(1))^n}$.
2. $\Psi(G) \geq exp_2[2h(G) - n\, log_2\, e - o(n)]$.

3. *$h(G) \geq (n/2)log\ d$, so that the minimum number of Hamiltonian cycles in G exceeds* $(\frac{d}{e+o(1)})^n$.

Turning to bipartite graphs, let $\sigma_2^2(G)$ denote the minimum degree sum of a pair of nonadjacent vertices from different partite sets. Using this, the following was shown, extending earlier work of Moon and Moser [163].

Theorem 48 [83] *If $G = (X, Y)$ is a balanced bipartite graph of order $2n$, with $n \geq 128k^2$ such that $\sigma_2^2(G) \geq n+2k-1$, then G contains k edge disjoint Hamiltonian cycles.*

Given an abelian group $\mathcal{G}$ and $S \subset V(\mathcal{G})$, the Cayley graph $CAY(\mathcal{G}; S)$ may be thought of as the graph whose vertices are the elements of the group $\mathcal{G}$ with g adjacent to h if and only if $h = gs$ for some $s \in S$. The folklore conjecture that all finite Cayley graphs are Hamiltonian has received a great deal of attention over the years, but remains an open problem. Witte and Gallian [209] wrote a survey in 1984. The amount of information here deserves its own survey. I shall limit myself to the following problem.

Alspach [10] gave the following conjecture concerning Cayley graphs.

Conjecture 6 [10] *Every connected Cayley graph on an abelian group has a Hamiltonian decomposition* (*where a single 1-factor is allowed when the degree is odd*).

The conjecture is settled affirmatively when the Cayley graph is regular of degree is 5 or less. For regular Cayley graphs of degree 6, when the group is odd, there is also a Hamiltonian decomposition (see [207]) into three Hamiltonian cycles. Some other partial results exist for degree 6 and little is known for degree 7 or higher.

Let $F(q)$ denote the finite field of order q where q is prime and $q \equiv 1(mod\ 4)$. The Paley graph $P(q)$ is a Cayley graph with S the set of quadratic residues in $F(q)$.

Theorem 49 [12] *All Paley graphs are Hamiltonian decomposable.*

Finally, in the area of graph products, the following results were shown. The tensor product (sometimes called the direct product) of graphs G and H is the graph $G \otimes H$ with vertex set $V(G) \times V(H)$ and edge set $\{(u, x)(v, y) \mid uv \in E(G)$ and $xy \in E(H)\}$.

Theorem 50 [17] *If $r, s \geq 3$, then $K_r \otimes K_s$ has a Hamiltonian cycle decomposition.*

Strengthening this result, it is shown in [160] that the tensor product of two regular complete multipartite graphs is Hamiltonian decomposable. Thus, the previous theorem is a corollary of this result. Earlier, in [159] it was shown that for $m \geq 3$ the tensor product of $K_{r,r}$ with K_m is Hamiltonian decomposable.

In 1988, Bermond [23] conjectured that if a graph G is Hamiltonian decomposable, then its line graph $L(G)$ is Hamiltonian decomposable. Pike [174] consider this conjecture and showed the following:

Theorem 51 [174] *Every bipartite Hamiltonian decomposable graph G with connectivity $\kappa(G) = 2$ has a Hamiltonian decomposable line graph.*

It has long been known that K_n has a Hamiltonian decomposition when n is odd, and when n is even the Hamiltonian decomposition leaves a 1-factor. In [147] they take the next natural step and show that for any 2-factor U of K_n (n even), there exists a 3-factor T of K_n such that $E(U) \subset E(T)$ and $K_n - E(T)$ admits a Hamiltonian decomposition. Following in this line, the question of Hamiltonian decompositions in complete multipartite graphs with certain leaves was also considered in [148], [178], and [38].

The classic work of Thomason [200] showed that the number of pairs of Hamiltonian cycles in a graph, with a pair of such cycles, is even and at least four. He further showed that the number of Hamiltonian cycles containing any two edges was also even. Skupien [192] considered such questions including pairs of Hamiltonian cycles (traceable pairs) and showed the following:

Theorem 52 [192]

1. *For each integer $n \geq 3$, there is a n-vertex 4-regular multigraph M_n with exactly four Hamiltonian cycles.*
2. *For each integer $n \geq 3$, there is an n-vertex multigraph N_n which has exactly two traceable pairs.*
3. *Each multigraph $M \neq C_2$ has an even number of traceable pairs.*
4. *For each $n \geq 5$, there is a simple n-vertex graph with exactly four traceable pairs in which endvertices of the paths make up a 3-element set.*
5. *For each integer $n \geq 9$, there is a simple graph on n vertices which has precisely 16 Hamiltonian pairs.*

5 Distributing Vertices on the Cycle

In this section we examine a series of results designed to control the placement of a set of vertices on a Hamiltonian cycle so that certain distances are maintained between these vertices. We begin with an interesting result from Kaneko and Yoshimoto [122] which started this line of investigation.

Theorem 53 [122] *Let G be a graph of order n with $\delta(G) \geq n/2$, and let d be a positive integer such that $d \leq n/4$. Then, for any vertex subset A with $|A| \leq n/2d$, there is a Hamiltonian cycle C such that $dist_C(u, v) \geq d$ for any $u, v \in A$.*

This result is sharp as can be seen from the graph $2K_{\frac{n}{2}-1} + K_2$. When all vertices of A are placed in one of the copies of $K_{\frac{n}{2}-1}$, then the bound becomes clear. Sárkozy and Selkow [189] showed more could be said.

Theorem 54 [189] *There are $\omega, n_0 > 0$ such that if G is a graph with $\delta(G) \geq n/2$ on $n \geq n_0$ vertices, d is an arbitrary integer with $3 \leq d \leq \omega n/2$ and S is an arbitrary subset of $V(G)$ with $2 \leq |S| = k \leq \omega n/2$, then for every sequence of integers with $3 \leq d_i \leq d$, and $1 \leq i \leq k-1$, there is a Hamiltonian cycle C of G and an ordering of the vertices of S, $a_1, a_2, \ldots, a_k$ such that the vertices of S are encountered in this order on C and we have*

$$|dist_C(a_i, a_{i+1}) - d_i| \leq 1,$$

for all but one $1 \leq i \leq k-1$.

The limiting factor in Theorem 53 is connectivity. Thus, it was natural to ask if more connectivity would allow better distribution of the vertices. In fact, much more can be said.

Theorem 55 [77] *Let $t \geq 3$ be an integer and let $0 < \epsilon < 1/(2t)$. For $n \geq 7t^6 \times 10^{10}/\epsilon^6$, let G be a graph of order n having $\delta(G) \geq n/2$ and $\kappa(G) \geq 2\lfloor t/2 \rfloor$. For every $X = \{x_1, x_2, \ldots, x_t\} \subset V(G)$, there exists a Hamiltonian cycle H in G such that $dist_H(x_i, x_j) \geq (1/t - \epsilon)n$ for all $1 \leq i < j \leq t$. Furthermore, the minimum degree and connectivity conditions are sharp.*

Actually, the vertices can be distributed pretty much as we wish.

Theorem 56 [77] *Let $t \geq 3$ be an integer and let $\epsilon, \gamma_1, \gamma_2, \ldots, \gamma_t$ be positive real numbers having $\sum_{i=1}^{t} \gamma_i = 1$ and $0 < \epsilon < min\{\gamma_i^2/2\}$. For $n \geq 7t^{12} \times 10^{10}/\epsilon^6$, let G be a graph of order n having $\delta(G) \geq (n + t - 1)/2$ or $\delta(G) \geq n/2$ and $\kappa(G) \geq 3t/2$. For every $X = \{x_1, x_2, \ldots, x_t\} \subset V(G)$, there exists a Hamiltonian cycle H containing the vertices of X in order such that*

$$(\gamma_i - \epsilon)n \leq dist_H(x_i, x_{i+1}) \leq (\gamma_i + \epsilon)n$$

for all $1 \leq i \leq t$. Furthermore, the minimum degree and connectivity conditions are sharp.

Very recently the question of exact placement for small sets of vertices was investigated. The following conjecture of Enomoto [65] is the spark for this work.

Conjecture 7 *If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2+1$, then for any pair of vertices x, y in G, there is a Hamiltonian cycle C of G such that $dist_C(x, y) = \lfloor n/2 \rfloor$.*

The following result of Faudree et al. [80] deals with a pair of vertices at a precise distance on a Hamiltonian cycle.

Theorem 57 [80] *Let $k \geq 2$ be a fixed positive integer. If G is a graph of order $n \geq 6k$ and $\delta(G) \geq (n + 2)/2$, then for any vertices x and y, G has a Hamiltonian cycle C such that $dist_C(x, y) = k$.*

This was generalized in [78].

Theorem 58 [78] *Given a set of $k - 1$ integers $\{p_1, p_2, \ldots, p_{k-1}\}$ and a fixed set of k vertices $\{x_1, x_2, \ldots, x_k\}$ in a graph G of sufficiently large order n with $\delta(G) \geq (n + 2k - 2)/2$, then there is a Hamiltonian cycle C such that $dist_C(x_i, x_{i+1}) = p_i$ for $1 \leq i \leq k - 1$.*

6 Placing Elements on Cycles

Besides finding ways to distribute elements on cycles at certain distances, the last two decades have seen a great many developments concerning placing various collections of elements on cycles, whether it be on a Hamiltonian cycle or on some special collections of cycles. The elements placed could be a specific set of vertices, edges, paths or some combination of these elements. There are many such results now. The interested reader should see [103] for a more general survey. I will concentrate on only a few recent results here.

Recall, a subset $X \subseteq V(G)$ is called cyclable if there is a cycle in G containing all the vertices of X. This clearly generalizes the idea of a Hamiltonian cycle. For earlier results of this type see [103].

By considering a generalized Fan-type [66] degree condition (as the distance between u and v may not be two), Sakai [186] improved on an earlier result due to Egawa et al. [62].

Theorem 59 [186] *Let k, d be integers with $d \geq k \geq 3$. Let G be a k-connected graph of order at least $2d$ and let $X \subseteq V(G)$ with $|X| \leq k$. If max $\{deg\ u, deg\ v\} \geq d$ for any nonadjacent distinct vertices u, v of G, then G is X-cyclable and the cycle has length at least $2d$.*

An alternate connectedness measure is toughness and, of course, there are results relating toughness with cycles containing prescribed vertices. In [154], the following Fan-type result was given.

Theorem 60 *If G is a 1-tough graph of order n and $X \subseteq V(G)$ such that $\sigma_3(G) \geq n$ and, for all $x, y \in X$, $dist(x, y) = 2$ implies max $\{deg\ x, deg\ y\} \geq \frac{n-4}{2}$, then G is X-cyclable.*

In discussions with Saito (personnel communication) the following (probably very difficult) problem emerged.

Problem 1 For each real number r, $0 < r \leq 1$, does there exist a function $f(r)$ so that any $\lfloor rn \rfloor$ vertices of an $f(r)$-tough graph lie on a common cycle?

Clearly, when $r = 1$, this is the well-known toughness problem of Chvátal [49] for Hamiltonian graphs. It is also natural to wonder about an edge analogue to this problem.

The use of forbidden subgraphs is a well established tool in the study of cycles. But, until recently, little had been done in this area concerning our topic. Recently, Fujisawa et al. [94] found a generalization of the results in [61] on $\{claw, net\}$-free graphs. To understand the work in [94] we need the following definitions.

Let G be a graph and $S \subseteq V(G)$. An induced subgraph F is called an *S-claw* if F satisfies the following properties:

1. F consists of three paths P_1, P_2, P_3 such that they have only one common vertex x and $V(F) = V(P_1) \cup V(P_2) \cup V(P_3)$.
2. For any $i \in \{1, 2, 3\}$, the end vertex of P_i which is not x is contained in S.
3. For any $i \in \{1, 2, 3\}$, the internal vertices of P_i are contained in $V(G) - S$.
4. $E(F) = E(P_1) \cup E(P_2) \cup E(P_3)$.

Similarly, an induced subgraph $F' \subseteq G$ is called an *S-net* if F' satisfies the following properties:

1. F' contains a triangle T with $V(T) = \{x_1, x_2, x_3\}$.
2. There exist three vertex disjoint paths P_1, P_2, P_3 such that x_i is an end vertex of P_i and $V(F') = V(P_1) \cup V(P_2) \cup V(P_3)$.
3. For any $i \in \{1, 2, 3\}$, the end vertex of P_i which is not x_i is contained in S.
4. For any $i \in \{1, 2, 3\}$, the internal vertices of P_i are contained in $V(G) - S$.
5. $E(F) = E(P_1) \cup E(P_2) \cup E(P_3) \cup E(T)$.

Under these definitions, it is clear that a $\{V(G)$-claw, $V(G)$-net $\}$-free graph is a {claw, net}-free graph.

Theorem 61 [94] *Let G be an $\{S$-claw, S-net $\}$-free graph, $S \subseteq V(G)$.*

1. *If G is connected, then G contains a path P such that $S \subseteq V(P)$.*
2. *If G is 2-connected, then G contains a cycle C such that $S \subseteq V(C)$.*

It is natural to wonder if other forbidden subgraph results can also be extended in this manner. The above authors have some results in this direction, in particular, involving P_6. Also, can an edge version also be found by considering the end edges of S-claws (defined in a similar manner) and S-nets, where now S is a set of disjoint edges?

A closure type approach has also been used to find cycles containing specific elements. We say a vertex x is $*$-*eligible* if

1. x is not the center of a claw,
2. $G[N(x)]$ is not a complete graph,
3. there is a tree T such that,
   (a) $N(x) \subseteq V(T) \subseteq N^2(x)$,
   (b) for any $s \in S(T) = \{s \in V(T) \mid deg_T(s) \geq 2\}$, the set $N(s) - N[x]$ induces a clique (possibly empty),
   (c) $V(T) - S(T) \subseteq N(x)$.

The local completion of G at x, denoted G^*_x, is the graph obtained from G by adding to $G[N(x)]$ all missing edges.

Now, let $H \subseteq V(G)$ be an arbitrary set of vertices and let $cl^*_H(G)$ be the graph obtained from G by recursively performing the local completion at the $*$-eligible vertices in H (this is clearly a local specialization of the standard closure of claw-free graphs). Using the above, Čada et al. [40] obtained the following stability result and subsequent theorem.

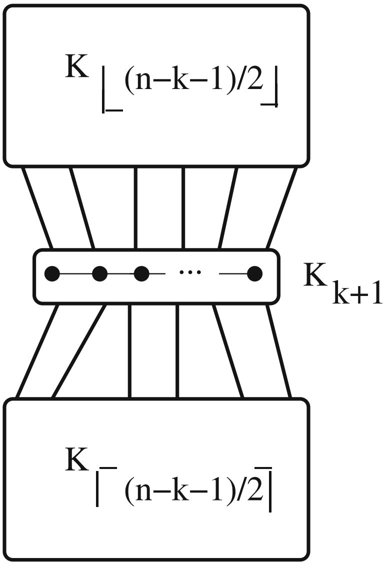
Theorem 62 *Let G be a graph, $S \subseteq V(G)$, $S \neq \emptyset$, and let k be an integer, $1 \leq k \leq |S|$. Let $H \subseteq V(G)$ be an arbitrary set of vertices. Then G contains a cycle C with $|V(C) \cap S| \geq k$ if and only if $cl^*_H(G)$ contains a cycle C' with $|V(C') \cap S| \geq k$. Hence, for $H, S \subseteq V(G)$, S is cyclable in G if and only if S is cyclable in $cl^*_H(G)$.*

Theorem 63 *Let G be a 2-connected graph of order $n \geq 33$ and let $S \subseteq V(G)$, $S \neq \emptyset$ be such that*

1. *no vertex in $S \cup N(S)$ is a claw center,*
2. *$\sigma_3(G) \geq n - 2$.*

Then, S is cyclable in G.

Fig. 1 The sharpness example for Theorem 64



Classic density is a common attack for cycle problems and placing elements on cycles is no exception. One of the oldest results of this kind is due to Pósa [175] (for minimum degree), and it was extended to σ_2 by Kronk [135]. Here a (k, t, s)-linear forest is a forest with exactly k edges, t total paths, with s of the paths being single vertices. When the number of single vertex paths in F is not critical, we will denote F simply as a (k, t)-linear forest. Further, a graph G is (k, t)-Hamiltonian if for each (k, t)-linear forest F of G, there is a Hamiltonian cycle of G containing the linear forest F.

Theorem 64 [135,175] *Let $0 \leq t \leq k$ be integers and G a graph of order n. If $\sigma_2(G) \geq n + k$, then for any $(k, t, 0)$-linear forest F, there is a Hamiltonian cycle of G that contains the linear forest F. Also, the σ_2 bound is sharp with respect to general n and general $(k, t, 0)$-linear forests.*

Sugiyama [194] generalized this result in the following:

Theorem 65 *Let G be a graph on $n \geq 5$ vertices and S a set of $m \geq 0$ edges inducing a linear forest in G. If $\sigma_2(G) \geq n + m$, then for every $t = 0, 1, 2, \dots, m$ there is a Hamiltonian cycle C_t in G such that $|E(C_t) \cap S| = t$.*

The graph of Fig. 1 shows that Theorem 64 is sharp in some sense. The forest F is a single path of length k in the K_{k+1}. However, for forests other than a single path, it may not be sharp as was shown in [76].

Theorem 66 [76] *Let G be a graph of order n. Let k, t and n be positive integers with $2 \leq k + t \leq n$ and let F be a (k, t)-linear forest. If*

1. $\sigma_2(G) \geq n + k$ *when $F = P_{k+1} \cup (t - 1)K_1$, and*

2. *$\sigma_2(G) \geq n + k - \epsilon(k, n)$ otherwise,*

then G is (k, t)-Hamiltonian, where $\epsilon(n, k) = 1$ if $2|(n-k)$ and $\epsilon(n, k) = 0$ otherwise. Furthermore, the condition on σ_2 is sharp.

In this result, the sharpness of part 2 is demonstrated by the graph obtained from the join of F and the complete bipartite graph $H = K_{(n-k+1+\epsilon)/2,(n-k-2t-1-\epsilon)/2}$. The bipartite graph has path cover number in excess of t and hence with the unbalanced nature of H, there is no Hamiltonian cycle containing the forest F.

In the following, let $\sigma_k(S, G)$ denote the minimum degree sum, in G, of k independent vertices of the vertex subset S. Let $\alpha(S, G)$ be the number of vertices of a maximum independent set in $G[S]$. Also, let $\delta(S, G)$ be the minimum degree, in G, of the vertices in S. In [112] 3-connected graphs were studied.

Theorem 67 *Let G be a 3-connected graph of order n and S a subset of $V(G)$.*

1. *If $\sigma_4(S, G) \geq n + 2\alpha(S, G) - 2$, then S is cyclable.*
2. *If $\sigma_4(S, G) \geq n + 2\delta(S, G)$ and deg $v \geq n/2$ for every $v \in S - (N(w) \cup \{w\})$, where $w \in S$ and deg $w = \delta(S, G)$, then S is cyclable in G.*
3. *If $\sigma_2(S, G) \geq n/2 + \delta(S, G)$, then S is cyclable.*

Standard edge density conditions have also been used to place edges on cycles. If F is a 1-factor of G and there exists a Hamiltonian cycle in G containing all edges of F, then we say G is F-Hamiltonian. An early result of this type is due to Häggkvist [110].

Theorem 68 *Let G be a graph, $|V(G)| = n \geq 4$, n even. If $\sigma_2(G) \geq n + 1$, then for any 1-factor F, G is F-Hamiltonian.*

A classic result due to Las Vergnas [146] made the natural transition to bipartite graphs.

Theorem 69 *Let $G = (A \cup B, E)$ be a bipartite with $|A| = |B| = n \geq 2$. If for each pair u, v of nonadjacent vertices with $u \in A$ and $v \in B$ we have deg $u+$deg $v > n+1$, then for any 1-factor F of G, G is F-Hamiltonian.*

Yang [210] provided a true edge density result as well. Suppose K_6 has vertex set $\{y_1, \ldots, y_6\}$. Let $S_1 = K_6$ minus the edges $\{y_1y_2, y_1y_4, y_2y_3, y_3y_4\}$. It is easy to see that if F is the matching $\{y_1y_3, y_2y_4, y_5y_6\}$, then S_1 is not F-Hamiltonian.

Theorem 70 *Let G be a graph on n vertices ($n \geq 4$, n even). If $\delta(G) \geq 2$ and $|E(G)| \geq \frac{(n-1)(n-2)}{2} + 1$, then for any 1-factor F of G, G is F-Hamiltonian if and only if $G \ncong S_1$.*

Yang [210] also provided a bipartite version of the above result.

An extension of the idea of cyclable sets is the following. A graph G is said to be *S-pancyclable* if for every integer l, $3 \leq l \leq |S|$, there is a cycle in G that contains exactly l vertices of S. An Ore-type result in this direction is the following:

Theorem 71 [81] *If G is a graph of order n and $\sigma_2(G) \geq n$, then either G is S-pancyclable or else n is even, $S = V(G)$ and $G = K_{n/2,n/2}$, or $|S| = 4$, $G[S] = K_{2,2}$ and the structure of G is well characterized.*

In [1] bipartite graphs were considered.

Theorem 72 *Let G be a 2-connected balanced bipartite graph of order $2n$ and bipartition (X, Y). Let S be a subset of X of cardinality at least 3. Then if the degree sum of every pair of nonadjacent vertices $x \in S$ and $y \in Y$ is at least $n + 3$, then G is S-pancyclable.*

It is also natural to expect that a closure property would apply to problems of this general type. Again in [40] this was considered. Of course, their first interest was in the stability of S-cyclability and S-pancyclability.

Theorem 73 *Let G be a graph of order n, let $S \subseteq V(G)$, $S \neq \emptyset$, and let k be an integer, $1 \leq k \leq |S|$. Let $u, v \in V(G)$ be such that $uv \notin E(G)$ and $deg\ u + deg\ v \geq n$. Then G contains a cycle C with $|V(C) \cap S| \geq k$ if and only if $G' = G + uv$ contains a cycle C' with $|V(C') \cap S| \geq k$.*

Now the *k-closure* of G is that graph obtained from G by recursively joining pairs of nonadjacent vertices x, y satisfying $deg\ x + deg\ y \geq k$ until no such pair remains. We denoted the resulting graph $C_k(G)$.

Theorem 74 [40] *Let G be a graph of order n, let $S \subseteq V(G)$, $|S| \geq 3$, and let $u, v \in V(G)$ be such that $uv \notin E(G)$ and*

$$deg\ u + deg\ v \geq n + |S| - 3.$$

Then S is pancyclable in G if and only if S is pancyclable in $G + uv$. Hence, S is pancyclable in G if and only if S is pancyclable in $C_{n+|S|-3}(G)$.

In [40], they then localize the closure as follows. For $S \subseteq V(G)$ and any integer k, define the (k, S)-closure of G denoted $C_k^S(G)$, as the graph obtained by recursively adding all missing edges uv with $deg\ u + deg\ v \geq k$, $u, v \in S$. The closure $C_k^S(G)$ is uniquely determined and if G is large while S is small, it is somewhat easier to handle. For $S \subset V(G)$, we say the *S-length* of a cycle in G is the number of vertices of S that the cycle contains. Then the *S-circumference* of G is the maximum S-length.

Theorem 75 [40] *Let G be a graph of order n and let $S \subseteq V(G)$, $|S| \geq 3$. Then*

1. *the S-circumference of G equals the circumference of $C_n^S(G)$,*
2. *S is cyclable in G if and only if S is cyclable in $C_n^S(G)$,*
3. *S is pancyclable in G if and only if S is pancyclable in $C_{n+|S|-3}^S(G)$.*

Next we consider another old property. A graph G of order n is said to be *vertex pancyclic* if for any vertex x, there is a cycle in G of length ℓ containing x, for each ℓ, $3 \leq \ell \leq n$. Bondy [30] initiated the study of pancyclic and vertex pancyclic graphs and he showed that if $\delta(G) \geq (n + 1)/2$, then G is vertex pancyclic. Many results concerning pancyclic graphs are based upon edge density conditions. We shall not address them here. The interested reader should see [101] and [102].

Instead, we wish to consider the next natural question: What about sets of more than one vertex?

Clearly, we cannot place k vertices on a 3-cycle when $k > 3$. Thus, we must adjust our idea of what pancyclic means. Recently, two approaches to this question appeared. The first approach we consider is due to Goddard [100].

Definition 1 For $k \geq 2$ we say G is *k-vertex pancyclic* if every set S of k vertices is in a cycle of every possible length. Further, G is *set-pancyclic* if G is k-vertex pancyclic for all $k \geq 2$.

Now by "possible length", Goddard means at least k + the path cover number of $G[S]$, where the path cover number of $G[S]$ is the least number of paths that cover all the vertices of $G[S]$. This is easily seen to be a reasonable range, since if $G[S]$ has path cover number t, then at least t new vertices will be needed to link the paths (containing our k vertices) into a cycle. Goddard [100] was able to show the following.

Theorem 76 *If G has order n and $\delta(G) \geq (n+1)/2$, then G is set pancyclic.*

At essentially the same time a second approach was developed in [79]. To understand this result, we need to develop some notation.

Definition 2 Let $k \geq 0$, $s \geq 0$, and $t \geq 1$ be fixed integers with $s \leq t$ and G a graph of order n. For an integer m with $k + t \leq m \leq n$, a graph G is *(k, t, s, m)-pancyclic* if for each (k, t, s)-linear forest F, there is a cycle C_r of length r in G containing F for each $m \leq r \leq n$.

With this, the following was shown in [79] (as well as the corresponding δ result). Note, these conditions were shown to be sharp.

Theorem 77 *Let $1 \leq t \leq m \leq n$ be integers, and G be a graph of order n. The graph G is $(0, t, t, m)$-pancyclic if $\sigma_2(G)$ satisfies any of the following conditions:*

1. $\sigma_2(G) \geq n$ *if* $m = n$,
2. $\sigma_2(G) \geq \lfloor (4n+1)/3 \rfloor$ *if* $t = 1$ *and* $m = 3$,
3. $\sigma_2(G) \geq 2n - 3$ *if* $t = 2$ *or* 3 *and* $m = 3$,
4. $\sigma_2(G) \geq 2n - m$ *if* $t = 3$ *and* $m = 4$ *or* 5,
5. $\sigma_2(G) \geq 2n - 2\lceil (m-1)/2 \rceil - 1$ *if* $4 \leq t \leq m < 2t, n > m$,
6. $\sigma_2(G) \geq n + 1$ *if* $t \geq 1,\ m \geq \max\{4, 2t\}, n > m$.

Also, all of the conditions on $\sigma_2(G)$ are sharp.

In yet another direction, the idea of placing vertices in a particular order was introduced in [169]. A graph is k-ordered (Hamiltonian) if for every ordered sequence of k vertices there is a (Hamiltonian) cycle in the graph that encounters the vertices of the sequence in the given order. Early Dirac and Ore-type results for k-ordered Hamiltonian graphs were given in [128] and [68] and the Ore result was extended in [75].

A slight improvement in the minimum degree condition is possible when considering graphs with larger connectivity.

Theorem 78 [47] *Let G be a graph on n vertices with minimum degree $\delta(G) \geq n/2$. Let $k \leq n/176$ be an integer. If G is $3\lfloor k/2\rfloor$-connected, then G is k-ordered Hamiltonian.*

The connectivity bound is best possible, as illustrated by the following graph G. Let L, M, R be complete graphs with $|R| = \lfloor k/2\rfloor, |M| = 2\lfloor k/2\rfloor - 1, |L| = n - |M| - |R|$. Let G' be the union of these three graphs, adding all possible edges containing vertices of M. Let $x_i \in L$ if i is odd, and let $x_i \in R$ otherwise. Add all edges $x_i x_j$ whenever $|i - j| \notin \{0, 1, k-1\}$, and the resulting graph is G. The degree sum condition is satisfied and G is $(\lfloor 3k/2\rfloor - 1)$-connected. But there is no cycle containing the x_i in the proper order, since such a cycle would contain $2\lfloor k/2\rfloor$ paths through M.

Order properties can be applied to more than vertex sets. For $k \geq 0$ and $0 \leq s \leq t$ fixed integers, a graph G of order n is (k, t, s)-*ordered Hamiltonian* if there is a Hamiltonian cycle C that contains any linear forest with k edges, t paths and with s of the paths being single vertices and respecting the order of the paths. The graph is *strongly* (k, t, s)-*ordered* if both the order of the paths and an orientation of the paths is respected.

Theorem 79 [45] *If $0 \leq s \leq t \leq k$ are fixed integers, and G is a graph of order $n \geq max\,\{178t + k, 8t^2 + k\}$ with*

1. $\sigma_2(G) \geq n + k - 3$ *if* $s = 0, t \geq 3$,
2. $\sigma_2(G) \geq n + k + s - 4$ *if* $0 < 2s \leq t, t \geq 3$,
3. $\sigma_2(G) \geq n + k + (t - 9)/2$ *if* $2s > t \geq 3$,
4. $\sigma_2(G) \geq n + k - 2$ *if* $s \leq 1, t = 2$,
5. $\sigma_2(G) \geq n + k - 1$ *if* $s = 0, t = 1$,
6. $\sigma_2(G) \geq n$ *if* $s = t \leq 2$,

then G is strongly (k, t, s)-ordered Hamiltonian.

The sharpness of this result for case (1) is shown by the following graph. Let G consist of three complete graphs: $A = K_{\frac{n-k+2}{2}}, K = K_{k-2}, B = K_{\frac{n-k+2}{2}}$. Add all edges between A and K and all edges between K and B. The degree sum condition is just missed and G is not $(k, t, 0)$-ordered. The ordered linear forest F is placed so that x_1, the first vertex of the first path, is in A and y_k, the last vertex of the last path, is in B and $k - 2$ intermediate vertices of F are placed in K (recall F has $k + t$ vertices and $t \geq 3$ here). Similar graphs exist for the other cases.

The situation for minimum degree was considered by Faudree and Faudree [67].

Theorem 80 *Let $k \geq 1$ and $0 \leq s < t$ be integers, and G a graph of sufficiently large order n. The graph G is strongly (k, t, s)-ordered Hamiltonian if $\delta(G)$ satisfies any of the following conditions:*

1. $\delta(G) \geq (n + k + t - 3)/2$ *when* $t \geq 3$,
2. $\delta(G) \geq (n + k)/2$ *when* $t \leq 2$.

Also, all the conditions on $\delta(G)$ are sharp.

Faudree and Faudree [67] also considered the not strong case, where the bounds are slightly different.

Theorem 81 *Let $k \geq 1$ and $0 \leq s \leq t$ be integers and G a graph of sufficiently large order n. The graph G is (k, t, s)-ordered Hamiltonian if $\sigma_2(G)$ satisfies any of the following conditions:*

1. $\sigma_2(G) \geq n + k + t - 5$ *when* $s = 0$ *and* $t \geq 5$,
2. $\sigma_2(G) \geq n + k + t - 4$ *when* $s = 0$ *and* $t = 4$,
3. $\sigma_2(G) \geq n + k + t + s - 6$ *when* $0 < 2s \leq t$, $s \geq 3$, *and* $t \geq 6$,
4. $\sigma_2(G) \geq n + k + t - 3$ *when* $0 < 2s \leq t$, $s = 1, 2$, $t \geq 3$ *or* $s = 0$, $t = 3$,
5. $\sigma_2(G) \geq n + k + (3t - 9)/2 - \lceil 4(1 - s/t) \rceil$ *when* $3 \leq t < 2s$ *and* $(s, t) \neq (3, 5)$ *or* $(2, 3)$,
6. $\sigma_2(G) \geq n + k + t - 3$ *when* $s = 3$ *and* $t = 5$ *or* $s = 2$ *and* $t = 3$,
7. $\sigma_2(G) \geq n + k$ *when* $t \leq 2$.

Also, all of the conditions on $\sigma_2(G)$ are sharp.

Their corresponding minimum degree result is the following:

Theorem 82 *Let $k \geq 1$ and $0 \leq s \leq t$ be integers and G a graph of sufficiently large order n. The graph G is (k, t, s)-ordered Hamiltonian if $\delta(G)$ satisfies any of the following conditions:*

1. $\delta(G) \geq (n + k + t - 5)/2$ *when* $s = 0$ *and* $t \geq 5$,
2. $\delta(G) \geq (n + k + t - 4)/2$ *when* $s = 1$ *and* $t \geq 4$ *or* $s = 0$ *and* $t = 4$,
3. $\delta(G) \geq (n + k + t - 3)/2$ *when* $1 < s < t$ *and* $t \geq 3$ *or* $s = 0, 1$ *and* $t = 3$.
4. $\delta(G) \geq (n + k)/2$ *when* $t \leq 2$.

Also, all of the conditions on $\delta(G)$ are sharp.

This work was extended to the generalized pancyclic case in [79]. Here a graph is (k, t, s, m)-pancyclic if for any (k, t, s)-linear forest F and each integer r, $m \leq r \leq n$, there is a cycle of length r containing F. If the paths of F are required to appear in a specific order we say the graph is (k, t, s, m)-pancyclic ordered.

Theorem 83 *Let $4 \leq t \leq m \leq n$ be positive integers and let G be a graph of order n. Then G is $(0, t, t, m)$-pancyclic ordered if $\sigma_2(G)$ satisfies any of the following conditions:*

1. $\sigma_2(G) \geq 2n - 3$ *when* $t \leq m < \lfloor 3t/2 \rfloor$,
2. $\sigma_2(G) \geq 2n - 4$ *when* $\lfloor 3t/2 \rfloor \leq m < \lceil (5t - 2)/3 \rceil$,
3. $\sigma_2(G) \geq 2n - 5$ *when* $\lceil (5t - 2)/3 \rceil \leq m < 2t$,
4. $\sigma_2(G) \geq n + 4t - m - 6$ *when* $2t \leq m \leq (5t - 3)/2$,
5. $\sigma_2(G) \geq n + (3t - 9)/2$ *when* $m > (5t - 3)/2$.

Also, all of the conditions on $\sigma_2(G)$ are sharp.

Minimum degree conditions vary here and are found in [74].

Theorem 84 [74] *Let $4 \leq t \leq m \leq n$ be positive integers, and let G be a graph of sufficiently large order n. The graph G is $(0, t, t, m)$-pancyclic ordered if $\delta(G)$ satisfies any of the following conditions (where $\epsilon_n = n - 2\lfloor n/2 \rfloor$):*

1. $\delta(G) = n - 1$ *when* $t \leq m < \lfloor 3t/2 \rfloor$,
2. $\delta(G) \geq n - 2$ *when* $\lfloor 3t/2 \rfloor \leq m < 2t$,
3. $\delta(G) \geq n/2 + 2$, *when* $m = 10$ *or* 11, $t = 5$ *and* n *even.*
4. $\delta(G) \geq n/2 + 7/2$, *when* $m = 12$, $t = 6$ *and* n *odd.*
5. $\delta(G) \geq \lceil n/2 \rceil + \lfloor t/2 \rfloor + p$ *when* $m = 3t - 2p - 6 - \epsilon_n$ *for* $-1 < p \leq (t - 6 - \epsilon_n)/2$
6. $\delta(G) \geq \lceil n/2 \rceil + \lfloor t/2 \rfloor - 1$ *when* $m \geq \max\{2t, 3t - 4 - \epsilon_n\}$, *unless* $m = 11$, $t = 5$ *and* n *even.*

The idea of placing elements on cycles was reversed in [85]. Here the idea of F-avoiding Hamiltonian graphs is introduced. Let G be a graph with subgraph H. If G contains a Hamiltonian cycle C such that $E(C) \cap E(H)$ is empty, then C is an H-avoiding Hamiltonian cycle. For any graph F, if G contains an H-avoiding Hamiltonian cycle for every subgraph H of G that is isomorphic to F, then G is F-avoiding Hamiltonian. In particular, the following was shown:

Theorem 85 [85]

1. *Let* $k \geq 0$ *be an integer and let* G *be a graph on* $n \geq 2k + 3$ *vertices with* $\sigma_2(G) \geq n + 2k - 1$. *If* F *is a graph with maximum degree at most* k, *then* G *is* F*-avoiding Hamiltonian, or there is some subgraph* H *of* G *that is isomorphic to* F *and either* $G - E(H)$ *is a butterfly, or* $K_{\frac{n-1}{2}, \frac{n+1}{2}} \subseteq G - E(H) \subseteq K_{\frac{n-1}{2}} + \bar{K}_{\frac{n+1}{2}}$.
2. *If* G *has order* $n \geq 3$ *and* $|F| \leq n/2$ *and maximum degree* k, *then* $\sigma_2(G) \geq n + k$ *ensures that* G *is* F*-avoiding Hamiltonian.*
3. *If the bound on* F *from* (2) *is removed, then* $\sigma_2(G) \geq n + 2k$ *is required for* G *to be* F*-avoiding Hamiltonian.*
4. *If* $\delta(G) \geq n/2$ *and* E_1 *is a set of edges with* $|E_1| \leq \frac{n-3}{4}$, *then* G *contains a Hamiltonian cycle that avoids* E_1.

7 Spectral Attacks

Spectral theory has been used to determine many interesting results about graphs. Thus, it is no surprise that it can be applied to Hamiltonian questions as well. This new approach has seen some fine work recently.

In order to proceed, we specify notation. Let $A(G)$ be the adjacency matrix of the graph G, let $D(G)$ be the degree matrix of G, i.e., the matrix with the degrees of the vertices down the main diagonal and zeros elsewhere, let the Laplacian of G be $L(G) = D(G) - A(G)$, and let $Q(G) = D(G) + A(G)$.

We extend our concept of a graph by allowing *free edges*, which are edges with only one end vertex. In this case the degree of a vertex counts both the ordinary and free edges incident with the vertex. However, the free edges do not appear in the adjacency matrix.

The *subdivision graph* of G, denoted $S(G)$, is the graph obtained from G by subdividing each edge of G. Let $C_{2n,l}$ denote the cycle C_{2n} with l free edges added to every second vertex of C_{2n}.

The *eigenvalues of a graph* are the eigenvalues of the adjacency matrix of that graph, unless otherwise stated. For a graph G we denote the eigenvalues of G as $\lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_n(G)$.

Theorem 86 [162] *Let G be a k-regular graph of order n. If G is not Hamiltonian, then for $i = 1, 2, \ldots, n$, $\lambda_i(L(S(G)) \leq \lambda_i(L(C_{2n,k-2}))$.*

In 1995, van den Heuvel [205] consider necessary conditions for Hamiltonian cycles using the Laplacian and the matrix $Q(G)$ defined above.

Theorem 87 [205] *Let G be a graph of order n and size m. If G is Hamiltonian, then for $i = 1, 2, \ldots, n$, $\lambda_i(L(C_n)) \leq \lambda_i(L(G))$ and $\lambda_i(Q(C_n)) \leq \lambda_i(Q(G))$. In addition, if $m < 2n$, then for $i = m - n + 1, \ldots, n$ we have*

$$\lambda_{i-m+n}(L(G)) \leq \lambda_i(L(C_n)) \leq \lambda_i(Q(G))$$

and

$$\lambda_{i-m+n}(Q(G)) > \lambda_i(Q(C_n)) \leq \lambda_i(L(G)).$$

Krivelevich and Sudokov [134] used the second largest (in absolute value) eigenvalue.

Theorem 88 [134] *If the second largest absolute value of an eigenvalue λ of the adjacency matrix of a d-regular graph satisfies*

$$\lambda \leq c\frac{(log\ log\ n)^2}{log\ n\ (log\ log\ log\ n)}d$$

for a constant c and n sufficiently large, then G is Hamiltonian.

Butler and Chung [39] showed that if the nontrivial eigenvalues of the Laplacian are sufficiently close to the average degree, then the graph is Hamiltonian. The proof is algorithmic of complexity $n^{cln\ n}$.

Theorem 89 [39] *Let G be a graph of order n and average degree d and $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of the Laplacian of G. If there is a constant c so that*

$$|d - \lambda_i| \leq c\frac{(log\ log\ n)^2}{log\ n\ (log\ log\ log\ n)}d,$$

for $i \neq 1$ and n sufficiently large, then G is Hamiltonian.

Fiedler and Nikiforov [87] used the largest eigenvalue (spectral radius) of the adjacency matrix to determine if the graph contains a spanning path or cycle.

Theorem 90 [87] *Let G be a graph of order n and let $\mu(G)$ be the largest eigenvalue of the adjacency matrix of G. Then,*

1. *If $\mu(G) \geq n - 2$, then G is traceable unless G is the disjoint union of K_{n-1} and a vertex.*
2. *If $\mu(G) > n - 2$, then G is Hamiltonian unless G is K_{n-1} with a pendent edge.*

3. *If* $\mu(\bar{G}) \leq \sqrt{n-1}$, *then G is traceable unless G is the disjoint union of* K_{n-1} *and a vertex.*
4. *If* $\mu(\bar{G}) \leq \sqrt{n-2}$, *then G is Hamiltonian unless G is* K_{n-1} *with a pendent edge.*

In [190] the authors study singular graphs (in which the zero eigenvalue belongs to the spectrum of the adjacency matrix), core graphs (the eigenvector corresponding to the zero eigenvalue has no zero entries) and (κ, τ)-regular sets. A (κ, τ)-regular set is a subset of vertices inducing a κ-regular subgraph such that every vertex not in the subset has τ neighbors in it. In particular, they show that if zero is a main eigenvalue (whose corresponding eigenvector is not orthogonal to the all 1's vector) of the subdivision of a graph G, then G is not Hamiltonian.

Knowing the graph has certain properties, such as being Hamiltonian can be of value as well.

Theorem 91 [212] *The spectral radius of a Hamiltonian planar graph of order* $n \geq 4$ *is at most* $2+\sqrt{3n-11}$ *and the spectral radius of outerplanar graphs of order* $n \geq 6$ *is at most* $2\sqrt{2}+\sqrt{n-5}$.

Finally, Krivelevich [132] considered the number of Hamiltonian cycles. Here an (n, d, λ)-graph is one that is d-regular on n vertices, all of whose nontrivial eigenvalues are at most λ.

Theorem 92 [132] *If G is an* (n, d, λ)-*graph and the following conditions are satisfied:*

1. $\frac{d}{\lambda} \leq (log\ n)^{1+\epsilon}$ *for some constant* $\epsilon > 0$;
2. $log\ d\ log\ \frac{d}{\lambda} >> log\ n$,

then the number of Hamiltonian cycles in G is $n!\ (\frac{d}{n})^n\ (1+o(1))^n$.

8 Hypergraphs

For many years there was little activity on Hamiltonian cycles in hypergraphs. This was probably due to the weakly structured form of cycle that was in use. A Hamiltonian cycle was a cyclic ordering of the vertices $(v_1, v_2, \ldots, v_n)$ such that for $i = 1, 2, \ldots, n$, there exist distinct edges E_i such that $\{v_i, v_{i+1}\} \subset E_i$. Under this definition, the following Dirac-type result was shown. By a k-uniform hypergraph (or k-graph for short) we mean a hypergraph with each of its edges containing k vertices.

Theorem 93 [24] *Let H be a k-uniform hypergraph on* $n \geq k+1$ *vertices. If deg* $v \geq \binom{n-2}{k-1} + k - 1$, *for every vertex v in H, then H contains a Hamiltonian cycle.*

In 1978, Bermond [22] showed the existence of a Hamiltonian decomposition of K_n^3 (the complete 3-uniform hypergraph on n vertices) for $n \equiv 2\ mod\ 3$ and $n \equiv 4\ mod\ 6$. In 1994, Verrall [206] provided a decomposition of K_n^3 for $n \equiv 1\ mod\ 6$ and when $n \equiv 0\ mod\ 3$ for the hypergraph $K_n^3 - I$ where I is a 1-factor, thus completing the problem.

However, in 1999, Katona and Kierstead [124] gave a more structured definition of a cycle in a hypergraph, which eventually led to the following idea. In a k-uniform hypergraph $H = (V, E)$, of order n, suppose that $1 \leq \ell \leq k - 1$. An *ℓ-overlapping Hamiltonian cycle C* is a collection of edges of H such that for some cyclic ordering of the vertices of H, every edge of C consists of k consecutive vertices of the ordering and for every pair of edges E_i, E_{i+1} in C, we have that $|E_i \cap E_{i+1}| = \ell$. Thus, in an ℓ-overlapping Hamiltonian cycle, the sets $E_i \setminus E_{i+1}$ partition V into sets of size $k - \ell$. Hence, there are $n/(k - \ell)$ edges in an ℓ-overlapping Hamiltonian cycle. Further, the two extreme cases are when $\ell = k - 1$ and we say the cycle is *tight* and when $\ell = 1$ where we say the cycle is *loose*. Further, let $\delta_{k-1}(H)$ be the minimum number of edges containing a fixed set of $k - 1$ vertices, taken over all such sets.

In [124], Katona and Kierstead showed that in a k-uniform hypergraph H, if $\delta_{k-1}(H) \geq (1 - 1/2k)n - k + 4 - 5/2k$, then H will contain a tight Hamiltonian cycle. They also suggested that $\delta_{k-1} \geq \frac{n-k+2}{2}$ should suffice. Rödl et al. [179,180] nearly showed that their suggestion was correct, showing that for $\gamma > 0$ and $k \geq 3$ and n sufficiently large, $\delta_{k-1}(H) \geq n/2 + \gamma n$ implies H contains a tight Hamiltonian cycle. They later strengthened their result for the $k = 3$ case.

Theorem 94 [181]

1. *Let H be a 3-uniform hypergraph on n vertices, where n is sufficiently large. If $\delta_2(H) \geq \lfloor \frac{n}{2} \rfloor$, then H has a tight Hamiltonian cycle. Moreover, for every n there exists an order n 3-uniform hypergraph H_n such that $\delta_2(H_n) = \lceil \frac{n}{2} \rceil - 1$ and H_n does not have a Hamiltonian cycle.*
2. *If $\delta_2(H) \geq n/2 - 1$, then H has a Hamiltonian path, and moreover, for every n there exists an order n 3-uniform hypergraph J_n such that $\delta_2(J_n) = \lceil n/2 \rceil - 2$ and J_n does not have a Hamiltonian path.*

Turning to loose Hamiltonian cycles, Kühn and Osthus [137] provided the following Dirac-type bound.

Theorem 95 [137] *For every $\gamma > 0$ there exists an n_0 such that every 3-uniform hypergraph H of even order $n \geq n_0$ with $\delta_2(H) \geq (1/4 + \gamma)n$ contains a loose Hamiltonian cycle.*

They also showed that this result was best possible up to the error term γn. They further conjectured that $\delta_{k-1}(H) \geq (\frac{1}{2(k-1)} + o(1))n$ would imply the existence of a loose Hamiltonian cycle in a k-uniform hypergraph. This conjecture was verified for ℓ-cycles with $\ell < k/2$ by Hán and Schacht [111].

Theorem 96 [111] *For all integers $k \geq 3$ and $1 \leq \ell < k/2$ and every $\gamma > 0$ there exists an n_0 such that every k-uniform hypergraph H of order n where $(k - \ell)|n$ and with $\delta_{k-1}(H) \geq (1/2(k - \ell) + \gamma)n$ contains a Hamiltonian ℓ-cycle.*

For the case $\ell = 1$, this bound was also proven in [127] using the hypergraph Blow-up lemma developed by Keevash [126]. The work in [111] used an absorption technique developed by Rödl et al. [179–181].

A hypergraph H is called k-edge Hamiltonian if by a removal of any k edges, a Hamiltonian hypergraph is obtained. Frankl and Katona [90] considered the problem of determining the minimum number of edges in a k-edge Hamiltonian r-uniform hypergraph of order n. They showed the following:

Theorem 97 [90]

1. *There is a 1-edge Hamiltonian 3-uniform hypergraph on n vertices and size at least* $11n/6 + o(n)$.
2. *Every 1-edge Hamiltonian 3-uniform hypergraph on* $n \geq 5$ *vertices has at least* $14n/9$ *edges.*
3. *There is a 2-edge Hamiltonian 3-uniform hypergraph of order n and size at least* $13n/4 + o(n)$.
4. *The size of any k-edge Hamiltonian 3-uniform hypergraph of order n is at least* $S(k)/n$, *where* $S(k)$ *is the minimum size of a graph which contains a path of order 4 after deletion of any k edges.*
5. *There is a 1-edge Hamiltonian r-uniform hypergraph of order n and size* $\frac{4r-1}{2r}n + o(n)$.
6. *The size of any 1-edge Hamiltonian -uniform hypergraph on* $n \geq 6$ *vertices is at least* $3n/2$.

Now let $H^{(k)}_{n,p}$ denote the random k-uniform hypergraph of order n where each k-tuple is an edge with probability p. In 2010, Frieze [91] showed that there is an absolute constant $K > 0$ such that if $p \geq \frac{K\ Log\ n}{n^2}$, then

$$\lim_{\substack{n\to\infty \\ 4|n}} Pr(H^{(3)}_{n,p} \text{ contains a loose Hamiltonian cycle}) = 1.$$

This was extended by Dudek and Frieze [57] to k-uniform hypergraphs. They showed that if $k \geq 3$ and if $pn^{k-1}/log\ n$ tends to infinity together with n, and $2(k-1)|n$, then H contains a loose Hamiltonian cycle asymptotically almost surely (a.a.s.). This was subsequently improved in [59] to the divisibility condition $(k-1)|n$, which is best possible. Loose Hamiltonian cycles were also the goal in [127] who showed the following theorem.

Theorem 98 [127] *For all* $k \geq 3$ *and any* $\nu > 0$, *there exists* n_0 *so that if* $n > n_0$, *then any k-uniform hypergraph with*

$$\delta_{k-1} > \left(\frac{1}{2(k-1)} + \nu\right) n$$

contains a loose Hamilton cycle.

Dudek and Frieze [58] continued their study by showing a sharp threshold of e/n for the existence of tight Hamiltonian cycles in random k-uniform hypergraphs, for all $k \geq 4$. Thus, if $p \geq (1+\epsilon)e/n$, then a.a.s. $H^{(k)}_{n,p}$ contains a tight Hamiltonian cycle. When $k = 3$, they showed that $1/n$ is an asymptotic threshold.

Asymptotic thresholds for the existence of ℓ-overlapping Hamiltonian cycles for $2 \leq \ell \leq k-2$ were also shown. Table 1 below summarizes these results.

Table 1 Hamiltonicity thresholds

ℓ	k	Order of magnitude of p	Divisibility requirements
$\ell = 1$	$k = 3$	$\frac{log\ n}{n^2}$ [91][59]	$2\|n$
$\ell = 1$	$k \geq 4$	$\omega(n)\frac{log\ n}{n^{k-1}}$ [57][59]	$(k-1)\|n$
$\ell = 2$	$k \geq 3$	$\omega(n)\frac{1}{n^{k-2}}$ [58]	$(k-2)\|n$
$k > \ell \geq 3$		$\frac{1}{n^{k-\ell}}$ [58]	$(k-\ell)\|n$
$\ell = k-1 \geq 3$		e/n is the sharp threshold [58]	no requirement

Interesting work on the extremal number for a Hamiltonian cycle has also been done. By an ℓ-tight Hamiltonian cycle in a k-graph H we mean a spanning sub-k-graph whose vertices can be cyclically ordered in such a way that the edges are segments of that ordering and every two consecutive edges intersect in exactly ℓ vertices. Here we denote an ℓ-tight Hamiltonian cycle in a k-graph H on n vertices by $C_n^{(k,\ell)}$, Katona and Kierstead [124] were the first to study the appearance of a $C_n^{(k,k-1)}$ in k-graphs, i.e., to bound the extremal number $ex(n, C_n^{(k,k-1)})$. They showed that for all integers k and n with $k \geq 2$ and $n \geq 2k-1$,

$$ex(n, C_n^{(k,k-1)}) \geq \binom{n-1}{k} + \binom{n-2}{k-2}.$$

Tuza [204] improved the lower bound for general k and tight Hamiltonian cycles to

$$ex(n, C_n^{(k,k-1)}) \geq \binom{n-1}{k} + \binom{n-1}{k-2}$$

if a Steiner system $S(k-2, 2k-3, n-1)$ exists.

Glebov et al. [99] further strengthened the bound. An ℓ-tight k-uniform t-path, denoted by $P_t^{(k,l)}$ is a k-graph on t vertices, $(k-\ell) \mid (t-\ell)$, such that there exists an ordering of the vertices in such a way that two consecutive edges intersect in exactly ℓ vertices. Their extremal number relies on the extremal number of $P(k,\ell) = P^{(k-1,\ell-1)}_{\lfloor k/(k-1)\rfloor(k-\ell)+(\ell-1)}$.

Theorem 99 [99] *For any $k \geq 2$, $\ell \in \{0, \ldots, k-1\}$, there exists an n_0 such that for any $n \geq n_0$ and $(k-\ell) \mid n$,*

$$ex(n, C_n^{(k,\ell)}) = \binom{n-1}{k} + ex(n-1, P(k,l)).$$

They also describe the extremal hypergraph and find a Dirac-type bound for Hamiltonian cycles.

8.1 Coloring Hamiltonian Cycles

To ensure the presence of properly colored or rainbow colored subhypergraphs, it is necessary to consider restricted colorings in some manner. We say a coloring is r-*bounded* if every color is used at most r times, and is r-*degree bounded* if the hypergraph induced by any single color has maximum degree at most r. Rainbow colored (every edge is a different color) Hamiltonian cycles have been well-studied in graphs (see [102]). But until recently very little was known about properly colored or rainbow Hamiltonian cycles in colored k-uniform hypergraphs for $k \geq 3$.

In [60], Dudek et al. provided the following two results:

Theorem 100 [60] *For every* $1 \leq \ell < k$ *there is a constant* $c = c(k, \ell)$ *such that if* n *is sufficiently large and* $k - \ell$ *divides* n*, then any* $cn^{k-\ell}$ *-degree bounded coloring of* $K_n^{(k)}$ *yields a rainbow copy of an* ℓ*-overlapping Hamiltonian cycle,* $C_n^{(k)}(\ell)$.

Theorem 101 [60] *For every* $1 \leq \ell < k$*, there is a constant* $c' = c'(k, \ell)$ *such that if* n *is sufficiently large and* $k - \ell$ *divides* n*, then any* $c'n^{k-\ell}$*-degree bounded coloring of* $K_n^{(k)}$ *yields a properly colored copy of* $C_n^{(k)}(\ell)$.

It was noted in [60] that both of these results are optimal for loose cycles up to the values of c and c'. It was also conjectured that these results are similarly optimal for all $2 \leq \ell < k$.

Dudek and Ferrara [56] extended these two results by demonstrating that for appropriate c, a cn^{k-1}-bounded coloring of $K_n^{(k)}$ assures a rainbow ℓ-overlapping Hamiltonian cycle, provided some additional conditions are placed on the number of edges of any color that contain a given ℓ-subset of vertices. We say a coloring of a hypergraph H is (a, r)-*bounded* if for each color i, every set of a vertices in $V(H)$ is contained in at most r edges of color i. Thus, an r-bounded coloring is thus $(0, r)$-bounded and an r-degree bounded coloring is $(1, r)$-bounded.

Theorem 102 [56] *For every* $1 \leq \ell < k$*, there is a constant* $c = c(k, \ell)$ *such that if* n *is sufficiently large and* $k - \ell$ *divides* n*, then any* $(0, cn^{k-1})$*- and* $(l, cn^{k-\ell})$*-bounded colorings of* $K_n^{(k)}$ *yields a rainbow copy of* $C_n^{(k)}(\ell)$.

Theorem 103 [56] *For every* $1 \leq \ell < k$*, there is a constant* $c = c(k, \ell)$ *such that if* n *is sufficiently large and* $k - \ell$ *divides* n*, then any* $(\ell, c'n^{k-\ell})$*-bounded coloring of* $K_n^{(k)}$ *yields a properly colored copy of* $C_n^{(k)}(\ell)$.

Both of these results are optimal up to the choices of c and c' as any subset of ℓ vertices is contained in at most $O(n^{k-\ell})$ edges. Theorem 102 is also optimal in that we cannot relax the condition that the coloring is $O(n^{k-1})$-bounded.

As noted in [57], it would be of interest to obtain corresponding results for much more general classes of hypergraphs. For graphs, such results were obtained by Böttcher et al. [31]. Their work relies on a framework developed by Lu and Székely [156] that is based on the Lovasz Local Lemma. Unfortunately, it is not clear how to obtain such results for hypergraphs.

9 Surfaces

There is a long history of interest in finding Hamiltonian cycles in graphs embedded on surfaces. In 1880, Tait [195] conjectured that every 3-connected planar graph is Hamiltonian. If true, this would have implied the four color theorem. However, Tutte [203] constructed a counter-example.

A triangulation of a closed surface is simply an embedding of a graph on the surface so that each face is a triangle and so that any two faces share at most one edge. Whitney [208] proved that every 4-connected planar triangulation has a Hamiltonian cycle. Tutte [203] extended this to show every 4-connected planar graph is Hamiltonian. Thomassen [199] further generalized this to every 4-connected planar graph is Hamiltonian connected. Thomas and Yu [196] proved another generalization.

Theorem 104 [196] *Let G be a graph obtained from a 4-connected planar graph by deleting at most two vertices. Then G is Hamiltonian.*

Chen [41] relaxed the conditions in a different manner.

Theorem 105 [41] *Any maximal planar graph with only one separating triangle is Hamiltonian.*

Helden [114] also relaxed the conditions.

Theorem 106 [114] *Every plane triangulation with at most two separating triangles is Hamiltonian.*

Helden and Vieten [115] gave conditions under which a maximal planar graph would have a Hamiltonian cycle containing any two boundary edges. They also extended Whitney's theorem to maximal planar graphs with exactly three separating triangles.

In 1988, Malkevitch [158] made the following conjecture.

Conjecture 8 [158] *Every 4-connected planar graph on n vertices is pancyclic if it contains a cycle of length 4.*

This conjecture has been verified for cycle length $n-1$ (by Nelson (see [199]), for length $n-2$ (Thomas and Yu [196]) and length $n-3$ (Sanders [187]). Chen, Fan and Yu [43] proved the cases $n-4$, $n-5$ and $n-6$. The case $n-7$ was established in [53], where it was also shown that if G is a 4-connected planar graph and $u \in V(G)$, then there exists a set $X \subset V(G)$ such that $u \in X$, $|X| = 6$, and $G - X$ is Hamiltonian when $|V(G)| \geq 9$.

A famed conjecture due to Barnette [18] is the following:

Conjecture 9 (The Cubic Planar Graph Conjecture) [18] *Every 3-connected 3-regular bipartite planar graph is Hamiltonian.*

This conjecture is known to be true for graphs up to order 66 [116]. Florek [89] proved the following:

Theorem 107 [89] *If G is a 3-connected 3-regular bipartite graph with a 2-factor F that consists only of facial 4-cycles, then the following are satisfied:*

1. *If an edge is chosen on a face and this edge is in F, there is a Hamiltonian cycle containing all other edges of this face.*
2. *If any face is chosen, there is a Hamiltonian cycle which avoids all edges of this face which are in F.*
3. *If any two edges are chosen on the same face, there is a Hamiltonian cycle through one and avoiding the other.*
4. *If any two edges are chosen which are an even distance apart on the same face, there is a Hamiltonian cycle which avoids both.*

For toroidal graphs, Grünbaum [109] and Nash-Williams [167] conjectured the following:

Conjecture 10 *Every 4-connected toroidal graph is Hamiltonian.*

Thomas and Yu [197] gave strong evidence the conjecture may be true.

Theorem 108 [197] *Let G be a 5-connected toroidal graph. Then every edge of G is contained in a Hamiltonian cycle.*

Note, the above does not hold for 4-connected toroidal graphs (see [199]). Thomas, Yu and Zang [198] recently gave more evidence in favor of the conjecture by showing:

Theorem 109 [198] *Every 4-connected toroidal graph has a Hamiltonian path.*

The radial graph of a map G is a bipartite quadrangulation obtained from the face subdivision of G by removing all edges of G. Nakamoto and Ozeki [165] showed the following:

Theorem 110 *[165]*

1. *The radial graph $R(Q)$ of a bipartite quadrangulation Q on the torus is Hamiltonian.*
2. *The map obtained from a closed 2-cell embedding G on the torus by taking a radial graph twice, (i.e. $R(R(G))$) is Hamiltonian.*

They also conjectured the following:

Conjecture 11 [165] *The radial graph of any quadrangulation Q on the torus is Hamiltonian.*

Very recently, in [95], the following was shown, which essentially shows the conjecture is true.

Theorem 111 [95] *Every quadrangulation map Q on the torus with no contractible 2-cycle has $R(Q)$ Hamiltonian.*

They further showed:

Theorem 112 [95]

1. *Let G be a 3-connected balanced bipartite graph which is embeddable in the torus. If one of the partite sets consists only of vertices of degree four, then G is Hamiltonian.*
2. *Let G be a 4-connected toroidal graph. If the toughness of G is exactly one, then G is Hamiltonian.*

Brunet et al. [37] considered the Klein Bottle.

Theorem 113 [37] *Every 5-connected triangulation of the Klein Bottle is Hamiltonian.*

Kawarabayashi [125] speculates that 4-connected might be enough.

Biebighauser and Ellingham [26] considered prisms. The prism over a graph G is the cartesian product of G with K_2. We say G is prism-Hamiltonian if $G \times K_2$ is Hamiltonian. In [26], the authors prove that the prism of any triangulation of the plane, projective plane, torus or Klein bottle contains a Hamiltonian cycle. They also show that the prism of every 4-connected triangulation of a surface with sufficiently large representativity and the prism of a 3-connected bipartite planar graph are Hamiltonian. This last result lends support to the following conjecture of Rosenfeld and Barnette [182].

Conjecture 12 [182] *Every 3-connected planar graph is prism-Hamiltonian.*

Note, Yu [211] showed that a 5-connected triangulation of a surface of Euler genus g and representativity $r(G) \geq 96(2^g - 1)$ is Hamiltonian. Kawarabayashi [125] conjectures that the condition of "triangulation" may not be needed, i.e., that perhaps the result would hold for any 5-connected graph on a surface of Euler genus g and sufficiently large representativity.

10 Random and Pseudo-Random Graphs

For results on multiple Hamiltonian cycles, see Sect. 4.

Lee and Sudakov [149] combined random and density ideas.

Theorem 114 [149] *For every $\epsilon > 0$, there is a constant $C = C(\epsilon)$ such that for $p \geq (C\ log\ n)/n$ asymptotically almost surely every subgraph with minimum degree at least $(1/2 + \epsilon)np$ is Hamiltonian.*

This result answered a question of Sudakov and Vu [193]. The constant $1/2$ and the range of p are both asymptotically best possible.

Another area that has blossomed is the study of resilience. Let $\mathcal{P}$ be a monotone increasing graph property. Counting the minimum number of edges one needs to remove from a graph G in order to destroy property $\mathcal{P}$ is called the global resilience of G with respect to $\mathcal{P}$ (or the edit distance of G with respect to $\mathcal{P}$). For some properties like being Hamiltonian, removing all edges incident to a vertex of minimum degree destroys the property and thus, yields a trivial upper bound on the global resilience.

We seek better control. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sequences of n numbers. Write $a \leq b$ if $a_i \leq b_i$ for every $1 \leq i \leq n$. Given a graph on vertex set $[n]$, we denote its degree sequence by $deg_G = (deg_G(1), \ldots, deg_G(n))$.

Let $k = (k_1, k_2, \ldots, k_n)$ be a sequence of integers. For an increasing monotone graph property $\mathcal{P}$ we say the graph $G = ([n], E)$ is k-resilient with respect to $\mathcal{P}$ if for every subgraph $H \subset G$ such that $deg_H(i) \leq k_i$ for every $1 \leq i \leq n$, the graph $G - H$ possesses property $\mathcal{P}$.

The local resilience of G with respect to $\mathcal{P}$ is the minimum value of the maximum degree of a non-k-resilient sequence and we denote this parameter

$$r_l(G, \mathcal{P}) = min \{ r \ : \ \exists H \subset G \text{ such that } \Delta(H) = r \text{ and } G - H \notin \mathcal{P}\}.$$

If G is Hamiltonian we will say $G \in \mathcal{H}AM$. Sudakov and Vu [193] were one of the first to study local resilience with respect to being Hamiltonian. Let $G(n, p)$ denote the usual binomial random graph introduced by Erdös and Rényi.

Theorem 115 [193] *For all $p > log^4 n/n$, the local resilience of $G(n, p)$ with respect to being Hamiltonian is $(1/2 + o(1))np$.*

Ben-Shimon et al. [21] studied this idea for the property of being Hamiltonian. For every constant $\epsilon > 0$ and $t > 0$ define the (not necessarily integral) sequence $\bar{d} = (\bar{d}_1, \ldots, \bar{d}_n)$ as follows:

1. $\bar{d}_v = d_v - 2$ if $deg\ v < t$:
2. $\bar{d}_v = d_v(1/3 - \epsilon)$ otherwise.

Theorem 116 [21] *For every $\epsilon > 0$ and $p \geq \frac{ln\ n\ +\ ln\ ln\ n + \omega(1)}{n}$, with high probability $G = ([n], E) \in G(n, p)$ with degree sequence d is $\bar{d}(\frac{np}{100}, \epsilon)$-resilient with respect to being Hamiltonian.*

Theorem 117 [21] *If $\frac{ln\ n + ln\ ln\ n + \omega(1)}{n} \leq p \leq \frac{1.02 ln\ n}{n}$ and $G \in G(n, p)$, then with high probability $r_l(G, \mathcal{H}AM) = \delta - 1$.*

Theorem 118 [21] *For every $\epsilon > 0$ there exists a constant $C = C(\epsilon) > 0$ such that if $p \geq \frac{C\ ln\ n}{n}$, then with high probability*

$$r_l(G(n, p), \mathcal{H}AM) \geq \frac{np}{3}(1 - \epsilon).$$

An interesting and fairly new class of random graphs has received considerable attention recently. Let n, m be positive integers, $0 \leq p \leq 1$. The random intersection graph $G_{n,m,p}$ is a probability space over the set of graphs on the vertex set $\{1, 2, \ldots, n\}$ where each vertex is assigned a random subset from a fixed set of m elements. Each edge arises between two vertices when their sets have at least one element in common. Each random subset assigned to a vertex is determined by

$$Pr[\text{vertex } i \text{ chooses element } j] = p$$

with these events mutually independent. This model was introduced by Karoński et al. [123]. Hamiltonicity was studied first in [63].

Theorem 119 [63] *Let* $m = \lfloor n^\alpha \rfloor$*, and* C_1, C_2 *be sufficiently large constants. If* $p \geq C_1 \frac{\log n}{m}$ *for* $0 < \alpha < 1$ *or* $p \geq C_2 \sqrt{\frac{\log n}{m}}$ *for* $\alpha > 1$ *then almost all* $G_{n,m,p}$ *are Hamiltonian. The bounds are asymptotically tight.*

The same authors extend their work in [64]. Here they study hamiltonicity of the random intersection graph in the natural setting $m = \lceil n^\alpha \rceil$ and establish a tight threshold $p = p(n, m)$ for hamiltonicity of $G_{n,m,p}$. This threshold is shown to satisfy $p = \frac{\log n}{m}$ for $\alpha \in (0, 1)$ and $p = \sqrt{\frac{\log n}{nm}}$ for $\alpha > 1$. Related results (somewhat more accurate) were obtained by Rybarczk [185].

A somewhat different model studied independently in [27] and [168] is the following: Consider a collection of n independent random subsets of $[m] = \{1, 2, \ldots, m\}$ that are uniformly distributed in the class of subsets of size d. Call any two subsets adjacent whenever they intersect. This defines a graph called the uniform random intersection graph and we denote this by $G_{n,m,d}$. In [27] they fix $d = 2, 3, \ldots$ and study when as $n, m \to \infty$, the graph $G_{n,m,d}$ contains a Hamiltonian cycle. They show that

$$Pr(G_{n,m,d} \in \mathcal{H}AM) = o(1) \quad \text{for} \quad d^2 nm^{-1} - ln\, ln\, m \to -\infty$$

and

$$Pr(G_{n,m,d} \in \mathcal{H}AM) = 1 - o(1) \quad \text{for} \quad 2nm^{-1} - ln\, m - ln\, ln\, m \to \infty.$$

Turning to random bipartite graphs (see also Sect. 10), Greenhill et al. [108] considered random bipartite regular graphs. Let $\mathcal{B}_{n,d}$ be the space of bipartite graphs on $2n$ labelled vertices $\{1, 2, \ldots, 2n\}$ which are d-regular, with each graph being equally likely to be chosen. They prove a conjecture of Robinson and Wormald [177] that the probability that a graph in $\mathcal{B}_{n,d}$ has $\lfloor d/2 \rfloor$ edge disjoint Hamiltonian cycles tends to 1 as $n \to \infty$. Further, if n is odd, the edges not in cycles form a perfect matching.

Frieze and Krivelevich [92] showed that for every constant $0 < p < 1$, with high probability almost all edges of $G(n, p)$ can be packed into edge disjoint Hamiltonian cycles. Further, they make the following conjecture.

Conjecture 13 [92] *With high probability a random graph* $G(n, m)$ *with* n *vertices and* m *edges contains* $\lfloor \delta/2 \rfloor$ *edge disjoint Hamiltonian cycles.*

Knox et al. [129] extended the work of Frieze and Krivelevich [92] to essentially the entire range of p.

Theorem 120 [129] *For any* $\epsilon > 0$*, there exists a constant* C *such that if* $p \geq (C\, log\, n)/n$*, then with high probability* $G(n, p)$ *contains* $(1 - \epsilon)np/2$ *edge disjoint Hamiltonian cycles.*

Krivelevich and Samotji [133] considered the polylogarithmic range for p.

Theorem 121 [133] *There exists a positive constant* ϵ *such that the following is true. Assume that* $log\, n/n \leq p(n) \leq n^{-1+\epsilon}$ *and* $G \in G(n, p)$*. Then* G *asymptotically almost surely (a.a.s) contains a collection of* $\lfloor \delta(G)/2 \rfloor$ *edge disjoint Hamiltonian cycles.*

Glebov and Krivelevich [98] were interested in estimating the number of Hamiltonian cycles in $G(n, p)$ when it is a.a.s Hamiltonian. They showed the following:

Theorem 122 [98]

1. *Let* $G \in G(n, p)$ *with* $p \geq \frac{ln\ n+ln\ ln\ n+\omega(1)}{n}$. *Then the number of Hamiltonian cycles is* $n!p^n(1-o(1))^n$ *a.a.s.*
2. *In the random graph process, at the very moment the minimum degree becomes two, the number of Hamiltonian cycles becomes* $(\frac{ln\ n}{e})^n(1-o(1))^n$ *a.a.s*

After [129], Kühn and Osthus [138] proved the conjecture in the range $p(n) \geq 1 - log^9\ n/n^{1/4}$. This work resolves the conjecture.

In [93] it is shown that if H is a 3-uniform hypergraph on n vertices and H satisfies a certain pseudo-randomness condition, and if n is divisible by 4, then H contains a collection of edge disjoint tight Hamiltonian cycles which cover almost all the edges of H. This has consequences for the random 3-uniform hypergraph $H_{n,p,3}$ which is obtained by including each of the $\binom{n}{3}$ possible edges with probability p, independently. Then if $\epsilon^4 5np^{16} \gg log^{21}\ n$ and n is divisible by 4, then with high probability $H_{n,p,3}$ contains a collection of edge disjoint tight Hamiltonian cycles which together cover all but at most $\epsilon^{1/15}\ n$ of the edges. These results have been extended by Bal and Frieze [15] as follows. To do this we need the following definition.

Definition 3 We say an n-vertex k-graph H is (ϵ, p)-regular if the following holds: Let $d \in \{1, 2, \ldots, \ell\}$ and let $s \in \{1, 2, \ldots, 2z+2\}$, where $z = \lceil (k-\ell)/\ell \rceil$. Given any s distinct $(k-d)$-sets $A_1, A_2, \ldots, A_s$ such that $|\cup_i\ \ A_i| \leq k + 2q$, where $(q = \ell z)$, there are $(1 \pm \epsilon)\frac{n^d}{d!}p^s$ sets of d vertices, D, such that all of $A_1 \cup D, \ldots, A_s \cup D$ are edges of H.

Note that (ϵ, p)-regular hypergraphs include random hypergraphs.

Theorem 123 [15] *Let* k *and* $\ell < k/2$ *be given. Let* $\alpha = 1/(9 + 7z^3)$. *Suppose that* n *is a sufficiently large multiple of* $2q$ *and that* ϵ, n *and* p *satisfy*

$$\epsilon^{16z+12}np^{8z} \gg log^{8z+5}n.$$

Let H *be an* (ϵ, p)*-regular* k*-graph with* n *vertices. Then* H *contains a collection of edge disjoint Hamiltonian* ℓ*-cycles that contain all but at most an* ϵ^α*-fraction of the edges.*

In [13] the question of a Hamiltonian decomposition of the complete k-uniform hypergraph of order n is discussed. The problem is connected to large sets of designs and several approaches are explored, including clique-finding techniques and difference patterns. The paper concludes with a table of results for $5 \leq n \leq 16$ and $2 \leq k \leq 14$.

Recently, Ben-Shimon et al. [21] extended the range of p for graphs in $G(n, p)$, using their work on k-resilience.

Theorem 124 [21] *If* $p \leq \frac{1.02\ ln\ n}{n}$, *then with high probability,* $G \in G(n, p)$ *contains* $\lfloor \frac{\delta(G)}{2} \rfloor$ *edge disjoint Hamiltonian cycles.*

Denote by $\mathcal{H}_\delta$ the property that a graph G contains $\lfloor \frac{\delta(G)}{2} \rfloor$ edge disjoint Hamiltonian cycles. Frieze and Kivelevich [92] showed that if $p \leq \frac{(1+o(1))ln\ n}{n}$, then with high probability $G(n, p) \in \mathcal{H}_\delta$. They conjectured that this property is typical for the entire range of p.

Conjecture 14 [92] *For every* $0 \leq p(n) \leq 1$*, with high probability* $G(n, p)$ *has the* $\mathcal{H}_\delta$ *property.*

Clearly, Theorem 124 extended the range for p.

Answering and old question on G_{3-out} (a graph on n vertices in which each vertex chooses three neighbors uniformly at random) Bohman and Frieze [28] show that the probability that G_{3-out} is Hamiltonian goes to 1 as n tends to ∞.

Three slightly different models of random geometric graphs were also investigated for hamiltonicity. In each, points are randomly chosen within some square. Edges are then randomly placed according to some rule. (See [164,16] and [54].) Random threshold graphs were considered in [172]. Finally, random graphs with a given degree sequence were studied in [51].

11 Special Topics

This section contains a number of interesting results that do not fit under the conditions of the other sections.

We begin with an old conjecture due to Hendry. A graph is cycle extendable if for every nonHamiltonian cycle C, there exists a cycle C' such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. Also recall that a graph is chordal if every cycle of length at least 4 has a chord. Hendry conjectured that every Hamiltonian chordal graph is cycle extendable. Several result have been shown which lend support to this conjecture. Jiang [119] showed the following:

Theorem 125 [119] *Every planar Hamiltonian chordal graph is cycle extendable.*

Two different proofs of the result for interval graphs were produced at the same time.

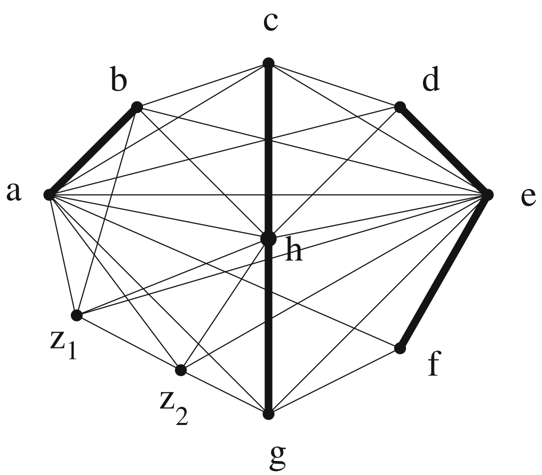
Theorem 126 [2,44] *Every Hamiltonian interval graph is cycle extendable.*

In [2] it is also shown that split graphs are cycle extendable as well as some subclasses of strongly chordal graphs. A spider is a tree of which one and only one vertex has degree exceeding 2. A spider intersection graph is that graph obtained from the intersection of subgraphs of a spider. Recently, these last two results were extended as follows:

Theorem 127 [3] *Every Hamiltonian spider intersection graph is cycle extendable.*

However, very recently, Lafond and Seamone [145] have produced a family of clever counterexamples to the Hendry conjecture. To understand the examples we need to define some terms. The clique sum of G and H (also called a clique pasting)

Fig. 2 The base graph H for counterexamples for Hendry's conjecture



is formed from the disjoint union of G and H by identifying pairs of vertices in these two cliques to form a single clique. The counterexamples to Hendry's conjecture are built from the graph H shown in Fig. 2.

Now the cycle $C : abchgfeda$ contains all the bold edges (called heavy edges). It is straightforward to verify that there is no extension of this cycle that contains all the heavy edges. It is also easy to see H is Hamiltonian. Now form graphs by pasting a clique on each heavy edge. Thus, the graph formed has order at least 15, will still be chordal and Hamiltonian, but since there is not extension of the cycle C, there exist nonHamiltonian cycles in this new graph that are not extendable. Thus, the Hendry conjecture is false.

A graph is called strongly chordal if it is chordal and every even cycle of length at least six has a chord that connects vertices at an odd distance from one another along the cycle. A graph G is fully cycle extendable if every vertex of G lies on a triangle and for every nonHamiltonian cycle C, there is a cycle C' in G such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. Lafond and Seamone [145] asked the following questions:

Question 2 [145]

1. *Is every strongly chordal graph fully cycle extendable?*
2. *Does there exist a value of $k > 2$ such that every k-connected Hamiltonian chordal graph is cycle extendable?*
3. *Does there exist a value of $t > 1$ such that every t-tough chordal Hamiltonian graph is cycle extendable?*

Recall that a k-tree is defined as follows: K_k is the smallest k-tree and a graph on at least $k + 1$ vertices is a k-tree if an only if it contains a simplicial vertex v with degree k such that $G - v$ is a k-tree.

Theorem 128 [35]

1. *Let $G \neq K_2$ be a k-tree. Then G is Hamiltonian if and only if G contains a 1-tough spanning 2-tree.*

2. *If $G \neq K_2$ is a $(k+1)/3$-tough k-tree ($k \geq 2$), then G is Hamiltonian.*

A graph is sep-chordal if it contains no separating chordless cycle of length at least four. The following generalized a result in [29].

Theorem 129 [97] *Every sep-chordal planar graph with toughness greater than one is Hamiltonian.*

The binding number of a graph G is defined as:

$$bind(G) = \min\{\frac{N(S)}{|S|} \mid \emptyset \neq S \subseteq V(G), N(S) \neq V(G)\}.$$

Woodal introduced binding number and showed that if $bind(G) \geq 3/2$, then G is Hamiltonian. This was generalized as follows:

Theorem 130 [19] *Let $b \leq 3/2$ and let G be a graph on $n \geq 3$ vertices such that $bind(G) \geq b$ and $\delta(G) \geq \frac{2-b}{3-b}n$. Then G is Hamiltonian unless $G = G_r + (r+1)K_2$ or $G = K_1 + 2K_r$ for $r \geq 2$, where G_r denotes an arbitrary graph on r vertices.*

Theorem 131 [19] *Let $b \leq 3/2$ and let G be a 2-connected graph on $n \geq 3$ vertices such that $bind(G) \geq b$ and $\delta(G) \geq \frac{2-b}{3-b}n$. Then G is pancyclic.*

References

1. Abderrezzak, M.E.K., Flandrin, E., Amar, D.: Cyclability and pancyclability in bipartite graphs. Discrete Math. **236**, 3–11 (2001)
2. Abueida, A., Sritharan, R.: Cycle extendability and Hamiltonian cycles in chordal graph classes. SIAM J. Discrete Math. **20**, 669–681 (2006)
3. Abueida, A., Busch, A., Sritharan, R.: Hamiltonian spider intersection graphs are cycle extendable (preprint)
4. Ainouche, A.: Dirac's type sufficient conditions for Hamiltonicity and pancyclicity. Graphs Combin. **25**, 129–137 (2009)
5. Ainouche, A.: Extensions of Bondy's theorem on cycles in 2-connected graphs. Ars Combin. **85**, 385–393 (2007)
6. Ainouche, A., Broersma, H.J., Veldman, H.J.: Remarks on Hamiltonian properties of claw-free graphs. Ars Combin. **29**, 110–121 (1990)
7. Ainouche, A., Kouider, M.: Hamiltonism and partially square graphs. Graphs Combin. **15**, 257–265 (1999)
8. Ainouche, A., Lapiquonne, S.: Hamiltonian connectedness and partially square graphs. Discrete Math. **306**, 1097–1104 (2006)
9. Ainouche, A., Schiermeyer, I.: 0-Dual closures for several classes of graphs. Graphs Combin. **19**, 297–307 (2003)
10. Alspach, B.: Research problem 59. Discrete Math. **50**, 115 (1984)
11. Alspach, B.: The wonderful Walecki construction. Bull. Inst. Combin. Appl. **52**, 7–20 (2008)
12. Alspach, B., Bryant, D., Dyer, D.: Paley graphs have Hamilton decompositions. Discrete Math. **312**, 113–118 (2012)
13. Bailey, R.F., Stevens, B.: Hamiltonian decompositions of complete k-uniform hypergraphs. Discrete Math. **310**, 3088–3095 (2010)
14. Babu, Ch.S., Diwan, A.A.: Subdivisions of graphs: a generalization of paths and cycles. Discrete Math. **308**, 4479–4486 (2008)
15. Bal, D., Frieze, A.: Packing tight Hamilton cycles in uniform hypergraphs. SIAM J. Discrete Math. **26**, 435–451 (2012)

16. Balogh, J., Bollobás, B., Krivelevich, M., Müller, T., Walters, M.: Hamilton cycles in random geometric graphs. Ann. Appl. Probab. **21**, 1053–1072 (2011)
17. Balakrishnan, R., Bermond, J.-C., Paulraja, P., Yu, M.-L.: On Hamilton cycle decompositions of the tensor product of complete graphs. Discrete Math. **268**, 49–58 (2003)
18. Barnette, D.: Conjecture 5. In: Tutte, W. (ed.) Recent Progress in Combinatorics, p. 343. Academic Press, New York (1969)
19. Bauer, D., Schmeichel, E.: Binding number, minimum degree, and cycle structure in graphs. J. Graph Theory **71**, 219–228 (2012)
20. Bedrosian, P.: Forbidden subgraphs and minimum degree conditions for Hamiltonicity. Ph.D. Thesis, Memphis State University (1991)
21. Ben-Shimon, S., Krivelevich, M., Sudakov, B.: On the resilience of hamiltonicity and optimal packing of Hamiltonian cycles in random graphs. SIAM J. Discrete Math. **25**, 1176–1193 (2011)
22. Bermond, J.-C.: Hamiltonian decompositions of graphs, directed graphs and hypergraphs. Advances in Graph Theory (Cambridge Combinatorial Conference, Trinity College, Cambridge, 1977). Ann. Discrete Math. **3**, 21–28 (1978)
23. Bermond, J.-C.: Problem 97. Discrete Math. **71**, 275 (1988)
24. Bermond, J.-C., Germa, A., Heydemann, M.C., Sotteau, D.: Hypergraphes hamiltoniens. Prob. Comb. Théorie Graph Orsay **260**, 39–43 (1976)
25. Bian, Q., Horn, P., Janiszewski, S., La Fleur, S., Gould, R.J., Wrayno, P.: 3-connected $\{K_{1,3}, P_9\}$-free graphs are Hamiltonian connected. Graphs Combin. **313**, 2772–2777 (2013)
26. Biebighauser, D.P., Ellingham, M.N.: Prism-Hamiltonicity of triangulations. J. Graph Theory **57**, 181–197 (2008)
27. Bloznelis, M., Radavičius, I.: A note on Hamiltonicity of uniform random intersection graphs. Lith. Math. J. **51**, 155–161 (2011)
28. Bohman, T., Frieze, A.: Hamilton cycles in 3-out. Random Struct. Algorithms **35**, 393–417 (2009)
29. Böhme, T., Harant, J., Tkáč, M.: More than one tough chordal planar graphs are Hamiltonian. J. Graph Theory **32**, 405–410 (1999)
30. Bondy, J.A.: Pancyclic graphs I. J. Combin. Theory Ser. B **11**, 80–84 (1971)
31. Böttcher, J., Kohayakawa, Y., Procacci, A.: Properly colored copies and rainbow copies of large graphs with small maximum degree. Random Struct. Algorithms **40**, 425–436 (2012)
32. Broersma, H., Kriesell, M., Ryjáček, Z.: On factors of 4-connected claw-free graphs. J. Graph Theory **37**, 125–136 (2001)
33. Broersma, H., Faudree, R.J., Huck, A., Trommel, H., Veldman, H.J.: Forbidden subgraphs that imply Hamiltonian connectedness. J. Graph Theory **40**, 104–119 (2002)
34. Broersma, H., Ryjacek, Z., Vrána, P.: How many conjectures can you stand? A survey. Graphs Combin. **28**, 57–75 (2012)
35. Broersma, H., Xiong, L., Yoshimoto, K.: Toughness and Hamiltonicity in k-trees. Discrete Math. **307**, 832–838 (2007)
36. Brousek, J.: Forbidden triples for Hamiltonicity. Discrete Math. **251**, 71–76 (2002)
37. Brunet, R., Nakamoto, A., Negami, S.: Every 5-connected triangulation of the Klein bottle is Hamiltonian. In: Proceedings of the 10th Workshop on Topological Graph Theory (Yokohama, 1998) Yokohama Mathematical Journal, vol. **47**, pp. 239–244 (1999)
38. Bryant, D., Leach, C.D., Rodger, C.: Hamilton decompositions of complete bipartite graphs with 3-factor leaves. Australas. J. Combin. **31**, 331–336 (2005)
39. Butler, S., Chung, F.: Small spectral gap in the combinatorial Laplacian implies Hamiltonian. Ann. Comb. **13**, 403–412 (2010)
40. Čada, R., Flandrin, E., Li, H., Ryjáček, Z.: Cycles through given vertices and closures. Discrete Math. **276**, 65–80 (2004)
41. Chen, C.: Any maximal planar graph with only one separating triangle is Hamiltonian. J. Comb. Optim. **7**, 79–86 (2003)
42. Chen, B., Zhang, S., Qiao, Q.: Hamilton cycles in claw-heavy graphs. Discrete Math. **309**, 2015–2019 (2009)
43. Chen, G., Fan, G., Yu, X.: Cycles in 4-connected planar graphs. Eur. J. Combin. **25**, 763–780 (2004)
44. Chen, G., Faudree, R.J., Gould, R.J., Jacobson, M.S.: Cycle extendability of Hamiltonian interval graphs. SIAM J. Discrete Math. **20**, 682–689 (2006)
45. Chen, G., Faudree, R.J., Gould, R.J., Jacobson, M.S., Lesniak, L., Pfender, F.: Linear forests and ordered cycles. Discussiones Mathematicae Graph Theory **24**, 359–372 (2004)

46. Chen, G., Gould, R.J.: Hamiltonian connected graphs involving forbidden subgraphs. Bull. Inst. Combin. Appl. **29**, 25–32 (2000)
47. Chen, G., Gould, R.J., Pfender, F.: New conditions for k-ordered Hamiltonian graphs. Ars Combin. **70**, 245–255 (2004)
48. Christofides, D., Kühn, D., Osthus, D.: Edge-disjoint Hamiltonian cycles in graphs. J. Combin. Theory Ser. B **102**, 1035–1060 (2012)
49. Chvátal, V.: Tough graphs and Hamiltonian circuits. Discrete Math. **5**, 215–223 (1973)
50. Chvátal, V., Erdös, P.: A note on Hamiltonian circuits. Discrete Math. **2**, 111–113 (1972)
51. Cooper, C., Frieze, A., Krivelevich, M.: Hamilton graphs in random graphs with a fixed degree sequence. SIAM J. Discrete Math. **24**, 558–569 (2010)
52. Cuckler, B., Kahn, J.: Hamiltonian cycles in Dirac graphs. Combinatorica **29**, 299–326 (2009)
53. Cui, Q., Hu, Y., Wang, J.: Long cycles in 4-connected planar graphs. Discrete Math. **309**, 1051–1059 (2009)
54. Diaz, J., Mitsche, D., Pérez, X.: Sharp threshold for Hamiltonicity of random geometric graphs. SIAM J. Discrete Math. **21**, 57–65 (2007)
55. Dirac, G.A.: Some theorems on abstract graphs. Proc. Lond. Math. Soc. **2**, 69–81 (1952)
56. Dudek, A., Ferrara, M.: Extensions of results on rainbow Hamilton cycles in uniform hypergraphs (preprint)
57. Dudek, A., Frieze, A.: Loose Hamiltonian cycles in random uniform hypergraphs. Electron. J. Combin. **18** (2011), Paper 48
58. Dudek, A., Frieze, A.: Tight Hamilton cycles in random uniform hypergraphs. Random Struct. Algorithms **42**, 374–385 (2012)
59. Dudek, A., Frieze, A., Loh, P.-S., Speiss, S.: Optimal divisibility conditions for loose Hamilton cycles in random hypergraphs. Electron. J. Combin. **19** (2012), Paper 44
60. Dudek, A., Frieze, A., Rucinski, A.: Rainbow Hamilton cycles in uniform hypergraphs. Electron. J. Combin. **19** (2012), Paper 46
61. Duffus, D., Gould, R.J., Jacobson, M.S.: Forbidden subgraphs and the Hamiltonian theme. In: Chartrand, Alavi, Goldsmith, Lesniak, Lick (eds.) The Theory and Applications of Graphs, (Kalmazoo, Mich., 1980), pp. 297–316. Wiley, New York (1981)
62. Egawa, Y., Glas, R., Locke, S.: Cycles and paths through specified vertices in k-connected graphs. J. Combin. Theory Ser. B **52**, 20–29 (1991)
63. Efthymiou, C., Spitakas, P.G.: On the existence of Hamiltonian cycles in random intersection graphs. Lecture Notes in Computer Science, pp. 690–701. Springer, Berlin (2005)
64. Efthymiou, C., Spirakas, P.G.: Sharp thresholds for hamiltonicity in random intersection graphs. Theor. Comput. Sci. **411**, 3714–3730 (2010)
65. Enomoto, H.: Personal communication
66. Fan, G.H.: New sufficient condition for cycles in graphs. J. Combin. Theory B **37**, 221–227 (1984)
67. Faudree, J.R., Faudree, R.J.: Hamiltonian cycles containing ordered linear forests. Bull. Inst. Combin. Appl. **5**, 78–104 (2009)
68. Faudree, J.R., Faudree, R.J., Gould, R.J., Jacobson, M.S., Lesniak, L.: On k-ordered graphs. J. Graph Theory **35**, 69–82 (2000)
69. Faudree, J.R., Faudree, R.J., Ryjáčk, Z., Vrána, P.: On forbidden pairs implying Hamilton-connectedness. J. Graph Theory **72**, 327–345 (2013)
70. Faudree, R.J., Gould, R.J.: Characterizing forbidden pairs for Hamiltonian properties. Discrete Math. **173**, 45–60 (1997)
71. Faudree, R.J., Gould, R.J., Jacobson, M.J., Lesniak, L.: Characterizing forbidden clawless triples implying Hamiltonian graphs. Discrete Math. **249**, 71–81 (2002)
72. Faudree, R.J., Gould, R.J., Jacobson, M.S.: Forbidden triples implying hamiltonicity: for all graphs. Discuss. Math. Graph Theory **24**, 47–54 (2004)
73. Faudree, R.J., Gould, R.J., Jacobson, M.S.: Potential forbidden triples implying Hamiltonicity: for sufficiently large graphs. Discuss. Math. Graph Theory **25**, 273–289 (2005)
74. Faudree, R.J., Gould, R.J., Jacobson, M.S., Lesniak, L.: Minimum degree and (k, m)-pancyclic ordered graphs. Graphs Combin. **21**, 197–211 (2005)
75. Faudree, R.J., Gould, R.J., Kostochka, A., Lesniak, L., Schiermeyer, I., Saito, A.: Degree conditions for k-ordered Hamiltonian graphs. J. Graph Theory **42**, 199–210 (2003)
76. Faudree, R.J., Gould, R.J., Jacobson, M.S.: Pancyclic graphs and linear forests. Discrete Math. (2013, to appear)

77. Faudree, R.J., Gould, R.J., Jacobson, M.S., Magnant, C.: Distributing vertices on Hamiltonian cycles. J. Graph Theory **69**, 28–45 (2012)
78. Faudree, R.J., Gould, R.J.: Precise location of vertices on Hamiltonian cycles. Discrete Math. **313**(23), 2772–2777 (2013)
79. Faudree, R.J., Gould, R.J., Jacobson, M.S., Lesniak, L.: Generalizing pancyclic and k-ordered graphs. Graphs Combin. **20**, 291–309 (2004)
80. Faudree, R.J., Lehel, J., Yoshimoto, K.: A note on locating pairs of vertices on a Hamiltonian cycle (preprint)
81. Favaron, O., Flandrin, E., Li, H., Tian, F.: An Ore-type condition for pancyclability. Discrete Math. **206**, 139–144 (1999)
82. Ferrara, M., Gould, R.J., Jacobson, M.S., Pfender, F., Powell, J., Whalen, T.: New Ore-type conditions for H-linked graphs. J. Graph Theory **71**, 69–77 (2012)
83. Ferrara, M., Gould, R.J., Tansey, G., Whalen, T.: Disjoint Hamiltonian cycles in bipartite graphs. Discrete Math. **309**, 3811–3820 (2009)
84. Ferrara, M., Gould, R.J., Tansey, G., Whalen, T.: On H-linked graphs. Graphs Combin. **22**, 217–224 (2006)
85. Ferrara, M., Jacobson, M.S., Harlan, A.: Hamiltonian cycles avoiding sets of edges in a graph. Australas. J. Combin. **48**, 191–203 (2010)
86. Ferrara, M., Magnant, C., Powell, J.: Pan-H-linked graphs. Graphs Combin. **26**, 225–242 (2010)
87. Fiedler, M., Nikiforov, V.: Spectral radius and hamiltonicity of graphs. Linear Algebra Appl. **432**, 2170–2173 (2010)
88. Flandrin, E., Li, H., Marczyk, A., Wožniak, M.: A note on pancyclism of highly connected graphs. Discrete Math. **286**, 57–60 (2004)
89. Florek, J.: On Barnette's conjecture. Discrete Math. **310**, 1531–1535 (2010)
90. Frankl, P., Katona, Gy.Y.: Extremal k-edge Hamiltonian hypergraphs. Discrete Math. **308**, 1415–1424 (2008)
91. Frieze, A.: Loose Hamilton cycles in random 3-uniform hypergraphs. Electron. J. Combin. **17**, #N283 (2010)
92. Frieze, A., Krivelevich, M.: On packing Hamilton cycles in ϵ-regular graphs. J. Combin. Theory B **94**, 159–172 (2005)
93. Frieze, A., Krivelevich, M., Loh, P.-S.: Packing tight Hamilton cycles in 3-uniform hypergraphs. Random Struct. Algorithms **40**, 269–300 (2012)
94. Fujisawa, J., Ota, K., Sugiyama, T., Tsugaki, M.: Forbidden subgraphs and the existence of paths and cycles through specified vertices. Discrete Math. **308**, 6111–6114 (2008)
95. Fujisawa, J., Nakamoto, A., Ozeki, K.: Hamiltonian cycles in bipartite toroidal graphs with a partite set of degree four vertices. J. Combin. Theory B **103**, 46–60 (2013)
96. Fujisawa, J., Yamashita, T.: Degree conditions on claws and modified claws for hamiltonicity of graphs. Discrete Math. **308**, 1612–1619 (2008)
97. Gerlach, T.: Toughness and hamiltonicity of a class of planar graphs. Discrete Math. **286**, 61–65 (2004)
98. Glebov, R., Krivelevich, M.: On the number of Hamilton cycles in sparse random graphs. SIAM J. Discrete Math. **27**, 27–42 (2013)
99. Glebov, R., Person, Y., Weps, W.: On extremal hypergraphs for Hamiltonian cycles. Eur. J. Combin. **33**, 544–555 (2012)
100. Goddard, W.: Minimum degree conditions for cycles including specified sets of vertices. Graphs Combin. **20**, 467–483 (2004)
101. Gould, R.J.: Updating the Hamiltonian problem—a survey. J. Graph Theory **15**, 121–157 (1991)
102. Gould, R.J.: Advances on the Hamiltonian problem—a survey. Graphs Combin. **19**, 7–52 (2003)
103. Gould, R.J.: A look at cycles containing specified elements of a graph. Discrete Math. **309**, 6299–6311 (2009)
104. Gould, R.J.: Graph Theory, Dover Publications, Inc., Mineola (2012)
105. Gould, R.J., Luczak, T., Pfender, F.: Pancyclicity of 3-connected graphs: pairs of forbidden subgraphs. J. Graph Theory **47**, 183–202 (2004)
106. Gould, R.J., Kostochka, A., Yu, G.: On minimum degree implying that a graph is H-linked. SIAM J. Discrete Math. **20**, 829–840 (2006)
107. Gould, R.J., Whalen, T.: Subdivision extendibility. Graphs Combin. **23**, 165–182 (2007)
108. Greenhill, C., Kim, J.H., Wormald, N.: Hamiltonian decompositions of random bipartite regular graphs. J. Combin. Theory Ser. B **90**, 195–222 (2004)

109. Grünbaum, B.: Polytopes graphs and complexes. Bull. Am. Math. Soc. **76**, 1131–1201 (1970)
110. Häggkvist, R.: On F-Hamiltonian graphs. In: Bondy, J.A., Murty, U.S.R. (eds) Graphs and Related Topics, pp. 219–231. Academic Press, New York (1979)
111. Hán, H., Schacht, M.: Dirac-type results for loose Hamilton cycles in uniform hypergraphs. J. Combin. Theory B **100**, 332–346 (2010)
112. Harkat-Benhamdine, A., Li, H., Tian, F.: Cyclability of 3-connected graphs. J. Graph Theory **34**, 191–203 (2000)
113. Hartke, S., Seacrest, T.: Random partitions and edge disjoint Hamiltonian cycles (preprint)
114. Helden, G.: Each maximal planar graph with exactly two separating triangles is Hamiltonian. Discrete Appl. Math. **155**, 1833–1836 (2007)
115. Helden, G., Vieten, O.: Hamiltonian cycles in maximal planar graphs. In: Cologne-Twente Workshop on Graphs and Combinatorial Optimization 71, Electronic Notes in Discrete Mathematics, vol. 25, Elsevier, Amsterdam (2006)
116. Holton, D.L., Manvel, B., McKay, B.D.: Hamiltonian cycles in cubic 3-connected bipartite graphs. J. Combin. Theory B **38**, 279–297 (1985)
117. Hu, Z., Tian, F., Bing, W.: Hamiltonian connectivity of line graphs and claw-free graphs. J. Graph Theory **50**, 130–141 (2005)
118. Jackson, B.: Edge-disjoint Hamiltonian cycles in regular graphs of large degree. J. Lond. Math. Soc. **19**, 13–16 (1979)
119. Jiang, T.: Planar Hamiltonian chordal graphs are cycle extendable. Discrete Math. **257**, 441–444 (2002)
120. Kaiser, T., Kriesell, M.: On the pancyclicity of lexicographic products. Graphs Combin. **22**, 51–58 (2006)
121. Kaiser, T., Vrána, P.: Hamilton cycles in 5-connected line graphs. Eur. J. Combin. **33**, 924–947 (2012)
122. Kaneko, A., Yoshimoto, K.: On a Hamiltonian cycle in which specified vertices are uniformly distributed. J. Combin. Theory Ser. B **81**, 100–109 (2001)
123. Karoński, M., Scheinerman, E., Singer-Cohen, K.: On random intersection graphs: the subgraph problem. Combin. Prob. Comput. **8**, 131–159 (1999)
124. Katona, G.Y., Kierstead, H.: Hamiltonian chains in hypergraphs. J. Graph Theory **30**, 205–212 (1999)
125. Kawarabayashi, K.: A survey on Hamiltonian cycles. Interdiscip. Inf. Sci. **7**, 25–39 (2001)
126. Keevash, P.: A hypergraph blow-up lemma. Random Struct. Algorithms **39**, 275–376 (2013)
127. Keevash, P., Kühn, D., Mycroft, R., Osthus, D.: Loose Hamilton cycles in hypergraphs. Discrete Math. **311**, 544–559 (2011)
128. Kierstead, H., Sárközy, G., Selkow, S.: On k-ordered Hamiltonian graphs. J. Graph Theory **32**, 17–25 (1999)
129. Knox, F., Kühn, D., Osthus, D.: Approximate Hamilton decompositions of random graphs. Random Struct. Algorithms **40**, 133–149 (2012)
130. Kostochka, A., Yu, G.: An extremal problem for H-linked graphs. J. Graph Theory **50**, 321–339 (2005)
131. Kriesell, M.: All 4-connected line graphs of claw-free graphs are Hamiltonian connected. J. Combin. Theory B **82**, 306–315 (2001)
132. Krivelevich, M.: On the number of Hamilton cycles in pseudo-random graphs. Electron. J. Combin. **19** (2012), Paper 24
133. Krivelevich, M., Samotij, W.: Optimal packings of Hamilton cycles in sparse random graphs. SIAM J. Discrete Math. **26**, 964–982 (2012)
134. Krivelevich, M., Sudakov, B.: Sparse pseudo-random graphs are Hamiltonian. J. Graph Theory **42**, 17–33 (2003)
135. Kronk, H.: A generalization of a theorem of Pósa. Proc. Am. Math. Soc. **21**, 77–78 (1969)
136. Kühn, D., Lapinskas, J., Osthus, D.: Optimal packings of Hamilton cycles in graphs of high minimum degree. Combin. Prob. Comput. **22**, 394–416 (2013)
137. Kühn, D., Osthus, D.: Loose Hamilton cycles in 3-uniform hypergraphs of high minimum degree. J. Combin. Theory Ser. B **96**, 767–821 (2006)
138. Kühn, D., Osthus, D.: Hamilton decompositions of regular expanders: a proof of Kelly's conjecture for large tournaments. Adv. Math. **237**, 62–146 (2013)
139. Kühn, D., Osthus, D.: A survey on Hamilton cycles in directed graphs. Eur. J. Combin. **33**, 750–766 (2012)
140. Kuipers, E.J., Veldman, H.: Recognizing claw-free Hamiltonian graphs with large minimum degree. Memo, 1437, Department of Applied Mathematics, University of Twente, Enschede (1998)

141. Lai, H.-J., Shao, Y., Yu, G., Zhan, M.: Hamiltonian connectedness in 3-connected line graphs. Discrete Appl. Math. **157**, 982–990 (2009)
142. Lai, H.-J., Shao, Y., Zhan, M.: Hamiltonicity in 3-connected claw-free graphs. J. Combin. Theory B **96**, 493–504 (2006)
143. Lai, H.-J., Shao, Y., Zhan, M.: Every 4-connected line graph of a quasi-claw-free graph is Hamiltonian connected. Discrete Math. **308**, 5312–5316 (2008)
144. Lai, H.-J., Xiong, L., Yan, H., Yan, J.: Every 3-connected claw-free Z_8-free graph is Hamiltonian. J. Graph Theory **64**, 1–11 (2010)
145. Lafond, M., Seamone, B.: Some Hamiltonian chordal graphs are not cycle extendable (preprint)
146. Las Vergnas, M.: Thesis, University of Paris, Paris (1972)
147. Leach, C.D., Rodger, C.: Hamilton decompositions of complete graphs with 3-factor leaves. Discrete Math. **279**, 337–344 (2004)
148. Leach, C.D., Rodger, C.: Hamilton decompositions of complete multipartite graphs with any 2-factor leave. J. Graph Theory **44**, 208–214 (2003)
149. Lee, C., Sudakov, B.: Dirac's theorem for random graphs. Random Struct. Algorithms **41**, 293–305 (2012)
150. Li, G., Lu, M., Liu, Z.: Hamiltonian cycles in 3-connected claw-free graphs. Discrete Math. **250**, 137–151 (2002)
151. Li, R.: Hamiltonicity of 3-connected quasi-claw-free graphs. Discrete Math. **265**, 393–399 (2003)
152. Li, R., Schelp, R.: Every 3-connected distance claw-free graph is Hamiltonian connected. Discrete Math. **268**, 185–197 (2003)
153. Li, M., Chen, X., Broersma, H.: Hamiltonian connectedness in 4-connected hourglass-free claw-free graphs. J. Graph Theory **68**, 285–298 (2011)
154. Li, J., Shen, R., Tian, F.: Cycles containing given subsets in 1-tough graphs. Ars Combin. **58**, 193–204 (2001)
155. Li, M.: Hamiltonian connected claw-free graphs. Graphs Combin. **20**, 341–362 (2004)
156. Lu, L., Székely, L.: Using Lovász local lemma in the space of random injections. Electron. J. Combin. **14**, 13 (2007) Paper 63
157. Luczak, T., Pfender, F.: Claw-free 3-connected P_{11}-free graphs are Hamiltonian. J. Graph Theory **47**, 111–121 (2004)
158. Malkevitch, J.: Polytopal graphs. In: Beineke, Wilson (eds.) Selected Topics in Graph Theory, vol. 3. Academic Press, New York, pp. 169–188 (1988)
159. Manikandan, R.S., Paulraja, P.: Hamiltonian decompositions of the tensor product of a complete graph and a complete bipartite graph. Ars Combin. **80**, 33–44 (2006)
160. Manikandan, R.S., Paulraja, P.: Hamilton cycle decompositions of the tensor product of complete multipartite graphs. Discrete Math. **308**, 3586–3606 (2008)
161. Matthews, M., Sumner, D.: Hamiltonian results in $K_{1,3}$-free graphs. J. Graph Theory **8**, 139–146 (1984)
162. Mohar, B.: A domain monotonicity theorem for graphs and hamiltonicity. Discrete Appl. Math. **36**, 169–177 (1992)
163. Moon, J., Moser, L.: On Hamiltonian bipartite graphs. Isr. J. Math. **1**, 163–165 (1963)
164. Müller, T., Pérez-Giménez, X., Wormald, N.: Disjoint Hamilton cycles in the random geometric graph. J. Graph Theory **68**, 299–322 (2011)
165. Nakamoto, A., Ozeki, K.: Hamiltonian cycles in bipartite quadrangulations on the torus. J. Graph Theory **69**, 143–151 (2012)
166. Nash-Williams, C.St.J.A.: Edge-Disjoint Hamiltonian Circuits in Graphs Having Sufficiently Large Valencies. Studies in Pure Mathematics. Academic Press, London, pp. 157–183 (1971)
167. Nash-Williams, C.St.J.A.: Unexplored and Semi-Explored Territories in Graph Theory. New Directions in Graph Theory. Academic Press, New York, pp. 169–176 (1973)
168. Nikoletseas, S., Raptopoulos, C., Spitakis, P.G.: On the independence number and Hamiltonicity of uniform random intersection graphs. Theor. Comput. Sci. **412**, 6750–6760 (2011)
169. Ng, L., Schultz, M.: k-ordered Hamiltonian graphs. J. Graph Theory **24**, 45–57 (1997)
170. Okol'nishnikova, E.A.: On the number of Hamiltonian cycles in dense Hamiltonian graphs. (Russ.) Mat. Tr. **8**, 199–206 (2005)
171. Ore, O.: A note on Hamiltonian circuits. Am. Math. Mon. **67**, 55 (1960)
172. Perez Reilly, E., Scheinerman, E.: Random threshold graphs. Electron. J. Combin. **16**, Paper 130 (2009)

173. Pfender, F.: Hamiltonicity and forbidden subgraphs in 4-connected graphs. J. Graph Theory **49**, 262–272 (2005)
174. Pike, D.A.: Hamilton decompositions of line graphs of some bipartite graph. Discuss. Math. Graph Theory **25**, 303–310 (2005)
175. Pósa, L.: On the circuits of finite graphs. Magyar Tud. Akad. Mat. Kutató Int. Közl. **8**, 355–361 (1963)
176. Qiao, S., Zhang, S.: Spanning cyclic subdivisions of vertex-disjoint cycles and chorded cycles in graphs. Graphs Combin. **28**, 277–285 (2012)
177. Robinson, R.W., Wormald, N.: Almost all regular graphs are Hamiltonian. Random Struct. Algorithms **5**, 363–374 (1994)
178. Rodger, C.: Hamiltonian decomposable graphs with specified leaves. Graphs Combin. **20**, 541–543 (2004)
179. Rödl, V., Rucinski, A., Szemerédi, E.: A Dirac-type theorem for 3-uniform hypergraphs. Combin. Probab. Comput. **15**, 229–251 (2006)
180. Rödl, V., Rucinski, A., Szemerédi, E.: An approximate Dirac-type theorem for k-uniform hypergraphs. Combinatorica **28**(2), 229–260 (2008)
181. Rödl, V., Ruciński, A., Szemerédi, E.: Dirac-type conditions for Hamiltonian paths and cycles in 3-uniform hypergraphs. Adv. Math. **227**, 1225–1229 (2011)
182. Rosenfeld, M., Barnette, D.: Hamiltonian circuits in certain prisms. Discrete Math. **5**, 389–394 (1973)
183. Ryjáček, Z.: On a closure concept in claw-free graphs. J. Combin. Theory B **70**, 217–224 (1997)
184. Ryjáček, Z., Vrána, P.: Line graphs of multigraphs and Hamilton-connectedness of claw-free graphs. J. Graph Theory **66**, 152–173 (2011)
185. Rybarczyk, K.: Sharp threshold functions for random intersection graphs via a coupling method. Electron. J. Combin. **18**, Paper 36 (2011)
186. Sakai, T.: Long paths and cycles through specified vertices in k-connected graphs. Ars Combin. **58**, 33–65 (2001)
187. Sanders, D.P.: On paths in planar graphs. J. Graph Theory **24**, 341–345 (1997)
188. Sárkozy, G., Selkow, S., Szemerédi, E.: On the number of Hamiltonian cycles in Dirac graphs. Discrete Math. **265**, 237–250 (2003)
189. Sárkozy, G., Selkow, S.: Distributing vertices along a Hamiltonian cycle in Dirac graphs. Discrete Math. **308**, 5757–5770 (2008)
190. Sciriha, I., Cardoso, D.M.: Necessary and sufficient conditions for a Hamiltonian graph. J. Combin. Math. Combin. Comput. **80**, 127–150 (2012)
191. Shepherd, F.B.: Hamiltonicity in claw-free graphs. J. Combin. Theory B **53**, 173–194 (1991)
192. Skupien, Z.: Sparse Hamiltonian 2-decompositions together with exact count of numerous Hamilton cycles. Discrete Math. **309**, 6382–6390 (2009)
193. Sudakov, B., Vu, V.: Local resilience of graphs. Random Struct. Algorithms **33**, 409–433 (2008)
194. Sugiyama, T.: Hamiltonian cycles through a linear forest. Sut. J. Math. **40**, 103–109 (2004)
195. Tait, P.G.: Remark on the coloring of maps. Proc. R. Soc. Edinb. **10**, 729 (1880)
196. Thomas, R., Yu, X.: 4-connected projective planar graphs are Hamiltonian. J. Combin. Theory B **62**, 114–132 (1994)
197. Thomas, R., Yu, X.: Five-connected toroidal graphs are Hamiltonian. J. Combin. Theory B **69**, 79–96 (1997)
198. Thomas, R., Yu, X., Zang, W.: Hamilton paths in toroidal graphs. J. Combin. Theory B **94**, 214–236 (2005)
199. Thomassen, C.: A theorem on paths in planar graphs. J. Graph Theory **9**, 169–176 (1983)
200. Thomason, A.G.: Hamiltonian cycles and uniquely edge colorable graphs. Ann. Discrete Math. **3**, 259–268 (1978)
201. Thomassen, C.: Reflections on graph theory. J. Graph Theory **10**, 309–324 (1986)
202. Tian, F., Wei, B.: Pancyclicity mod k of $K_{1,4}$-free graphs. Adv. Math. (China) **34**, 221–232 (2005)
203. Tutte, W.: A theorem on planar graphs. Trans. Am. Math. Soc. **82**, 309–324 (1956)
204. Tuza, Z.: Steiner system and large non-Hamiltonian hypergraphs. Matematiche **61**, 179–183 (2006)
205. van den Heuvel, J.: Hamilton cycles and eigenvalues of graphs. Linear Algebra Appl. **226–228**, 723–730 (1995)
206. Verrall, H.: Hamiltonian decompositions of complete 3-uniform hypergraphs. Discrete Math. **132**, 333–348 (1994)
207. Westland, E., Liu, J., Kreher, D.: 6-regular Cayley graphs on abelian groups of odd order are Hamiltonian decomposable. Discrete Math. **309**, 5106–5110 (2009)

208. Whitney, H.: A theorem on graphs. Ann. Math. **32**, 378–390 (1931)
209. Witte, D., Gallian, J.: A survey: Hamiltonian cycles in Cayley graphs. Discrete Math. **51**, 293–304 (1984)
210. Yang, Z.: Note on F-Hamiltonian graphs. Discrete Math. **196**, 281–286 (1999)
211. Yu, X.: Disjoint paths, planarizing cycles, and spanning walks. Trans. Am. Math. Soc. **349**, 1333–1358 (1997)
212. Zhou, J., Lin, C., Hu, G.: Spectral radius of Hamiltonian planar graphs and outerplanar graphs. Tsinghua Sci. Technol. **6**, 350–354 (2001)